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Modeling a case of herding behavior in a multi-player game

Guy Fayolle · Jean-Marc Lasgouttes

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Abstract The system mentioned in the title belongs to the family of the so-called massively multi-player online social games. It features a scoring system for the elements of the game that is prone to herding effects. We analyze in detail its stationary regime in the thermodynamic limit, when the number of players tends to infinity. In particular, for some classes of input sequences and selection policies, we provide necessary and sufficient conditions for the existence of a complete meanfield-like measure, showing off an interesting *condensation* phenomenon.

Keywords Thermodynamical limit · Condensation · Ergodicity · Transience · Non-linear differential system · Banach space

Mathematics Subject Classification 60J28 · 34L30 · 37C40

1 Description of the basic model

Ma Micro Planète [7] is a geolocalized massively multi-player online social game which entices players to use sustainable means of transport. At the heart of the game is a community-driven creation of a set of *points of interest* (POIs): interesting places, events, or even traffic incidents. The popularity of the POIs is assessed by scores. At his turn, the player is presented with a number of POIs to visit in his neighborhood, which have been selected through a mechanism that favors already popular ones. Then the selected POI has its score increased by a random value. It is not difficult to see that there is a risk of *herding* behavior, that would concentrate all the attention on a few places only. The aim of this paper is to present some results about the POIs dynamics

G. Fayolle (✉) · J.-M. Lasgouttes
INRIA Paris–Rocquencourt, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay cedex, France
e-mail: Guy.Fayolle@inria.fr

by means of a probabilistic model, and in particular to determine when a stationary solution exists.

Assume there are N players in the game and let $n_i(t; N)$ be the number of POIs having an integer score i at time t. The state of the system at time t can be described by the infinite sequence

$$\vec{R}(t; N) = \frac{1}{N}[n_1(t; N), \ldots, n_i(t; N), \ldots].$$

The dynamics of the model is

- POIs are created according to a Poisson process of rate λN. Each new POI is granted an initial score k with probability φ_k, $k \geq 1$, where the φ_k's form a proper probability distribution.
- For any $i \geq 1$, each POI with score i decreases its score by 1 after a random time exponentially distributed at rate μ, where μ is a strictly positive constant.
- The N players are all independent and visit POIs at instants forming a global Poisson process with rate αN. Upon being visited, a POI sees its score increased by $j \geq 1$ with probability θ_j, independently of any other event, where the sequence $\{\theta_j, j \geq 1\}$ forms a proper probability distribution. We shall implicitly take $\theta_0 = 0$, which is in no way a restriction, as it is simply tantamount to modifying the parameter α.
- When he decides to play, at the instant of a Poisson process of parameter αN, a player selects a POI having score $i \geq 1$ with probability $\pi_i(\vec{R}(t; N))$, written as a functional of the state of the system, with the normalizing condition

$$\sum_{i \geq 1} \pi_i(\vec{R}(t; N)) = 1.$$

In accordance with the description of the game, we shall essentially consider selection policies such that the probability π of selecting a POI, given its score i, is a *non-decreasing* function of i.

The following typical scoring rules are used in *Ma Micro Planète*.

- Each player starts a journey at rate α and selects a POI to visit with a probability proportional to its current score.
  - with probability p, the player adds new information to the POI, the score of which is increased by 15 points;
  - otherwise, the POI is merely visited and its score gains 5 points; this happens with probability $1 - p$.
- Each player creates POIs at rate λ and their initial score is set to 50.
- The POIs decrease their scores by 1 at a rate $\mu = 3$ per day.
- A POI disappears as soon as its score becomes zero or negative.
- The values of α, p and λ can only be measured empirically from concrete game statistics.

A way of describing this model might be in terms of queues. Consider a possibly infinite array of queues $M/M/1$ queues with service rate μ. New queues are created at rate λ and batches of customers join an existing queue with score $i \geq 1$ at rate $\alpha\pi_i(\vec{R}(t; N))$. Such systems have already been described, for example when the customers join the shortest of two queues [10]. The situation described in this paper is unusual due to the bias towards selecting the *longest* queue. This only makes sense because there is no cost associated to visiting a highly popular POI, whereas joining a long queue is expensive in terms of waiting time. A similar situation [9] occurs when customers, unsure of the quality of two restaurants, choose the busier one.

The model can have ramifications with other areas of interest, e.g., dynamics of populations, social networks, chemical or energy networks in physics.

