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How the augmented Lagrangian algorithm can deal with an infeasible convex quadratic optimization problem — Motivation, analysis, implementation

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A tribute to Michael James David POWELL (1936-2015) . . .

Since you ask me to mention a gratifying paper, let me pick "A method for nonlinear constraints in minimization problems", because it is regarded as one of the sources of the "augmented Lagrangian method", which is now of fundamental importance in mathematical programming. I have been very fortunate to have played a part in discoveries of this kind.

M.J.D. Powell [19; 2003]

Outline

A brief overview of numerical nonlinear optimization

The problem to solve

- A standard generic nonlinear optimization problem consists in

$$(P_{EI}) \quad \begin{cases} \inf_x \; f(x) \\ c_E(x) = 0 \\ c_I(x) \leqslant 0, \end{cases}$$

  where $f : \mathbb{R}^n \to \mathbb{R}$, $c_E : \mathbb{R}^n \to \mathbb{R}^{m_E}$, and $c_I : \mathbb{R}^n \to \mathbb{R}^{m_I}$ are smooth (possibly non convex) functions.

- Sometimes we will consider simplified a version (to avoid being cumbersome), namely

$$(P_I) \quad \begin{cases} \inf_x \; f(x) \\ c_I(x) \leqslant 0. \end{cases}$$

A primal algorithm gives priority to the *visible* or primal variables x.

Main ideas

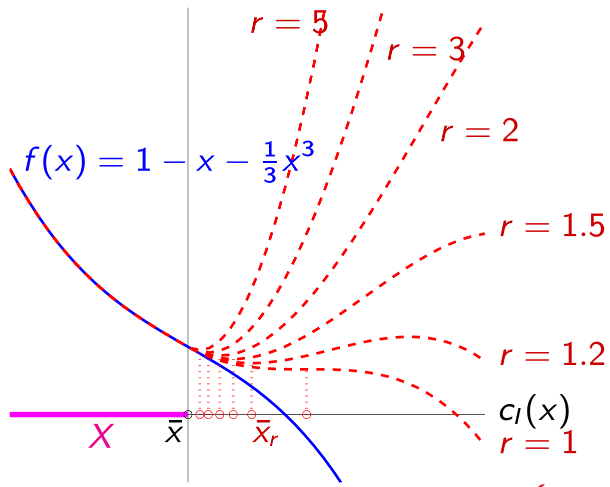
- penalize the constraints with penalty parameter $r \to$ (some limit),
- apply an unconstrained algorithm to solve the penalized problem.

Example 1: exterior penalization (quadratic penalization)

$$(P_I) \quad \begin{cases} \inf_x \ f(x) \\ c_I(x) \leqslant 0 \end{cases} \qquad \curvearrowright \qquad (P_{I,r}) \quad \inf_x \left(f(x) + \frac{r}{2}\|c_I(x)^+\|_2^2 \right).$$

Pros and cons

- ⊕ Easy to implement.
- ⊖ Sequence of problems to solve.
- ⊖ Ill-conditioning.

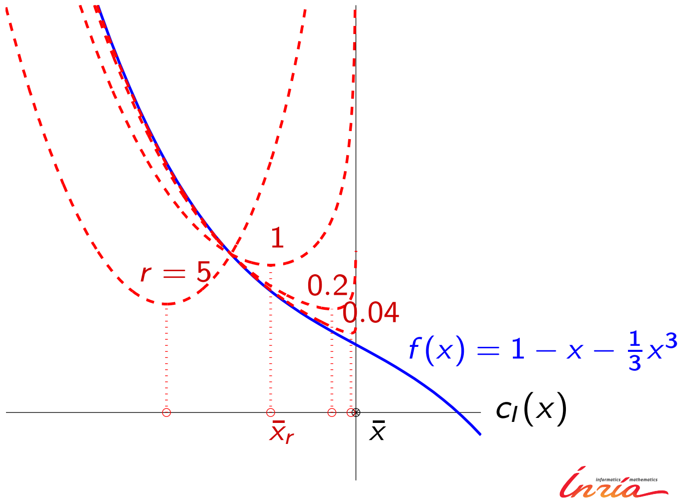


Example 2: interior penalization (interior point methods)

$$(P_I) \quad \begin{cases} \inf_x \ f(x) \\ c_I(x) \leqslant 0 \end{cases} \qquad \curvearrowright \qquad (P_{I,r}) \quad \inf_x \left(f(x) - r \sum_{i \in I} \log |c_i(x)| \right).$$

Pros and cons

- ⊕ Easy to implement.
- ⊖ Sequence of problems to solve.
- ⊖ Ill-conditioning.
- ⊕ Each problem $(P_{I,r})$ can be solved inexactly (a single Newton step, in linear optimization).



$\boxed{\text{A dual algorithm gives priority to the } \textit{hidden} \text{ or dual variables } \lambda.}$

- The *hidden* variables are revealed by the optimality conditions (= local description of optimality).
- If x_* is a *local* solution to (P_{EI}) (+ smoothness and qualification assumptions), there exist multipliers or dual variables $\lambda_* \in \mathbb{R}^m$ such that

$$(\text{KKT}) \quad \begin{cases} \nabla_x \ell(x_*, \lambda_*) = 0 \\ c_E(x_*) = 0 \\ 0 \leqslant (\lambda_*)_I \perp c_I(x_*) \leqslant 0. \end{cases}$$

  where
  - KKT = Karush-Kuhn-Tucker,
  - Lagrangian function $\ell(x, \lambda) = f(x) + \lambda^{\mathsf{T}} c(x) = f(x) + \sum_i \lambda_i c_i(x)$.

How to generate dual iterates?

- For some coupling function $\varphi : X \times \Lambda \to \mathbb{R}$, write (P_{EI}) as an infsup:

$$(P_{EI}) \qquad \inf_{x \in X} \sup_{\lambda \in \Lambda} \varphi(x, \lambda).$$

- The dual problem then reads

$$(D_{EI}) \qquad \sup_{\lambda \in \Lambda} \inf_{x \in X} \varphi(x, \lambda) = - \inf_{\lambda \in \Lambda} \underbrace{\left(\sup_{x \in X} -\varphi(x, \lambda) \right)}_{\delta(\lambda)}.$$

- Generate the dual iterates by minimizing on Λ the dual function

$$\lambda \in \Lambda \mapsto \delta(\lambda) := \sup_{x \in X} -\varphi(x, \lambda) \in \bar{\mathbb{R}}.$$

How to chose the coupling function φ?

- The problem (P_{EI}) must be identical to

$$\inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \Lambda} \varphi(x, \lambda).$$

- In some sense, (D_{EI}) must be "equivalent" to (P_{EI}).
  Ensured if a PD solution (x_*, λ_*) to (P_{EI}) is a saddle-point of φ:

$$\forall x \in \mathbb{R}^n, \quad \forall \lambda \in \Lambda : \quad \varphi(x_*, \lambda) \leqslant \varphi(x_*, \lambda_*) \leqslant \varphi(x, \lambda_*).$$

Lagrangian relaxation

- The problem (P_{EI}) can be written
$$\inf_{x\in\mathbb{R}^n}\ \sup_{\lambda\in\Lambda}\ \underbrace{f(x)+\lambda_E^{\mathsf{T}}c_E(x)+\lambda_I^{\mathsf{T}}c_I(x)}_{\ell(x,\lambda)},$$
where $\Lambda := \{\lambda\in\mathbb{R}^m : \lambda_I \geqslant 0\}$.
- Hence the dual problem (D_{EI}) consists in minimizing the dual function
$$\lambda\in\mathbb{R}^m \mapsto \delta(\lambda) := \left(\sup_{x\in\mathbb{R}^n} -\ell(x,\lambda)\right) + \mathcal{I}_\Lambda(\lambda) \in \overline{\mathbb{R}},$$
which is nonsmooth, convex, and closed (i.e., l.s.c.).
- Saddle-point at a KKT point (x_*,λ_*) if (P_{EI}) is convex.
- Typical (and difficult) algorithm: bundle method [17].

Augmented Lagrangian relaxation (multiplier method)

- For any $r > 0$, problem (P_I) can also be written $(c_I(x)+y=0,\ y\geqslant 0)$
$$\inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m_+}\ \sup_{\lambda\in\mathbb{R}^m}\ \underbrace{f(x)+\lambda^{\mathsf{T}}(c_I(x)+y)+\frac{r}{2}\|c_I(x)+y\|_2^2}_{\ell_r(x,y,\lambda)},$$
where ℓ_r is called the augmented Lagrangien.
- Hence the dual problem (D_{EI}) consists in minimizing the dual function
$$\lambda\in\mathbb{R}^m \mapsto \delta_r(\lambda) := \sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m_+} -\ell_r(x,y,\lambda), \qquad \boxed{\text{solution } (x_+,y_+)}$$
which is smooth $(C^{1,1})$, convex, and closed.
- Local saddle-point at a KKT+SOC2 point (x_*,λ_*) if r is large enough.
- Easy algorithm: $\boxed{\lambda_+ := \lambda + r\,[c_I(x_+)+y_+]}$ [16, 18, 21, 4, 1, 23, 24].

Outline of the augmented Lagrangian (AL) algorithm

One iteration: from $(\lambda_k, r_k) \in \mathbb{R}^m \times \mathbb{R}_{++}$ to (λ_{k+1}, r_{k+1}).

- Compute (if possible, exit otherwise)

$$(x_{k+1}, y_{k+1}) \in \underset{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m_+}{\arg\min} \ \ell_{r_k}(x, y, \lambda_k). \tag{1}$$

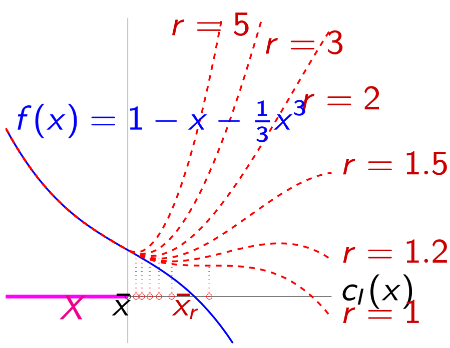
- Update the multipliers by $\lambda_{k+1} = \lambda_k + r_k[c_I(x_{k+1}) + y_{k+1}]$.
- Stop if $[c_I(x_{k+1}) + y_{k+1}] \simeq 0$.
- Update $r_k \curvearrowright r_{k+1}$ ...

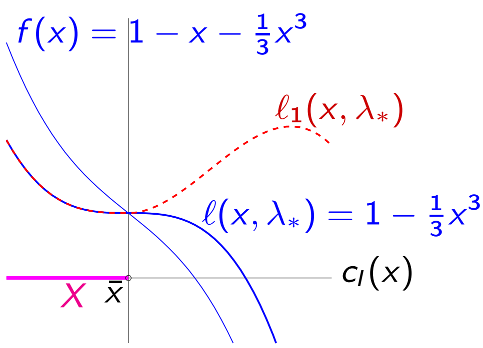
Pros and cons

- $\oplus$ Do not require convexity (but easier if (P_{EI}) is convex).
- $\oplus$ Convergence well understood if (P_{EI}) is convex.
- $\ominus$ A sequence of nonlinear optimization problems to solve in (1).
- $\ominus$ (1) sometimes difficult ($y \geqslant 0$, destroy decomposition, ill-conditioning).
- $\ominus$ Update of r_k is tricky.

Another point of view on the augmented Lagrangian

- The original idea [16, 18] was to penalize $\ell(\cdot, \lambda_*)$ instead of f because this yields
  - exactness (solving a single penalty problem),
  - better conditioning (r large but not infinite).





- Since λ_* is not known, an iterative process must generate $\lambda_k \to \lambda_*$ (by minimizing the dual function).

An important property of the AL algorithm, when (P_{EI}) is convex

$$\boxed{\text{AL algorithm = proximal algorithm on the dual function } \delta.}$$

- The proximal algorithm on the dual function δ computes λ_{k+1} from λ_k by

$$\lambda_{k+1} = \operatorname*{arg\,min}_{\lambda \in \mathbb{R}^m} \left(\delta(\lambda) + \frac{1}{2r_k} \|\lambda - \lambda_k\|^2 \right).$$

  Optimality: $\exists\, s_{k+1} \in \partial\delta(\lambda_{k+1})$ such that $0 = s_{k+1} + \frac{1}{r_k}(\lambda_{k+1} - \lambda_k)$ or

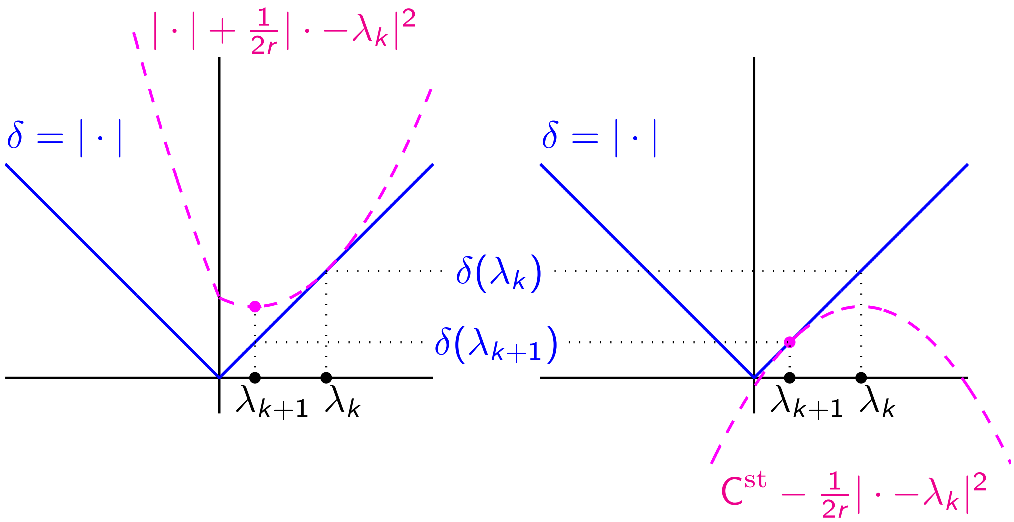
$$\lambda_{k+1} = \lambda_k - r_k s_{k+1}, \quad \text{for some } s_{k+1} \in \partial\delta(\lambda_{k+1}).$$

  Hence it is an implicit subgradient method (implicit Euler).

- One writes

$$\lambda_{k+1} = \operatorname{prox}_{\delta, r_k}(\lambda_k)$$

- With pictures:



Proposition (Rockafellar [22; 1973])

- *If* $\delta \in \overline{\mathrm{Conv}}(\mathbb{R}^m)$ *and* $r_k > 0$, *then*

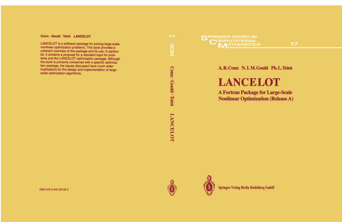
$$-\inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m_+} \ell_{r_k}(x,y,\lambda_k) = \inf_{\lambda\in\mathbb{R}^m} \left(\delta(\lambda) + \frac{1}{2r_k}\|\lambda-\lambda_k\|^2\right).$$

- *Any solution* (x_{k+1}, y_{k+1}) *to the problem in the LHS and the unique solution* λ_{k+1} *to the problem in the RHS are linked by*

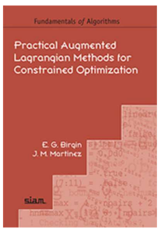
$$\left\{\begin{array}{l} \lambda_{k+1} = \lambda_k + r_k[c_I(x_{k+1}) + y_{k+1}] \\ -[c_I(x_{k+1}) + y_{k+1}] \in \partial\delta(\lambda_{k+1}). \end{array}\right.$$

Hence the multiplier computed by the AL algorithm is $\lambda_{k+1} = \mathrm{prox}_{\delta,r_k}(\lambda_k)$.

Codes implementing the AL for nonlinear optimization

- 

Lancelot
Conn, Gould et Toint [6; 1992]

- 

Algencan
Birgin et Martínez [2; 2014]

A primal-dual algorithm generates a PD sequence $\{(x_k, \lambda_k)\}$

- Consider the generic problem

$$(P_{EI}) \quad \begin{cases} \inf_x \ f(x) \\ c_E(x) = 0 \\ c_I(x) \leqslant 0, \end{cases}$$

- The classical primal-dual algorithm works on the first order optimality conditions directly

$$(\text{KKT}) \quad \begin{cases} \nabla_x \ell(x_*, \lambda_*) = 0 \\ c_E(x_*) = 0 \\ 0 \leqslant (\lambda_*)_I \perp c_I(x_*) \leqslant 0. \end{cases}$$

- "Linearization" gives the displacement (d, μ) of (x, λ):

$$(\text{KKT}') \quad \begin{cases} \nabla_x \ell(x_k, \lambda_k) + \nabla^2_{xx} \ell(x_k, \lambda_k) d + c'(x_k)^\mathsf{T} \mu = 0 \\ c_E(x_k) + c'_E(x_k) d = 0 \\ 0 \leqslant (\lambda_k + \mu)_I \perp (c_I(x_k) + c'_I(x_k) d) \leqslant 0. \end{cases}$$

- The system (KKT') is formed of the first order optimality conditions of the following osculating quadratic problem in d:

$$(\text{OQP}) \quad \begin{cases} \inf_d \ \nabla f(x_k)^\mathsf{T} d + \frac{1}{2} d^\mathsf{T} \nabla^2_{xx} \ell(x_k, \lambda_k) d \\ c_E(x_k) + c'_E(x_k) d = 0 \\ c_I(x_k) + c'_I(x_k) d \leqslant 0, \end{cases}$$

whose multipliers are $\lambda_k^{\mathrm{QP}} := \lambda_k + \mu$.

- One iteration of the local SQP/SQO algorithm: from (x_k, λ_k) to (x_{k+1}, λ_{k+1})
  - If possible, solve (OQP), to get d_k and λ_k^{QP}.
  - Update $x_{k+1} := x_k + d_k$ and $\lambda_{k+1} := \lambda_k^{\mathrm{QP}}$.

In the sequel:

> Analyse/implement an AL algorithm to the solve efficiently the OQP of the SQP algorithm.

The problem to solve

$$(P) \quad \left\{ \begin{array}{l} \inf_{x \in \mathbb{R}^n} q(x) \\ l \leqslant Ax \leqslant u, \end{array} \right. \tag{2}$$

where q is a convex quadratic function defined at $x \in \mathbb{R}^n$ by

$$q(x) = g^{\mathsf{T}} x + \frac{1}{2} x^{\mathsf{T}} H x$$

and

- $g \in \mathbb{R}^n$
- $H \succcurlyeq 0$ (NP-hard otherwise, (P) encompasses linear optimization),
- A is $m \times n$,
- $l, u \in \overline{\mathbb{R}}^m$ satisfy $l < u$.

Also equality constraints in all solvers.

The problem is polynomial and can be solved by

- active-set methods $\to$ probably non-polynomial,
- interior-point methods $\to$ polynomial,
- nonsmooth methods $\to$ polynomial on subclasses,
- other methods (including the augmented Lagrangian method).

Has this discipline been fully explored in the XXth century?

Observation 1. Odd behavior of Quadprog (Matlab). If the data is

$$g = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad x \geqslant \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

Quadprog-`active-set` answers

```
Exiting:  the solution is unbounded and at infinity;
Function value:   3.20000e+33
```

Very odd, since the problem has a *unique* solution, which is

$$x = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathrm{val}(P) = -1.5.$$

It is a benign flaw, since if $H \curvearrowright H + \varepsilon I$, Quadprog finds a near solution.

Convex quadratic optimization

Can one still make progress in convex quadratic optimization?

Quadprog-`reflective-trust-region` (default algorithm) answers

```
Optimization terminated:  relative function value changing by
 less than OPTIONS.TolFun.
Function value:  -1.5
```

Correct answer!

Conclusion: the good algorithm may depend on the problem.

Convex quadratic optimization

Can one still make progress in convex quadratic optimization?

Observation 2. On the *solvable* convex QPs of the CUTEst collection:

- first group: 138 problems, solvers in Fortran or C++,
- second group: 58 problems ($n \leqslant 500$), solver in Matlab.

Solvers	% failure	% too slow	% infeasibility	% other
Qpa (AS)	30 %	15 %	15 %	–
Qpb (IP)	20 %	5 %	2 %	13 %
Ooqp (IP)	54 %	1 %	12 %	41 %
Quadprog (AS)	33 %	12 %	19 %	2 %

- "too slow": requires more than 600 seconds,
- "infeasibility": wrong diagnosis of infeasibility,
- "other": "too small stepsize", "too small direction", "ill-conditioning", and "unknown".

The problem does not come from some very difficult QPs.
For example, on the CUTEst problem QSCTAP1 ($n = 480$, $n_b = 480$ lower bounds, $m_I = 180$ lower bounds, $m_E = 120$):

- Qpa claims that the problem is unbounded,
- Qpb claims that the problem has a solution,
- Ooqp claims that the problem is infeasible,
- Quadprog stops on a too large number of iterations ($\geqslant 10^4$).

$\Longrightarrow$ Still progress to do.

Observation 3 (more important).

Most (all?) solvers do not give appropriate information
when the QP is special, they just return a flag.

- Special means $\text{val}(P) \notin \mathbb{R}$ below:
  - $\text{val}(P) \in \mathbb{R} \iff$ the problem has a solution (Frank-Wolfe [10; 1956]),
  - $\text{val}(P) = -\infty \iff$ the problem is feasible and unbounded,
  - $\text{val}(P) = +\infty \iff$ the problem is infeasible.
- Appropriate means useful when the QP solver is used in the SQP algorithm for solving a nonlinear optimization problem.

Towards the AL algorithm

- The problem is transformed by using an auxiliary variable y:

$$(P) \quad \left\{ \begin{array}{l} \inf_{x\in\mathbb{R}^n} \ q(x) \\ l \leqslant Ax \leqslant u \end{array} \right. \qquad \curvearrowright \qquad (P') \quad \left\{ \begin{array}{l} \inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \ q(x) \\ Ax = y \\ l \leqslant y \leqslant u. \end{array} \right.$$

- Equality constraints penalized by the augmented Lagrangian

$$\ell_r(x, y, \lambda) := q(x) + \lambda^{\mathsf{T}}(Ax - y) + \frac{r}{2}\|Ax - y\|^2.$$

- At each iteration the AL algorithm [16, 18, 21, 4, 1, 23, 24; 1969-74] solves

$$\inf_{(x,y)\in\mathbb{R}^n\times[l,u]} \ell_r(x, y, \lambda). \tag{3}$$

- The AL algorithm makes sense if it is easier to solve (3) than (P).