In this work, we examine the system in the so-called *thermodynamic limit*, as $N \to \infty$, the existence of which is proved in Sect. 2, under general conditions, where the probability of increasing the score of a visited POI is a function of the state of the system. Section 3 is devoted to the stationary regime, for which complete (but surprising!) answers are given for various input sequences and selection policies, and the existence of the so-called herding behavior is connected to possible condensation phenomena. Section 4 presents miscellaneous simple case studies. The final Sect. 5 outlines a possible method of solution for general input sequences, and also discusses a brief list of unsolved questions.

2 The thermodynamical limit

To capture information on the behavior of the system under various game policies, we proceed by scaling and analyze the *thermodynamical limit*, when the number N of players becomes large. In mathematical terms, we consider the system when $N \to \infty$, for any fixed t.

2.1 Notation

It will be convenient to gather here most of the notational material concerning input sequences and mathematical symbols.

- $\mathcal{E} \stackrel{\text{def}}{=} \{\vec{y} = [y_1, \ldots, y_i, \ldots], y_i \in \mathbb{Q}\}$, where $\mathbb{Q}_+$ is the set of positive rational numbers. $\mathbb{R}_+^\infty$ being the set of non-negative real numbers, $\vec{e}_j$ will stand for the j-th unit vector of $\mathbb{R}_+^\infty$. We shall also denote by $\mathbb{L} \in \mathbb{R}_+^\infty$ the subset of bounded sequences with the norm $\|\vec{y}\| \stackrel{\text{def}}{=} \sum_{i\geq 1} |y_i| < M < \infty$.
- E being an arbitrary metric space, $\mathcal{B}(E)$ is the space of bounded real-valued Borel measurable functions on E, $\mathcal{C}(E) \subset \mathcal{B}(E)$ the Banach subspace of bounded continuous functions.
- We shall need the three following subspaces of $\mathcal{C}(\mathbb{L})$:

$$\mathcal{C}_1(\mathbb{L}) \stackrel{\text{def}}{=} \left\{ f : \sup_{j\geq 1} \left| \frac{\partial f}{\partial y_j} \right| \leq C \right\},$$

$$\mathcal{C}_2(\mathbb{L}) \stackrel{\text{def}}{=} \left\{ f : \sup_{j,k\geq 1} \left| \frac{\partial^2 f}{\partial y_j \partial y_k} \right| \leq H \right\},$$

C, H being arbitrary positive constants.

- For any complete separable metric space S, let $D_S[0,\infty]$ denote the space of right continuous functions $f:[0,\infty]\to S$ with left limits, endowed with the Skorokhod topology.

2.2 The main limit theorem

Clearly $\vec{R}(t;N)$ is a continuous time Markov chain, and its generator $G^{(N)}$, for $f\in\mathcal{C}_1(\mathbb{L})$, is the operator given by,

$$\begin{aligned}G^{(N)}f(\vec{y})&=\lambda N\sum_{j\geq 1}\varphi_j\left[f\left(\vec{y}+\frac{\vec{e}_j}{N}\right)-f(\vec{y})\right]\\&+\mu N\sum_{j\geq 1}y_j\left[f\left(\vec{y}+\frac{\vec{e}_{j-1}}{N}-\frac{\vec{e}_j}{N}\right)-f(\vec{y})\right]\\&+\alpha N\sum_{j\geq 1}\pi_j(\vec{y})\sum_{i\geq 1}\theta_i\left[f\left(\vec{y}-\frac{\vec{e}_j}{N}+\frac{\vec{e}_{i+j}}{N}\right)-f(\vec{y})\right].\end{aligned}\tag{2.1}$$

Throughout this study, the game will be subject to the following reasonably weak assumption.

Assumption (L) For all $\vec{r}\in\mathbb{L}$, the probability distribution $\pi:\mathbb{L}\to\mathbb{L}$, considered as an infinite vector $\{\pi_i(\vec{r}), i\geq 1\}$ with values in the Banach space $\mathbb{L}$ of absolutely summable sequences, will be supposed to be boundedly and twice continuously differentiable (see, for example, [1]). In this case, all partial derivatives $\frac{\partial\pi_i(\vec{y})}{\partial y_j}$, $j,k\geq 1$, $\vec{y}\in\mathbb{L}$, belong to $\mathcal{C}_1(\mathbb{L})$ and the famous Lipschitz condition is also satisfied.