The AL algorithm for a solvable convex QP

One iteration, from $(\lambda_k, r_k) \in \mathbb{R}^m \times \mathbb{R}_{++}$ to (λ_{k+1}, r_{k+1}):

- Compute (if possible, exit otherwise)

$$(x_{k+1}, y_{k+1}) \in \underset{(x,y)\in\mathbb{R}^n\times[l,u]}{\arg\min} \ \ell_{r_k}(x, y, \lambda_k).$$

- Update the multipliers

$$\lambda_{k+1} = \lambda_k - r_k s_{k+1}, \quad \text{where } s_{k+1} := y_{k+1} - Ax_{k+1}.$$

- Stop if

$$s_{k+1} \simeq 0.$$

- Update $r_k \curvearrowright r_{k+1} > 0$: $\rho_k := \|s_{k+1}\|/\|s_k\|$ and

$$r_{k+1} := \max\left(1, \frac{\rho_k}{\rho_{\mathrm{des}}}\right) r_k.$$

Interpretation of the AL algorithm

One iteration, from $(\lambda_k, r_k) \in \mathbb{R}^m \times \mathbb{R}_{++}$ to (λ_{k+1}, r_{k+1}):

- Compute (if possible, exit otherwise)

$$(x_{k+1}, y_{k+1}) \in \underset{(x,y)\in\mathbb{R}^n\times[l,u]}{\arg\min} \; \ell_{r_k}(x, y, \lambda_k).$$

- Update the multipliers

$$\boxed{\lambda_{k+1} = \lambda_k - r_k s_{k+1}, \quad \text{where } s_{k+1} := y_{k+1} - Ax_{k+1}.}$$

- Stop if

$$s_{k+1} \simeq 0.$$

- Update $r_k \curvearrowright r_{k+1} > 0$: $\rho_k := \|s_{k+1}\|/\|s_k\|$ and

$$r_{k+1} := \max\left(1, \frac{\rho_k}{\rho_{\rm des}}\right) r_k.$$

- The dual function $\delta : \mathbb{R}^m \to \overline{\mathbb{R}}$, defined at $\lambda \in \mathbb{R}^m$ by

$$\delta(\lambda) := - \inf_{(x,y)\in\mathbb{R}^n\times[l,u]} \left(q(x) + \lambda^{\mathsf{T}}(Ax - y)\right).$$

  - δ is convex, closed, and $\delta > -\infty$.
  - $\operatorname{dom} \delta \neq \varnothing \quad \Longleftrightarrow \quad \delta \not\equiv +\infty \quad \Longleftrightarrow \quad \delta \in \overline{\mathrm{Conv}}(\mathbb{R}^m)$.
  - Piecewise quadratic (quadratic on each orthant).

- If $(P) \equiv (P')$ has a solution:

$$0 \in \partial\delta(\bar\lambda) \quad \Longleftrightarrow \quad \bar\lambda \text{ is a dual solution to } (P').$$

- The AL algorithm looks for a

$$\bar\lambda \in \arg\min \delta.$$

AL iterates minimizing the dual function for a solvable QP

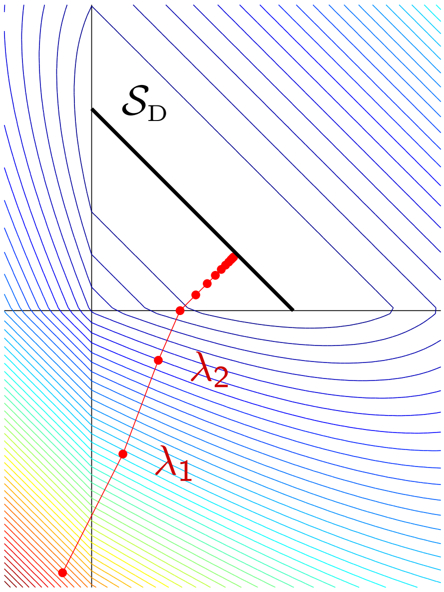
- δ is piecewise quadratic

$$\delta(\lambda) = \frac{1}{2}\lambda^{\mathsf{T}} S\lambda + (v+y_\lambda)^{\mathsf{T}}\lambda + \mathrm{C}^{\mathrm{st}}$$

- $\mathcal{S}_{\mathrm{D}} := \arg\min \delta$
- $\partial\delta(\lambda_{k+1})$ contains

$$\frac{\lambda_k - \lambda_{k+1}}{r_k} = y_{k+1} - Ax_{k+1}$$

- small r_k's in the figure



Motivation of the update rule of the penalty parameters

One iteration, from $(\lambda_k, r_k) \in \mathbb{R}^m \times \mathbb{R}_{++}$ to (λ_{k+1}, r_{k+1}):

- Compute (if possible, exit otherwise)

$$(x_{k+1}, y_{k+1}) \in \underset{(x,y)\in\mathbb{R}^n\times[l,u]}{\arg\min}\; \ell_{r_k}(x, y, \lambda_k).$$

- Update the multipliers

$$\lambda_{k+1} = \lambda_k - r_k s_{k+1}, \quad \text{where } s_{k+1} := y_{k+1} - Ax_{k+1}.$$

- Stop if

$$s_{k+1} \simeq 0.$$

- Update $r_k \curvearrowright r_{k+1} > 0$: $\rho_k := \|s_{k+1}\|/\|s_k\|$ and

$$\boxed{r_{k+1} := \max\left(1, \frac{\rho_k}{\rho_{\mathrm{des}}}\right) r_k.}$$

- The update rule of r_k is based on the following global linear convergence result [8; 2005].
  - If (P) has a solution, then the dual solution set $\mathcal{S}_\mathrm{D} \neq \varnothing$ and

$$\boxed{\begin{array}{c} \forall\, \beta > 0, \quad \exists\, L > 0, \quad \mathrm{dist}_{\mathcal{S}_\mathrm{D}}(\lambda_0) \leqslant \beta \quad \text{implies that} \\ \forall\, k \geqslant 1, \quad \|s_{k+1}\| \leqslant \min\left(1, \frac{L}{r_k}\right) \|s_k\|, \end{array}} \tag{4}$$

    where $s_k := y_k - Ax_k$.
  - (4) comes from a quasi-global error bound on the dual solution set $\mathcal{S}_\mathrm{D}$:

$$\boxed{\begin{array}{c} \text{for any bounded set } \mathcal{B} \subset \mathbb{R}^m, \text{ there is an } L > 0, \text{ such that} \\ \forall\, \lambda \in \mathcal{S}_\mathrm{D} + \mathcal{B} : \quad \mathrm{dist}_{\mathcal{S}_\mathrm{D}}(\lambda) \leqslant L \left(\inf_{s \in \partial\delta(\lambda)} \|s\| \right). \end{array}} \tag{5}$$

  - The Lipschitz constant L is difficult to deduce from the data ...

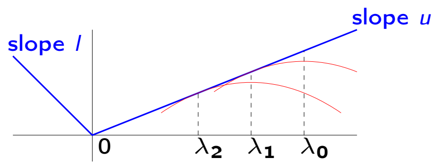
The Lipschitz constant L is difficult to deduce from the data ...

- Let $m = 1$ and $l < 0 < u$. Consider the problem

$$\left\{ \begin{array}{l} \inf 0 \\ l \leqslant 0x \leqslant u, \end{array} \right.$$

- The dual function reads

$$\delta(\lambda) = \left\{ \begin{array}{ll} l\lambda & \text{if } \lambda \leqslant 0 \\ u\lambda & \text{if } \lambda > 0. \end{array} \right.$$



- Hence $\mathcal{S}_\mathrm{D} = \{0\}$ and the quasi-global error bound reads

$$\forall\, B > 0, \quad \exists\, L > 0, \quad |\lambda| \leqslant B \quad \Longrightarrow \quad |\lambda| \leqslant \left\{ \begin{array}{ll} -Ll & \text{if } \lambda < 0 \\ 0 & \text{if } \lambda = 0 \\ Lu & \text{if } \lambda > 0. \end{array} \right.$$

- Therefore, for $\mathcal{B}$ fixed, $L \nearrow \infty$ when $l \nearrow 0$ or $u \searrow 0$ (fix λ in the error bound).

The rule of the *nonlinear* solver Algencan [2; 2014]:

$$r_0 = P_{[10^{-8},10^{+8}]}\left(10\,\frac{\max(1,|q(x_0)|)}{\max(1,\|Ax_0-y_0\|^2)}\right).$$

- Motivation: balancing the objective and constraint parts of the ℓ_2 penalty function.
- In the previous example, the rule yields (whatever is l and u):

$$r_0 = 10.$$

- It does not catch the following fact:

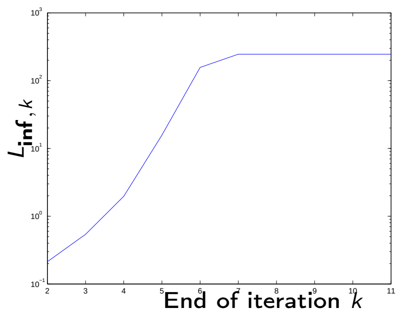
for some problems, the appropriate r depends on
the distance from the optimal constraint value $A\bar{x}$ to $[l,u]^c$.

In Oqla/Qpalm, L is guessed and r_k is set by the observation of $\rho_k := \|s_{k+1}\|/\|s_k\|$, thanks to the global linear convergence:

$$\forall\,\beta>0,\quad \exists\,L>0,\quad \operatorname{dist}_{\mathcal{S}_{\mathrm{D}}}(\lambda_0)\leqslant\beta \quad \text{implies that}$$
$$\forall\,k\geqslant 1,\quad \|s_{k+1}\|\leqslant\frac{L}{r_k}\,\|s_k\|.$$

- Lower bound of L:

$$L_{\mathrm{inf},k} := \max_{1\leqslant i\leqslant k}\ \rho_i r_i.$$



- Setting of r_{k+1}:

$$r_{k+1} = \frac{L_{\mathrm{inf},k}}{\rho_{\mathrm{des}}}.$$

- With $\rho_{\mathrm{des}} = 1/10$, convergence occurs in 10..15 AL iterations.

Effect of the update rule of r_k for infeasible QPs

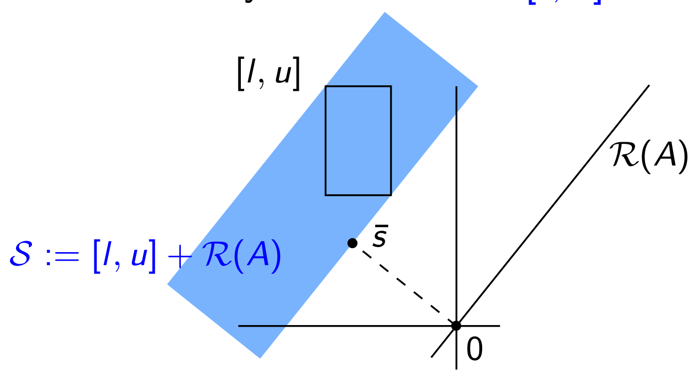
If the QP is infeasible:

- $\|s_k\| \searrow \sigma > 0$ and
$$\rho_k := \frac{\|s_{k+1}\|}{\|s_k\|} \to 1,$$
- the rule (increases r_k whenever $\rho_k > \rho_{\text{des}}$ [$\rho_{\text{des}} < 1$]) $\Longrightarrow r_k \nearrow \infty$,
- the AL subproblems become ill-conditioned,
- could stop when $r_k \geqslant \bar{r}$, but
  - difficult to find a universal threshold $\bar{r}$,
  - no information on the problem on return.

Can one have a global linear convergence when the QP is infeasible?

The smallest feasible shift

- It is always possible to find a shift $s \in \mathbb{R}^m$ such that
$$l \leqslant Ax + s \leqslant u \text{ is feasible for } x \in \mathbb{R}^n.$$
- These feasible shifts are exactly those in $\mathcal{S} := [l, u] + \mathcal{R}(A)$:



- The smallest feasible shift $\bar{s} := \arg\min\{\|s\| : s \in \mathcal{S}\}$.
$$\bar{s} = 0 \quad \Longleftrightarrow \quad \text{(P) is feasible.}$$

The closest feasible problem

The shifted QPs (feasible iff $s \in \mathcal{S}$, may be unbounded)

$$(P_s) \quad \begin{cases} \inf_x \, q(x) \\ l \leqslant Ax + s \leqslant u \end{cases} \qquad \text{and} \qquad (P'_s) \quad \begin{cases} \inf_x \, q(x) \\ Ax + s = y \\ l \leqslant y \leqslant u. \end{cases} \tag{6}$$

The closest feasible problems (feasible, may be unbounded)

$$(P_{\bar{s}}) \quad \begin{cases} \inf_x \, q(x) \\ l \leqslant Ax + \bar{s} \leqslant u. \end{cases} \qquad \text{and} \qquad (P'_{\bar{s}}) \quad \begin{cases} \inf_x \, q(x) \\ Ax + \bar{s} = y \\ l \leqslant y \leqslant u. \end{cases} \tag{7}$$

Claims clarified below ([26, 5])

- *The AL algorithm actually "solves" the closest feasible problem* $(P_{\bar{s}})$.
- *The speed of convergence is globally linear.*

When is the AL algorithm well defined?

Proposition ([5])

For the convex QP (2), the following properties are equivalent:

(*i*) $\operatorname{dom} \delta \neq \varnothing \qquad (\Longleftrightarrow \delta \not\equiv +\infty \Longleftrightarrow \delta \in \overline{\mathrm{Conv}}(\mathbb{R}^m))$,
(*ii*) *for some/any* $s \in \mathcal{S}$, *the shifted QP* (6) *is solvable,*
(*iii*) *for some/any* $r > 0$ *and* $\lambda \in \mathbb{R}^m$, *the AL subproblem* (3) *is solvable,*
(*iv*) *there is no* $d \in \mathbb{R}^n$ *such that* $g^{\mathsf{T}} d < 0$, $Hd = 0$, *and* $Ad \in [l, u]^\infty$.

- C^∞ denotes the asymptotic/recession cone of a convex set C.
- A direction like d in (*iv*) is called here an unboundedness direction.
- The failure of these conditions can be detected on the first AL subproblem (3), by finding a direction d such that

$$g^{\mathsf{T}} d < 0, \qquad Hd = 0, \qquad \text{and} \qquad Ad \in [l, u]^\infty.$$

- Fundamental assumption: (*i*)-(*iv*) holds from now on.

The AL algorithm

Convergence for an infeasible QP (val$(P) = +\infty$)

Feasibility and dual function

- No duality gap:

$$\boxed{\text{the QP is feasible} \quad \iff \quad \delta \text{ is bounded below.}}$$

  - [$\Rightarrow$] (contrapositive) true for any convex problem by weak duality.
  - [$\Leftarrow$] (contrapositive) $\delta \not\equiv +\infty$ and $\delta \to -\infty$ along $\bar{s} \neq 0$ ($\mathcal{S}$ is closed).

- Consequence for a convex QP:

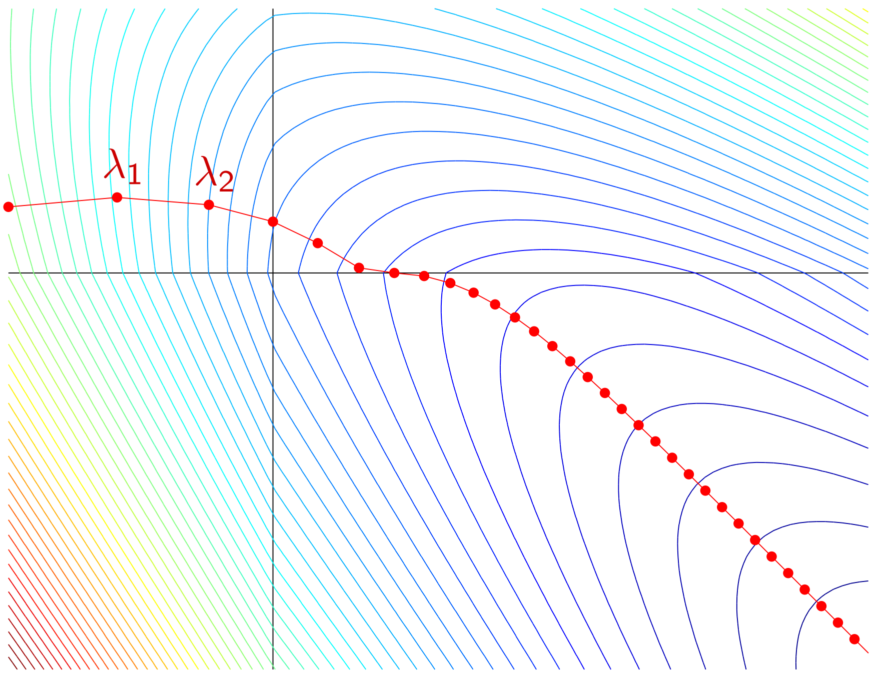
$$\begin{array}{lll} \text{the QP is infeasible} & \Longrightarrow & \delta \text{ is unbounded below} \\ & \Longrightarrow & \{\lambda_k\} \text{ blows up} \\ & & \text{(by the proximal interpretation).} \end{array}$$

- One can say more.

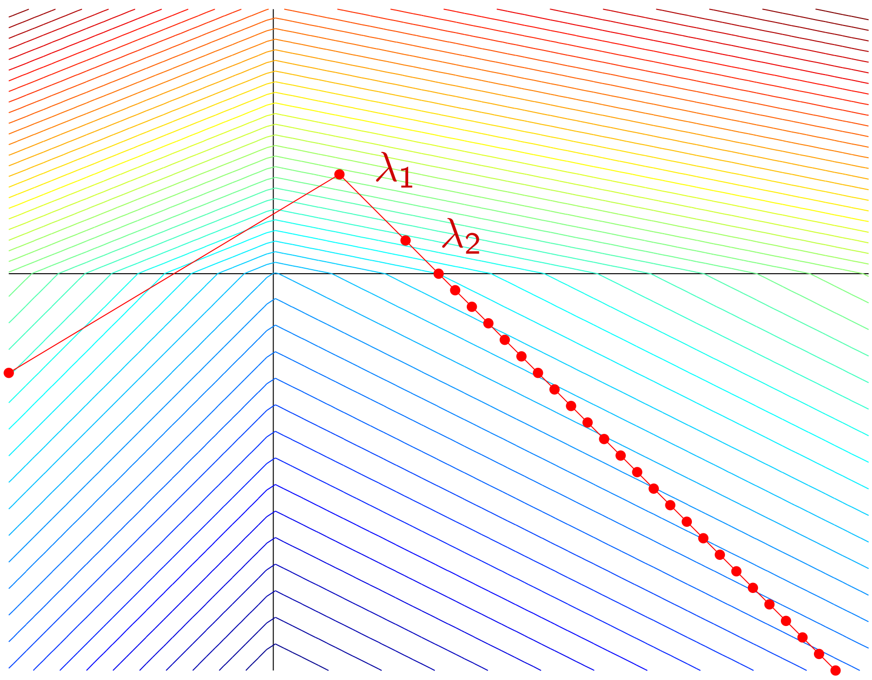
The AL algorithm

Convergence for an infeasible QP (val$(P) = +\infty$)

Level curves of the dual function δ (infeasible QP, $H \succ 0$)



Level curves of the dual function δ (infeasible QP, $H = 0$)



A surprising identity [5; 2015]

When dom $\delta \neq \varnothing$,

$$\boxed{\mathcal{S} = \mathcal{R}(\partial\delta).}$$

- Surprising since
  - $\mathcal{S}$ only depends on the constraints of the QP,
  - δ also depends on the objective of the QP.
- We already know that $\mathcal{S} \cap \mathcal{R}(\partial\delta) \neq \varnothing$:

$$\mathcal{S} = [l, u] + \mathcal{R}(A) \ni s_{k+1} := y_{k+1} - Ax_{k+1} \in \partial\delta(\lambda_{k+1}) \subset \mathcal{R}(\partial\delta).$$

When $\mathrm{dom}\,\delta \neq \varnothing$,

$$\boxed{\mathcal{S} = \mathcal{R}(\partial\delta).}$$

Proof

- The value function $v(s) := \inf\{q(x) : l \leqslant Ax + s \leqslant u,\ x \in \mathbb{R}^n\}$ verifies

$$\mathrm{dom}\, v = \mathcal{S} \qquad \text{and} \qquad \delta = v^*.$$

- No duality gap: $\mathrm{val}(P'_s) = \mathrm{val}(D'_s)$, which can be written

$$v = \delta^*.$$

Proof (continued)

- $[\mathcal{S} \subset \mathcal{R}(\partial\delta)]$ (Frank-Wolfe and constraint qualification)

$$\begin{aligned} s \in \mathcal{S} \quad &\Longrightarrow \quad (P'_s) \text{ has a primal-dual solution } ((x_s, y_s), \lambda_s) \\ &\Longrightarrow \quad (x_s, y_s) \in \arg\min\{\ell(x, y, \lambda_s) + \cancel{s^\mathsf{T}\lambda_s} : (x, y) \in \mathbb{R}^n \times [l, u]\} \\ &\Longrightarrow \quad (x_s, y_s) \in \arg\min\{\ell(x, y, \lambda_s) : (x, y) \in \mathbb{R}^n \times [l, u]\} \\ &\Longrightarrow \quad s = y_s - Ax_s \in \partial\delta(\lambda_s) \subset \mathcal{R}(\partial\delta). \end{aligned}$$

- $[\mathcal{S} \supset \mathcal{R}(\partial\delta)]$ ($\delta \not\equiv +\infty$, no duality gap)

$$\begin{aligned} s \in \mathcal{R}(\partial\delta) \quad &\Longrightarrow \quad s \in \partial\delta(\lambda) \text{ for some } \lambda \\ &\Longrightarrow \quad \lambda \in \partial\delta^*(s) = \partial v(s) \\ &\Longrightarrow \quad s \in \mathrm{dom}\, v = \mathcal{S}. \end{aligned}$$

Is the identity $\mathcal{S} = \mathcal{R}(\partial\delta)$ true for an arbitrary convex problem?

For an arbitrary convex function $\delta \in \overline{\mathrm{Conv}}(\mathbb{R}^m)$, there holds

$$\operatorname{ri}(\operatorname{dom}\delta^*) \subset \mathcal{R}(\partial\delta) \subset \operatorname{dom}\delta^*,$$

Taking the closure yields

$$\boxed{\operatorname{cl}\operatorname{dom}\delta^* = \operatorname{cl}\mathcal{R}(\partial\delta).}$$

The identity $\mathcal{S} = \mathcal{R}(\partial\delta)$ holds for a convex QP (with $\delta \not\equiv +\infty$) since

- $\delta^* = v$ (no duality gap) (not always true) $\implies$ $\operatorname{cl}\operatorname{dom} v = \operatorname{cl}\mathcal{R}(\partial\delta)$,
- $\operatorname{dom} v = \mathcal{S}$ (always true) $\implies$ $\boxed{\operatorname{cl}\mathcal{S} = \operatorname{cl}\mathcal{R}(\partial\delta),}$
- $\mathcal{S}$ is closed (not always true) $\implies$ $\mathcal{S} = \operatorname{cl}\mathcal{R}(\partial\delta)$,
- $\mathcal{R}(\partial\delta)$ is closed (not always true) $\implies$ $\mathcal{S} = \mathcal{R}(\partial\delta)$.