In particular the above assumption (**L**) holds for the forthcoming policy, denoted by $\mathbf{P}_{\vec{a}}$, which will be analyzed in Sect. 3,

$$\pi_i(\vec{r})=\frac{a_i r_i}{\sum_{j\geq 1}a_j r_j},$$

where the a_i's form a bounded sequence of positive numbers.

Then, with the notation $\theta_0=\pi_0(.)=0$, the following proposition holds.

Theorem 2.1 *Suppose that, as $N\to\infty$, $\vec{R}(0;N)$ converges in distribution to $\vec{R}(0)\in\mathbb{L}$. Then, the Markov process $\vec{R}(t;N)$ converges also weakly in $D[0,\infty]$ to a* deterministic *dynamical system $\vec{r}(.)\in\mathbb{R}_+^\infty$, which evolves according to the following infinite system of non-linear differential equations*

$$\frac{\mathrm{d}r_i(t)}{\mathrm{d}t}=\lambda\varphi_i+\mu\big[r_{i+1}(t)-r_i(t)\big]+\alpha\left[\sum_{j=0}^{j=i}\theta_j\pi_{i-j}(\vec{r}(t))-\pi_i(\vec{r}(t))\right],\tag{2.2}$$

*where $\pi_i(\vec{r}(t))$ denotes the probability that the currently visited POI has score i. Moreover, under Assumption (**L**) and for any given finite initial condition $\vec{r}(0) \in \mathbb{L}$, the system (2.2) admits a unique solution, for all $t \geq 0$.*

Proof The method to solve this weak convergence problem is somehow classical (see for example the footsteps of [2]), and relies on theoretical results of [3] quoted in Appendix. First, a second order Taylor's expansion in (2.1) yields immediately, for any $f \in \mathcal{C}_2(\mathbb{L})$,

$$G^{(N)} f(\vec{y}) = Gf(\vec{y}) + \mathcal{O}\left(\frac{1}{N}\right), \tag{2.3}$$

where G is the operator with domain $\mathcal{C}_1(\mathbb{L})$ satisfying

$$\begin{aligned} Gf(\vec{y}) = \lambda \sum_{j\geq 1} \varphi_j \frac{\partial f}{\partial y_j} + \mu \sum_{j\geq 1} y_j \left(\frac{\partial f}{\partial y_{j-1}} - \frac{\partial f}{\partial y_j}\right)(\vec{y}) \\ + \alpha \sum_{j\geq 1} \pi_j(\vec{y}) \sum_{i\geq 1} \theta_i \left(\frac{\partial f}{\partial y_{i+j}} - \frac{\partial f}{\partial y_j}\right)(\vec{y}). \end{aligned} \tag{2.4}$$

Indeed, taking for instance the second term in the right-hand side of (2.1), we have

$$\mu N \sum_{j\geq 1} y_j \left[f\left(\vec{y} + \frac{\vec{e}_{j-1}}{N} - \frac{\vec{e}_j}{N}\right) - f(\vec{y})\right] = \mu \sum_{j\geq 1} y_j \left(\frac{\partial f}{\partial y_{j-1}} - \frac{\partial f}{\partial y_j}\right)(\vec{y}) + \mathcal{R}_N,$$

where, by Taylor's formula,

$$\mathcal{R}_N = \frac{\mu}{N} \sum_{j\geq 1} y_j \left(\frac{\partial^2 f}{\partial y_{j-1}^2} + \frac{\partial^2 f}{\partial y_j^2} - 2\frac{\partial^2 f}{\partial y_j \partial y_{j-1}}\right)(\vec{x}_j),$$

and

$$\vec{x}_j = \vec{y}_j + \delta_j \left(\frac{\vec{e}_{j-1}}{N} - \frac{\vec{e}_j}{N}\right), \quad 0 < \delta_j < 1.$$

So, for any $f \in \mathcal{C}_2(\mathbb{L})$, we get the estimate

$$|\mathcal{R}_N| \leq \frac{4\mu M H}{N}.$$

The other terms in (2.1) could be dealt with in a similar way.