Convergence $s_k \to \bar{s}$ [26; 1987]

- Intuitive "proof"

$$\mathcal{S} = [l, u] + \mathcal{R}(A) \ni s_k := y_k - Ax_k \in \partial\delta(\lambda_k) \subset \mathcal{R}(\partial\delta).$$

  - Trust the proximal algo: $y_k - Ax_k \to$ the smallest element in $\mathcal{R}(\partial\delta)$.
  - Now $\mathcal{S} = \mathcal{R}(\partial\delta) \implies$ the smallest element in $\mathcal{R}(\partial\delta)$ is $\bar{s}$.
  - Hence $s_k := y_k - Ax_k \to \bar{s}$.

- Sketch of the proof [26] ($\{r_k\}$ is assumed bounded away from zero)
  - Let $\tilde{\mathcal{S}}_{\mathrm{D}}$ be the dual solution set of $(P_{\bar{s}})$.
  - Show first that $-\bar{s} \in \tilde{\mathcal{S}}_{\mathrm{D}}^{\infty}$.
  - Define $\{\mu_k\}$ by $\mu_0 \in \tilde{\mathcal{S}}_{\mathrm{D}}$ and $\mu_{k+1} := \mu_k - r_k\bar{s} \in \tilde{\mathcal{S}}_{\mathrm{D}}$.
  - Compare $\{\lambda_k\}$ and $\{\mu_k\}$: $\lambda_k - \mu_k = \lambda_{k+1} - \mu_{k+1} + r_k(s_{k+1} - \bar{s})$,

$$\|\lambda_k - \mu_k\|^2 \geqslant \|\lambda_{k+1} - \mu_{k+1}\|^2 + r_k^2\|s_{k+1} - \bar{s}\|^2.$$

  - Hence $s_k \to \bar{s}$.

Why $s_k \to \bar{s}$ implies that the AL algorithm solves the CFQP?

Since

$$(x, y) \in \underset{(x',y') \in \mathbb{R}^n \times [l,u]}{\arg\min} \ \ell_r(x', y', \lambda)$$
$$\text{and } Ax + \bar{s} = y$$

imply that (x, y) is a solution to the CFQP.

Global linear convergence $s_k \to \bar{s}$ [5]

$(P_{\bar{s}})$ with solution $\Rightarrow$ the dual solution set of $(P_{\bar{s}})$, namely

$$\tilde{\mathcal{S}}_{\mathrm{D}} := \{\lambda \in \mathbb{R}^m : \bar{s} \in \partial\delta(\lambda)\}$$

is nonempty and

$$\boxed{\begin{array}{c}\forall\, \beta > 0, \quad \exists\, L > 0, \quad \mathrm{dist}_{\tilde{\mathcal{S}}_{\mathrm{D}}}(\lambda_0) \leqslant \beta \quad \text{implies that} \\ \forall\, k \geqslant 1, \quad \|s_{k+1} - \bar{s}\| \leqslant \frac{L}{r_k} \|s_k - \bar{s}\|.\end{array}} \tag{8}$$

Comments:

- Similar to the solvable case, but with $s_k \curvearrowright s_k - \bar{s}$,
- $\bar{s}$ is not known $\Rightarrow$ more difficult to design an update rule for r_k: instead of $s_k - \bar{s}$, observe $s_k' := s_k - s_{k-1} \to 0$ globally linearly.

The AL algorithm

Convergence for an infeasible QP ($\mathrm{val}(P) = +\infty$)

Proof

- Let $\tilde\lambda \in \tilde{\mathcal{S}}_{\rm D}$, $\tilde\lambda_k := \lambda_k - r_k\bar s$, and subtract $\tilde\lambda + r_k\bar s$ from the iteration $\lambda_{k+1} = \lambda_k - r_k s_{k+1}$:

$$\lambda_{k+1} - \tilde\lambda + r_k \Big[\underbrace{(s_{k+1} - \bar s)}_{\in\partial\tilde\delta(\lambda_{k+1})} - \underbrace{0}_{\in\partial\tilde\delta(\tilde\lambda)}\Big] = \tilde\lambda_k - \tilde\lambda.$$

- Monotonicity of $\partial\tilde\delta(\cdot) = \partial\delta(\cdot) - \bar s$:

$$\forall\,\tilde\lambda \in \tilde{\mathcal{S}}_{\rm D}: \quad \|s_{k+1} - \bar s\| \leqslant \frac{1}{r_k}\|\tilde\lambda_k - \tilde\lambda\|.$$

- $\tilde\lambda \in \tilde{\mathcal{S}}_{\rm D}$ is arbitrary and $-\bar s \in \tilde{\mathcal{S}}_{\rm D}^{\infty}$:

$$\|s_{k+1} - \bar s\| \leqslant \frac{1}{r_k}\,\mathrm{dist}_{\tilde{\mathcal{S}}_{\rm D}}(\tilde\lambda_k) \leqslant \frac{1}{r_k}\,\mathrm{dist}_{\tilde{\mathcal{S}}_{\rm D}}(\lambda_k). \tag{9}$$

- Quasi-global error bound (5) on $\tilde{\mathcal{S}}_{\rm D}$:

$$\mathrm{dist}_{\tilde{\mathcal{S}}_{\rm D}}(\lambda_k) \leqslant L\|s_k - \bar s\|. \tag{10}$$

- (9) and (10) imply (8).

The AL algorithm

The revised AL algorithm

Set $\lambda_0 \in \mathbb{R}^m$, $r_0 > 0$, $\rho'_{\rm des} \in \,]0,1[$, and repeat for $k = 0, 1, 2, \ldots$

- Compute (if possible, exit with a direction of unboundedness otherwise)

$$(x_{k+1}, y_{k+1}) \in \underset{(x,y)\in\mathbb{R}^n\times[l,u]}{\arg\min}\ \ell_{r_k}(x, y, \lambda_k).$$

- Update the multipliers

$$\lambda_{k+1} = \lambda_k - r_k s_{k+1}, \quad \text{where } s_{k+1} := y_{k+1} - Ax_{k+1}.$$

- Stop if

$$A^{\mathsf T}(Ax_{k+1} - y_{k+1}) \simeq 0 \qquad \text{and} \qquad P_{[l,u]}(Ax_{k+1}) \simeq y_{k+1}.$$

- Update $r_k \curvearrowright r_{k+1} > 0$: $s'_k := s_k - s_{k-1}$, $\rho'_k := \|s'_{k+1}\|/\|s'_k\|$, and

$$r_{k+1} := \max\left(1, \frac{\rho'_k}{\rho'_{\rm des}}\right) r_k.$$

The SQP algorithm

The (LS-qN) SQP algorithm solves the nonlinear optimization problem

$$\begin{cases} \inf_x \ f(x) \\ l \leqslant c(x) \leqslant u, \end{cases} \tag{11}$$

as follows.

- It computes at the current iterate x the search direction d by solving the osculating quadratic problem (OQP)

  $$d \in \operatorname*{arg\,min}_{l' \leqslant Ad \leqslant u'} \left(g^\mathsf{T} d + \frac{1}{2} d^\mathsf{T} H d \right), \tag{12}$$

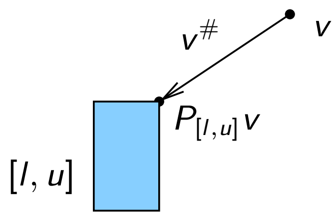
  with $g := \nabla f(x)$, H is a positive definite approximation of the Hessian of the Lagrangian of (11), $A := c'(x)$, $l' := l - c(x)$, and $u' := u - c(x)$.
- Then it computes a stepsize $\alpha > 0$ along d in order to decrease a merit function and takes as new iterate

  $$x_+ := x + \alpha\, d.$$

A classical merit function is

$$\begin{aligned} x \in \mathbb{R}^n \mapsto \Theta_\sigma(x) &= f(x) + \sigma \ \mathrm{dist}_{[l,u]}(c(x)) \\ &= f(x) + \sigma \, \|c(x)^\#\|, \end{aligned}$$

where $\sigma > 0$ and



$$v^\# := P_{[l,u]} v - v.$$

Using an unboundedness direction

If the closest feasible OQP is infeasible, the AL algorithm can return an unboundedness direction d, i.e., satisfying

$$g^{\mathsf{T}} d < 0, \quad Hd = 0, \quad \text{and} \quad Ad \in [l', u']^{\infty}.$$

Proposition

Let d be an unboundedness direction of the closest feasible OQP (14) at x. Then

$$(\|c(\cdot)^{\#}\|)'(x; d) \leqslant 0 \qquad \text{and} \qquad \Theta'_{\sigma}(x; d) < 0. \tag{13}$$

Again, a direction of unboundedness d of the closest feasible OQP allows the SQP algorithm to make a LS along it.

Using a solution to the closest feasible QP

If the OQP is infeasible, the AL algorithm solves instead the closest feasible OQP

$$d \in \underset{l' \leqslant Ad + \bar{s} \leqslant u'}{\arg\min} \left(g^{\mathsf{T}} d + \frac{1}{2} d^{\mathsf{T}} H d \right). \tag{14}$$

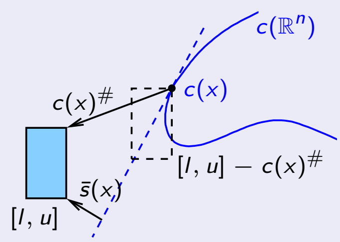
Proposition (link to make with [3; 1989])

If x is not a stationary point of the feasible problem

$$\begin{cases} \inf_y f(y) \\ l \leqslant c(y) + c(x)^{\#} \leqslant u, \end{cases}$$

if σ is large enough, if d solves (14), and if $H \succ 0$, then

$$\Theta'_{\sigma}(x; d) \leqslant -d^{\mathsf{T}} H d - \bar{\sigma} \left(\|c(x)^{\#}\| - \|\bar{s}(x)\| \right) < 0.$$



Hence a solution d to the closest feasible osculating QP allows the SQP algorithm to make a LS along it.

Oqla and Qpalm

Implementation of the revised AL algorithm in two solvers [12], soon freely available at https://who.rocq.inria.fr/Jean-Charles.Gilbert:

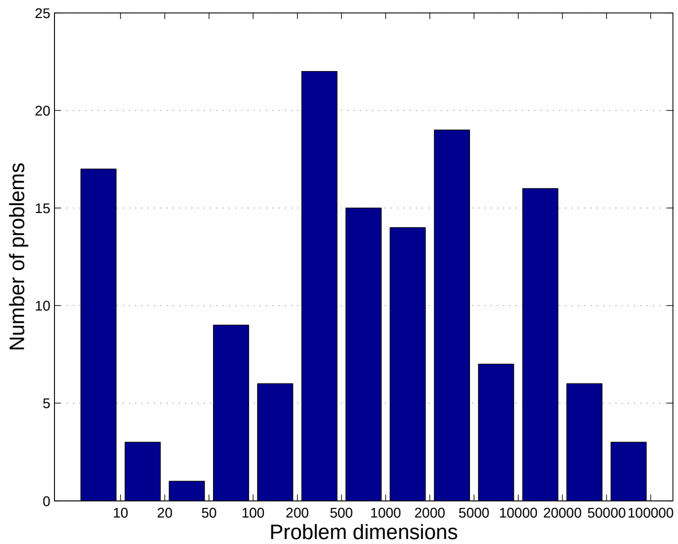
- Oqla
  - in C++,
  - fast execution, but slow implementation,
  - OO → easy to take into account new data structures, like Ooqp [11] (dense, sparse, ℓ-BFGS, ...),
  - AL subproblems solved by an active-set (AS) method,
  - more than 1 year of work for one engineer!
- Qpalm
  - in Matlab,
  - AL subproblems solved by an AS method,
  - fast implementation, easy to try new ideas, but slow execution.

Main objective of these tests: is it worth continuing working on the development of Oqla/Qpalm?

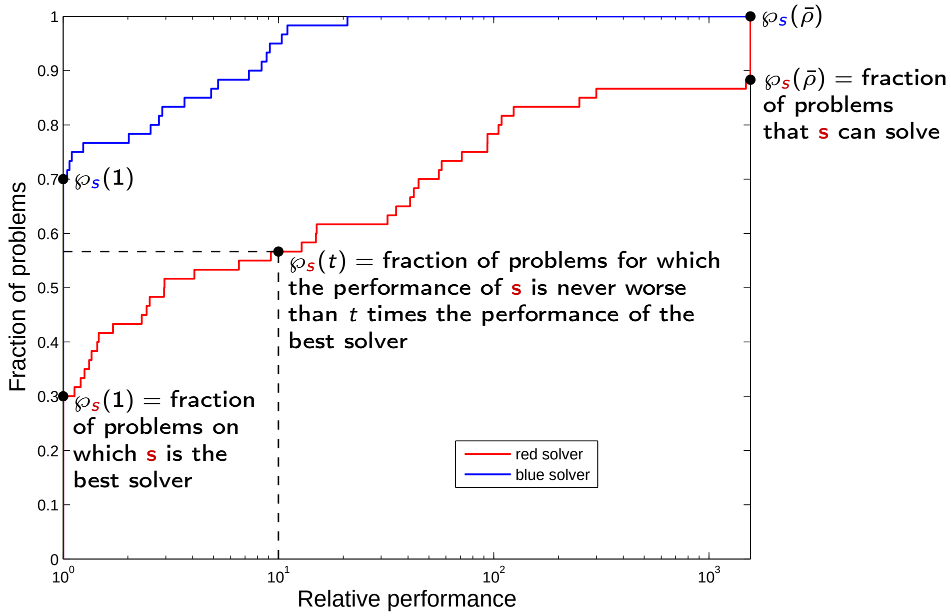
Selected Cutest problems

Comparison made on the Cutest collection of test-problems [15].

- 138 convex quadratic problems (all solvable, but 4?),
- 58 problems among them, with $n \leqslant 500$ (for comparison in Matlab).

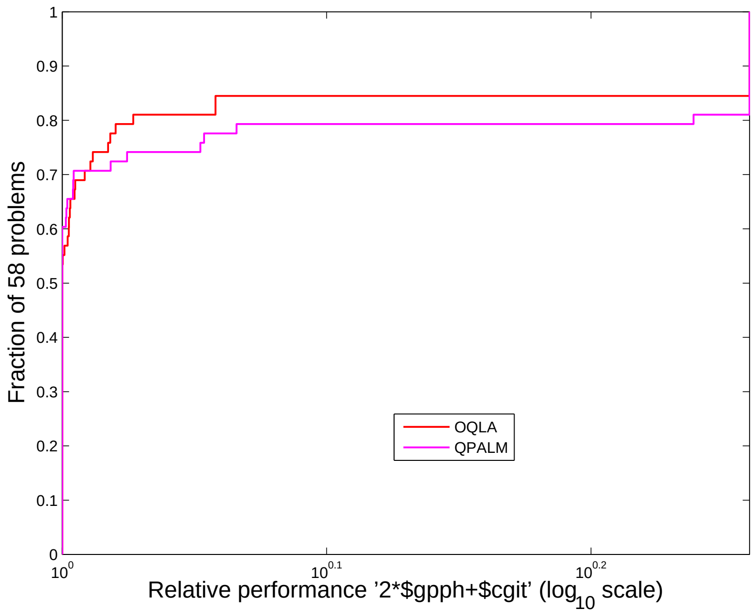


Reading performance profiles [9]



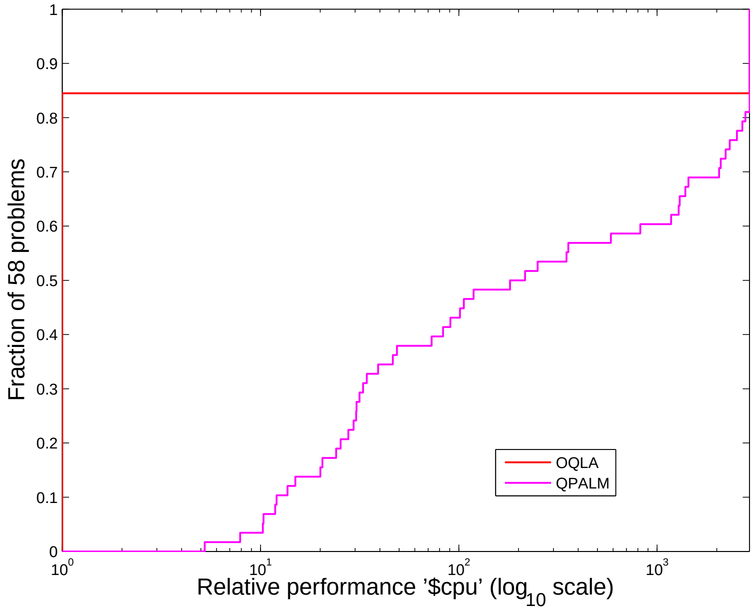
Performance profiles drawn with Libopt [13].

Comparison of Oqla and Qpalm on iteration counters



- Close to each other (see x-axis [$10^{0.05} \simeq 1.12$] and y-axis [even scores]).
- Difference in failures due to the slowness of Qpalm in Matlab (or still not clear).

Comparison of Oqla and Qpalm on CPU time



- Oqla (in C++) is 10..2000 times faster than Qpalm (in Matlab).

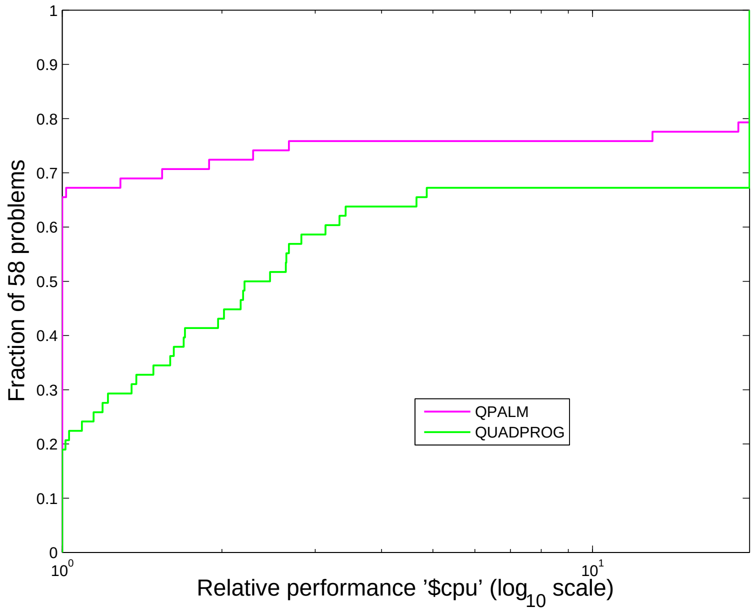
Two more codes, which use active-set methods:

- Quadprog
  - the standard QP solver of the Matlab optimization toolbox [25],
  - Options `'Algorithm'` $\to$ `'active-set'` and `'LargeScale'` $\to$ `'off'` $\Longrightarrow$ active-set method.
- Qpa
  - free code,
  - from the Galahad library [14],
  - in Fortran,
  - uses preprocessing and preconditioning?

Numerical results

Comparison with active-set methods

Comparison of Qpalm and Quadprog on CPU time

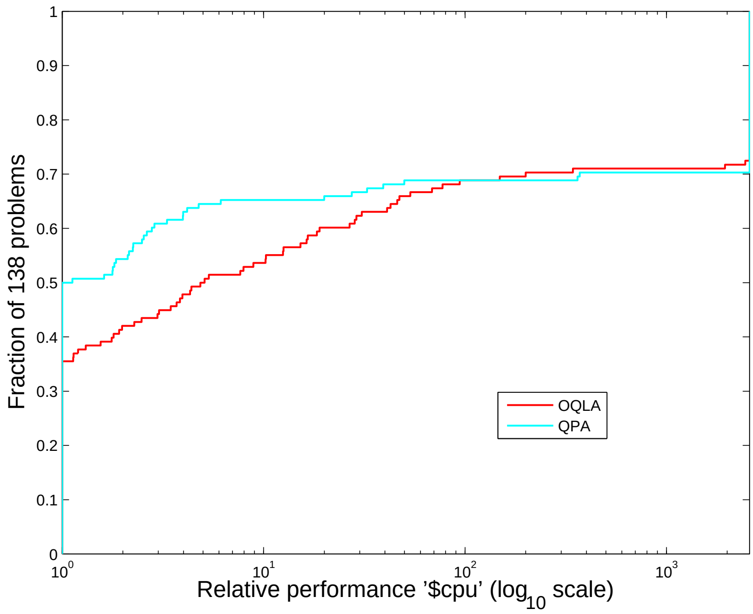


- Qpalm is often twice faster than Quadprog (but not always faster).
- Qpalm is more robust than Quadprog (81 % success to 67 %).
- Progress is still possible with Qpalm.

Numerical results

Comparison with active-set methods

Comparison of Oqla and Qpa on CPU time

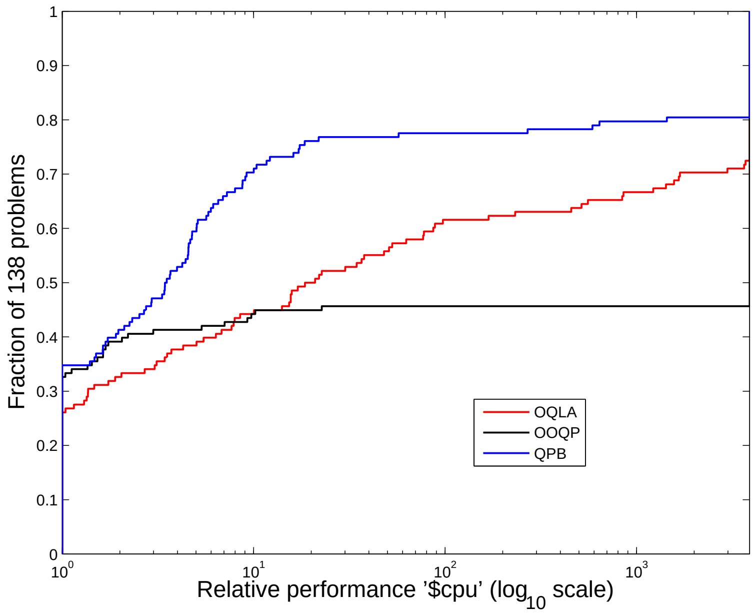


- Qpa is more often faster than Oqla, but not significantly.
- Oqla and Qpa have the same robustness (73 % and 71 % success respectively).
- Progress is still possible with Oqla.

Two more codes, which use interior-point methods:

- Ooqp
  - free code,
  - written by Gertz and Wright in 2003 [11],
  - to show the interest of an OO implementation.
- Qpb
  - free code,
  - from the Galahad library [14],
  - in Fortran,
  - uses preprocessing and preconditioning?