Then, for $\vec{r} \in \mathbb{L}$, denote by $U(\vec{r})$ the right-hand side of system (2.2). Under condition **L**, it is known (see, for example, [1,6]) that the solution $\vec{x} : [0, \infty) \times \mathbb{L} \to \mathbb{L}$ of the differential system

$$\begin{cases} \dfrac{\mathrm{d}\vec{x}(t,\vec{a})}{\mathrm{d}t} = U(\vec{x}(t,\vec{a})), \\ \vec{x}(0,\vec{a}) = \vec{a}, \end{cases} \tag{2.5}$$

is unique, for any initial condition $\vec{a} \in \mathbb{L}$ and all finite t. Therefore, for $f \in \mathcal{C}(\mathbb{L})$, one can define a one-parameter family $\{T(t), t \geq 0\}$, such that

$$T(t)f(\vec{a}) = f(\vec{x}(t,\vec{a})).$$

Since $\vec{x}(t+s,\vec{a}) = \vec{x}(t,\vec{x}(s,\vec{a}))$, it is not difficult to see that $\{T(t), t \geq 0\}$ is a strongly continuous contraction semigroup on $\mathcal{C}(\mathbb{L})$ with generator, say B, which is closed by Theorem 6.1. Under condition **L**, $U(.)$ admits continuous partial derivatives. Hence, the solution $\vec{x}(t,\vec{a})$ of system (2.5) is also twice differentiable with respect to $\vec{a}$ (see [1], Theorems 3.4.2 and 3.7.1, pp. 148 and 152 respectively). Then Theorem 6.3 shows that $\mathcal{C}_1(\mathbb{L})$ is a core for B and $B = \overline{G}$.

The final line of argument proceeds in two steps.

1. Choose $f \in \mathcal{C}_2(\mathbb{L})$, so that $T(t)f(\vec{r}(0)) \in \mathcal{C}_2(\mathbb{L})$. Since $\mathcal{C}_2(\mathbb{L})$ is dense in $\mathcal{C}(\mathbb{L})$, Theorem 6.3 entails that $\mathcal{C}_2(\mathbb{L})$ is a core for B.
2. Use Eq. (2.3) together with Theorem 6.4.

□

2.3 Some analytic facts

Let, for $|z| \leq 1$,

$$r(t,z) = \sum_{i=1}^{\infty} r_i(t) z^i, \quad \pi(t,z) = \sum_{i=1}^{\infty} \pi_i(\vec{r}(t)) z^i,$$
$$\varphi(z) = \sum_{i=1}^{\infty} \varphi_i z^i, \qquad \theta(z) = \sum_{i=0}^{\infty} \theta_i z^i.$$

In this respect, the model of *Ma Micro Planète* presented in Sect. 1 corresponds to

$$\theta(z) = pz^{15} + (1-p)z^5, \quad \varphi(z) = z^{50}.$$

Taking the summation over i in system (2.2), we get immediately

$$\frac{\mathrm{d}r(t,z)}{\mathrm{d}t} + \mu\left[1 - \frac{1}{z}\right] r(t,z) = \lambda\varphi(z) - \mu r_1(t) + \alpha[\theta(z) - 1]\pi(t,z). \tag{2.6}$$

The following proposition ensures in particular, for the class of policies considered in Sect. 2.2, the boundedness of the Cesáro limit of $r(t,1)$ as $t \to \infty$, which, from its mathematical definition, takes into account the POIs possibly going to infinity.

For any arbitrary positive function f, let f^* denote its ordinary Laplace transform

$$f^*(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} f(t)\mathrm{d}t, \quad \Re(s) \geq 0.$$

Proposition 2.2 *When* $\{\theta_k, k \geq 1\}$ *is a proper probability distribution, so that* $\theta(1) = 1$*, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t r_1(t)\mathrm{d}t = \frac{\lambda}{\mu}, \tag{2.7}$$

Moreover, if $\theta'(1).\varphi'(1) < \infty$*, and for any policy satisfying Assumption (**L**),*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t r(t,1)\mathrm{d}t = \frac{\alpha\underline{\pi}\theta'(1) + \lambda\varphi'(1)}{\mu} \leq \frac{\alpha\theta'(1) + \lambda\varphi'(1)}{\mu}, \tag{2.8}$$

where

$$\underline{\pi} = \lim_{s\to 0} s\pi^*\Big(s, \frac{\mu}{\mu+s}\Big) \leq 1.$$

Proof By simple algebra carried out on Eq. (2.6), we obtain

$$r^*(s,z) = \frac{1}{\mu\left(1-\frac{1}{z}\right)+s}\left[\frac{\lambda\varphi(z)}{s} + \alpha\big(\theta(z)-1\big)\pi^*(s,z) - \mu r_1^*(s) + r(0,z)\right]. \tag{2.9}$$