Comparison of Oqla, Ooqp, and Qpb on CPU time



- IP methods are clearly faster than our AL+AS method (in particular with Ooqp).
- Poor robustness of Ooqp $\Longrightarrow$ careful implementation yields much improvement?
- Oqla is located between Qpb and Ooqp in terms of robustness.

Behaviors in an SQP framework

- Recall that one iteration of the SQP algorithm computes a PD solution $(d^{\mathrm{QP}}, \lambda^{\mathrm{QP}})$ of the OQP

$$\min_{l' \leqslant Ad \leqslant u'} \left(g^{\mathsf{T}} d + \frac{1}{2} d^{\mathsf{T}} H d \right)$$

and then updates (locally) the PD variables (x, λ) by

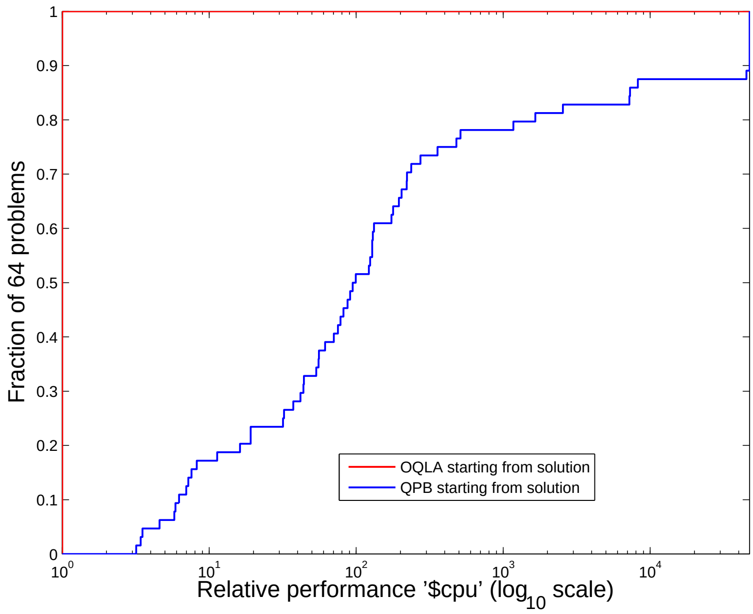
$$x_+ := x + d^{\mathrm{QP}} \qquad \text{and} \qquad \lambda_+ := \lambda^{\mathrm{QP}}.$$

- Close to the solution to the nonlinear problem, $x_+ \simeq x$ and $\lambda_+ \simeq \lambda$, therefore a good guess of the PD solution to the QP is available:

$$(0, \lambda).$$

- Hence, it makes sense to see how the QP solvers behave when the starting point is close to the solution to the QP.

Oqla vs. Qpb, starting from a primal-dual solution, on CPU time

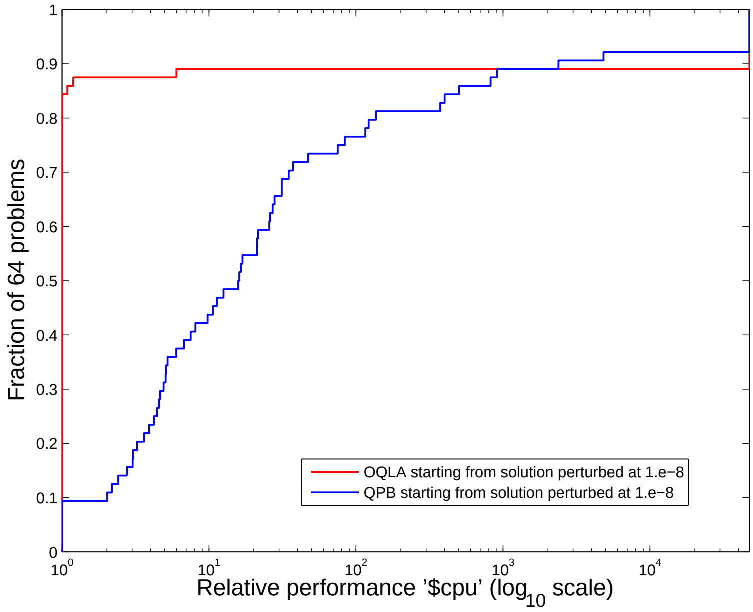


- Motivation: see whether Oqla can take advantage of a good starting point,
- 64 problems, for which an accurate primal-dual solution has been found,
- Qpb has no warm restart.

Numerical results

Comparison with interior-point methods

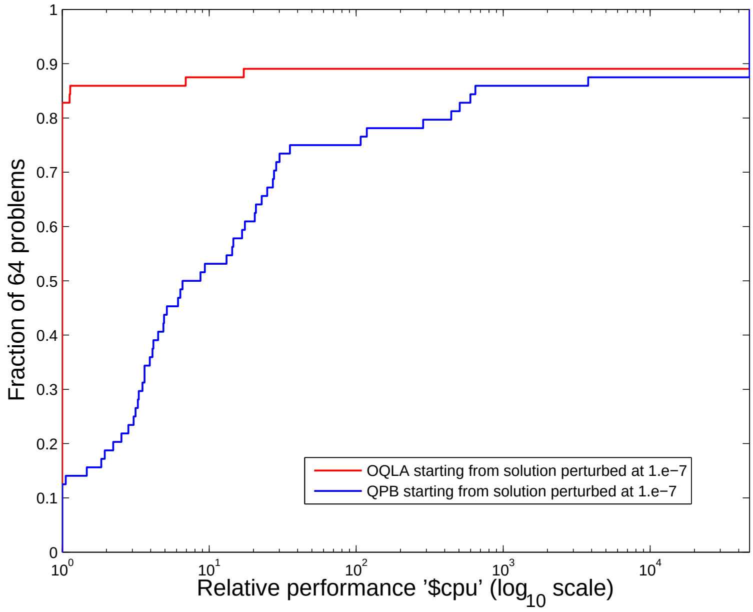
Oqla vs. Qpb, starting from a perturbed (10^{-8}) primal-dual solution



Numerical results

Comparison with interior-point methods

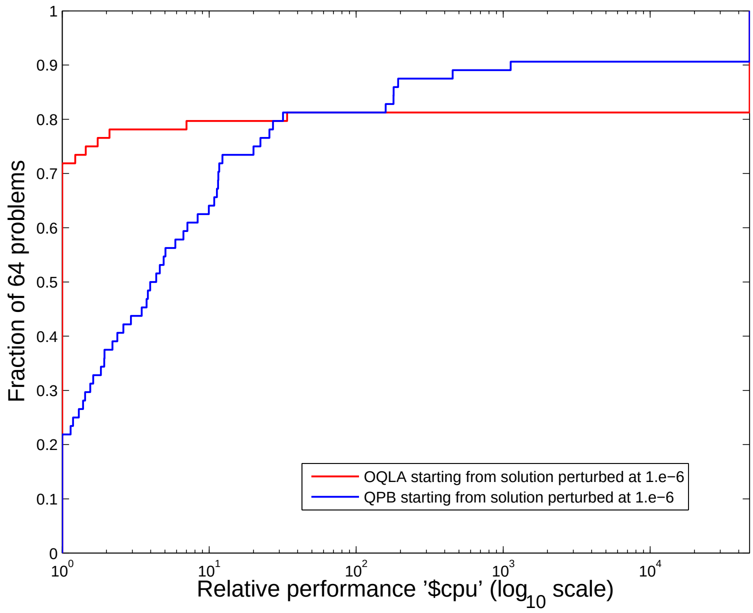
Oqla vs. Qpb, starting from a perturbed (10^{-7}) primal-dual solution



Numerical results

Comparison with interior-point methods

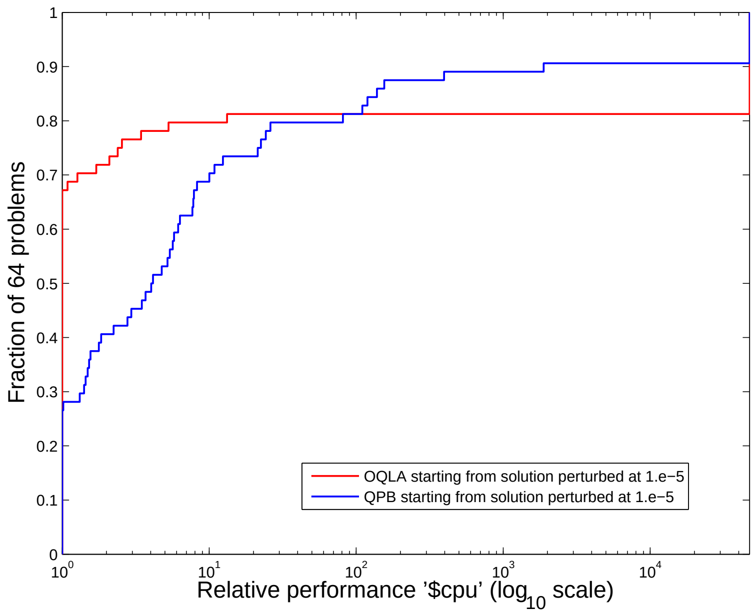
Oqla vs. Qpb, starting from a perturbed (10^{-6}) primal-dual solution



Numerical results

Comparison with interior-point methods

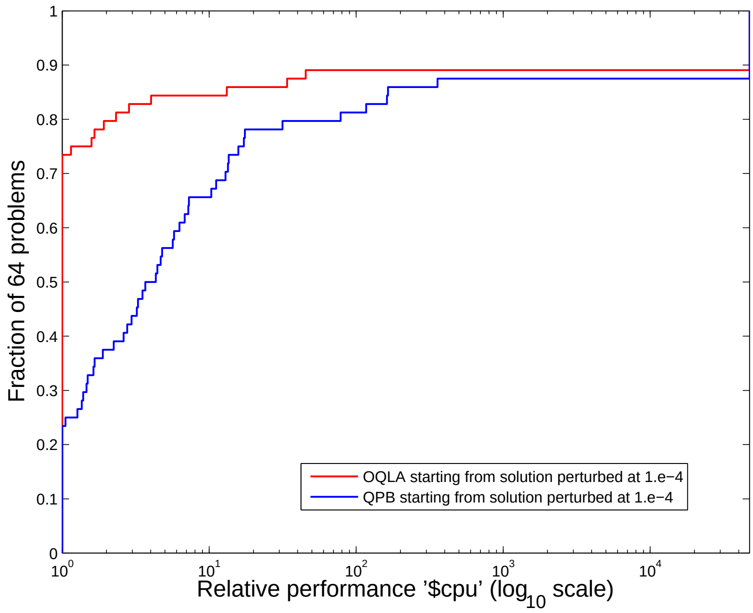
Oqla vs. Qpb, starting from a perturbed (10^{-5}) primal-dual solution



Numerical results

Comparison with interior-point methods

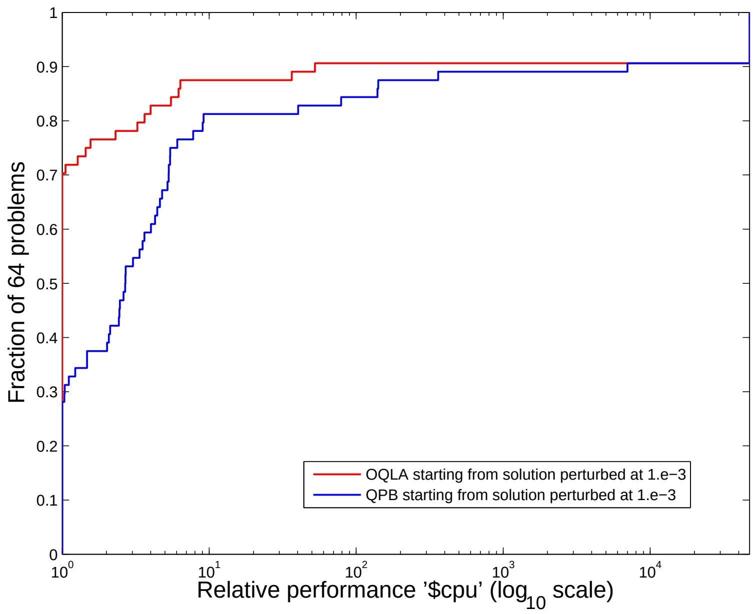
Oqla vs. Qpb, starting from a perturbed (10^{-4}) primal-dual solution



Numerical results

Comparison with interior-point methods

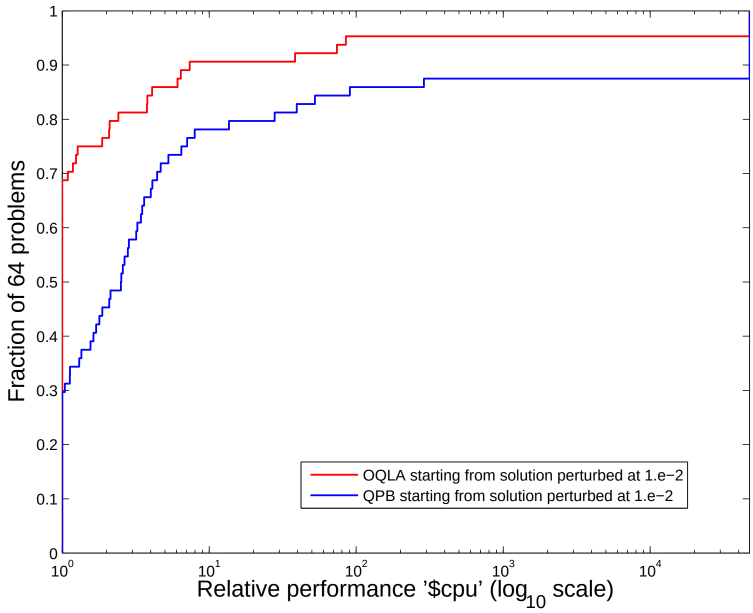
Oqla vs. Qpb, starting from a perturbed (10^{-3}) primal-dual solution



Numerical results

Comparison with interior-point methods

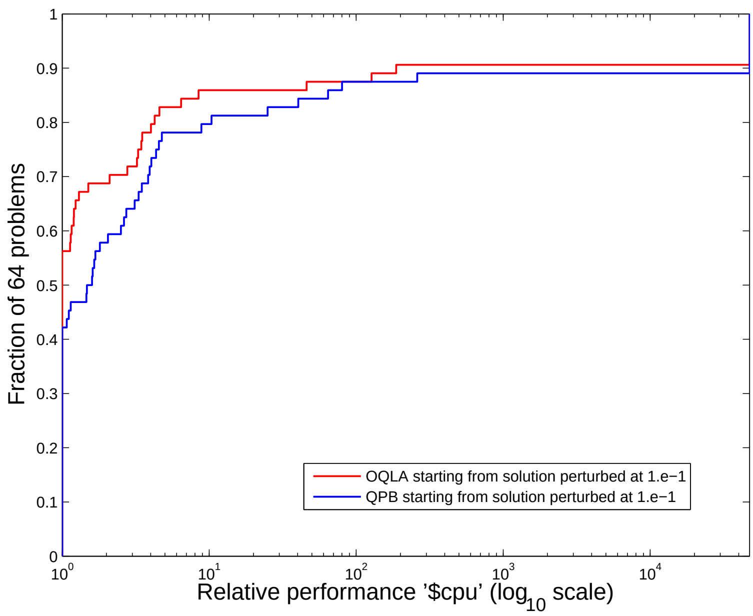
Oqla vs. Qpb, starting from a perturbed (10^{-2}) primal-dual solution



Numerical results

Comparison with interior-point methods

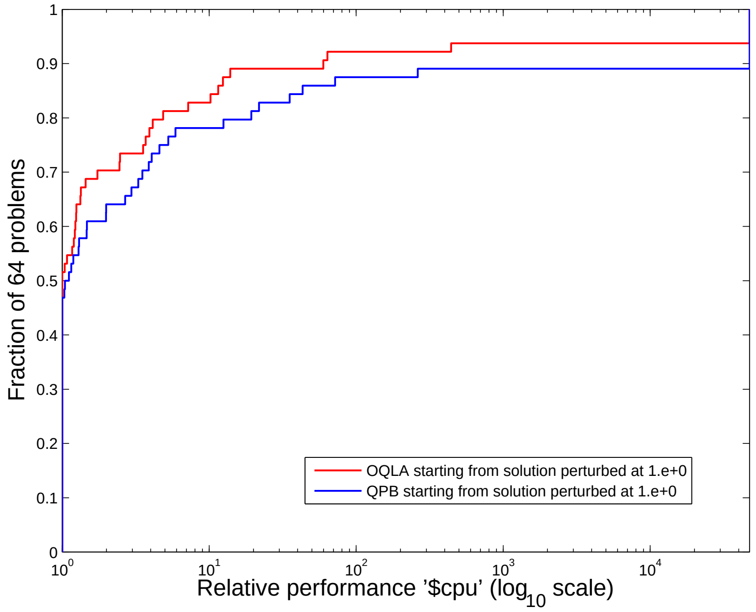
Oqla vs. Qpb, starting from a perturbed (10^{-1}) primal-dual solution



Numerical results

Comparison with interior-point methods

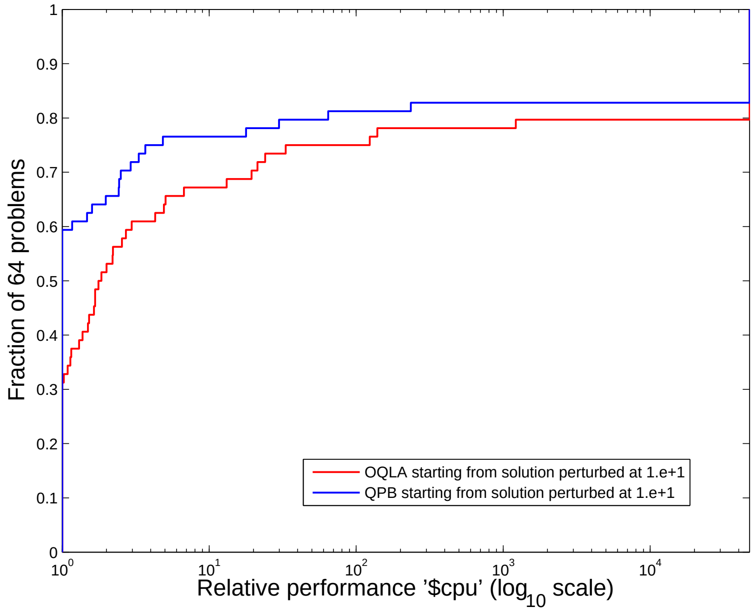
Oqla vs. Qpb, starting from a perturbed (10^0) primal-dual solution



Numerical results

Comparison with interior-point methods

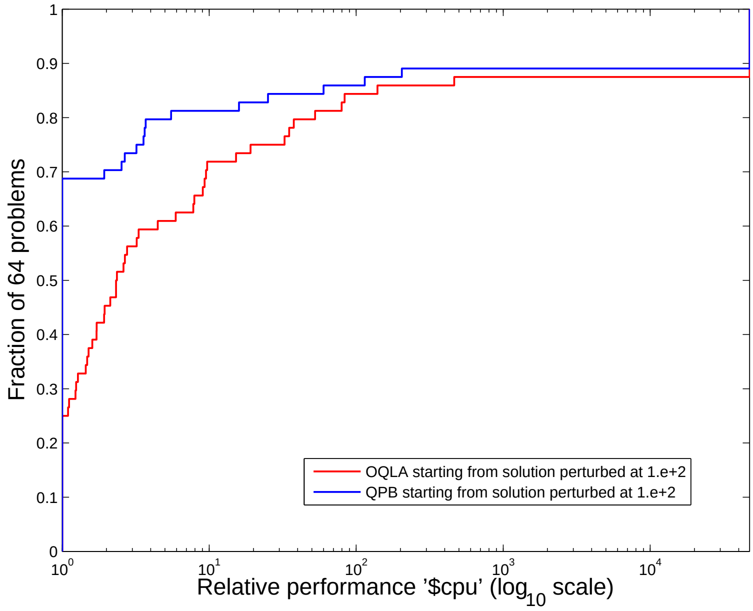
Oqla vs. Qpb, starting from a perturbed (10^1) primal-dual solution



Numerical results

Comparison with interior-point methods

Oqla vs. Qpb, starting from a perturbed (10^2) primal-dual solution



- Conclusion: for perturbations less than 100 %, the AL+AS solver Oqla is "more often better" than the IP solver Qpb.

Discussion and future work

Discussion

- Oqla/Qpalm give interesting answers on infeasbile or unbounded QPs.
- Oqla and Qpalm are not ridiculous, with respect to well established active-set solvers (Qpa), and sometimes clearly better (Quadprog).
- The present version of Oqla/Qpalm is not as efficient as the IP solver Qpb, but much more robust than Ooqp.
- Oqla/Qpalm can take advantage of an estimate of the solution (not the case of the other tested IP solvers) $\Longrightarrow$ nice for SQP.
- Still many possible improvements:
  - using preprocessing,
  - inexact minimization of the AL subproblems (3), while keeping the global linear convergence,
  - trying other solvers of the AL subproblems (3), like IP or Newton-min,
  - . . . .

Discussion and future work

Future work

- Can one preserve the global linear convergence of the AL algorithm when the AL subproblems (3) are solved inexactly?
- Try to use one (a few) interior point step(s) to solve the AL subproblems (3), in order to obtain polynomiality.
- Improve nonsmooth methods and use them to solve the AL subproblems (3), in order to gain in efficiency.
- Extend the result of Dean and Glowinski [7] to convex inequality constrained QP: for *stricty* convex QP with the *single equality constraint* $Ax = b$, the Lagrangian relaxation

$$\begin{aligned} x_k &= \arg\min_{x\in\mathbb{R}^n}\; q(x) + \lambda_k^{\mathsf{T}}(Ax-b) \\ \lambda_{k+1} &= \lambda_k + \alpha_k(Ax_k - b), \end{aligned}$$

where α_k is chosen is a compact of $]0, 2/\mu_1[$, generates iterates that converge globally linearly to the unique solution to the closest feasible problem

$$\left\{ \begin{array}{l} \inf_x \; q(x) \\ A^{\mathsf{T}}(Ax-b) = 0. \end{array} \right.$$

Discussion and future work

Future work (continued)

- Show the global linear convergence of an AL algorithm for the more general problem (+ constraint qualification):

$$\left\{ \begin{array}{l} \inf_{x\in\mathbb{E}} \; \langle g, x\rangle + \frac{1}{2}\langle Hx, x\rangle \\ Ax \in C \\ x \in X. \end{array} \right.$$

Two interesting instances:

- $\mathbb{E} = \mathbb{R}^n$, $C = [l, u]$, $X = \text{ball} \Longrightarrow$ trust region problem,
- $\mathbb{E} = \mathcal{S}^n$, $H = 0$, $C = \{b\}$, $X = \mathcal{S}^n_+ \Longrightarrow$ linear SDP problem.

The end

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