Furthermore, applying Cauchy's formula

$$r_1^*(s) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{r^*(s,y)\mathrm{d}y}{y^2}$$

in Eq. (2.9), where $\mathcal{C}$ stands for the unit circle centered at the origin, we get

$$\begin{aligned} r_1^*(s) &= \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{\lambda\varphi(y)\mathrm{d}y}{sy\,[(\mu+s)y-\mu]} \\ &\quad + \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{\big[\alpha(\theta(y)-1)\pi^*(s,y) - \mu r_1^*(s) + r(0,y)\big]\mathrm{d}y}{y[(\mu+s)y-\mu]}, \end{aligned}$$

which in turn, by a repeated (and elementary) application of Cauchy's residue theorem, allows us to extract

$$r_1^*(s) = \frac{\lambda\varphi\left(\frac{\mu}{\mu+s}\right)}{\mu s} + \frac{\alpha}{\mu}\left[\theta\left(\frac{\mu}{\mu+s}\right) - 1\right]\pi^*\left(s, \frac{\mu}{\mu+s}\right) + \frac{1}{\mu} r\left(0, \frac{\mu}{\mu+s}\right). \quad (2.10)$$

Hence, instantiating the latter expression of $r_1^*(s)$ in (2.9), we obtain finally

$$\begin{aligned} r^*(s,z) = \frac{1}{\mu\left(1-\frac{1}{z}\right)+s} \Bigg[& \frac{\lambda}{s}\left(\varphi(z) - \varphi\left(\frac{\mu}{\mu+s}\right)\right) \\ & + \alpha\big(\theta(z)-1\big)\pi^*(s,z) + \alpha\left(1-\theta\left(\frac{\mu}{\mu+s}\right)\right)\pi^*\left(s, \frac{\mu}{\mu+s}\right) \\ & + r(0,z) - r\left(0, \frac{\mu}{\mu+s}\right)\Bigg]. \end{aligned} \quad (2.11)$$

Take now $s > 0$ and let $z \to 1_-$ by real positive values in (2.11). This yields

$$r^*(s,1) = \frac{1}{s}\left[\frac{\lambda}{s}\left(1-\varphi\left(\frac{\mu}{\mu+s}\right)\right) + \alpha\left(1-\theta\left(\frac{\mu}{\mu+s}\right)\right)\pi^*\left(s, \frac{\mu}{\mu+s}\right)\right], \quad (2.12)$$

where we have used the fact that $\pi^*(s,1) = 1/s$, since $\pi(t,1) = 1, \forall t \geq 0$.

Moreover, $\pi^*\left(s, \frac{\mu}{\mu+s}\right) < \pi^*(s,1) = \frac{1}{s}$, for any $s > 0$, and we are now in a position to apply a Tauberian-type theorem for positive functions, see, for example, [4], just letting $s \to 0$ by positive values in (2.12), so that

$$\begin{aligned} \lim_{s\to 0} s r^*(s,1) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t r(t,1)\mathrm{d}t \\ &= \lim_{s\to 0}\left[\frac{\lambda}{s}\left(1-\varphi\left(\frac{\mu}{\mu+s}\right)\right) + \frac{\alpha}{s}\left(1-\theta\left(\frac{\mu}{\mu+s}\right)\right)s\pi^*(s,1)\right] \\ &= \frac{\alpha\underline{\pi}\theta'(1) + \lambda\varphi'(1)}{\mu}, \end{aligned}$$

which is exactly the asserted equality (2.8). It is worth noting that for the moment there is no need to assume the finiteness of the derivatives $\theta'(1)$ and $\varphi'(1)$, but this will be made more precise in Sect. 3.

As for (2.7), it is readily obtained from (2.10), using the inequality $s\pi^*\left(s, \frac{\mu}{\mu+s}\right) \leq 1$, and again the above Tauberian theorem. The proof of the proposition is concluded. □

3 System behavior as $t \to \infty$

In the sequel, we shall focus our attention solely on the deterministic dynamical system $\vec{r}(t)$ obtained in Theorem 2.1. Of special interest will be the non-degenerate stationary limits

$$\vec{r} = \lim_{t\to\infty} \vec{r}(t), \quad \text{with } r_i(t) \to r_i, \forall i \geq 1,$$

and

$$\overline{\pi} \stackrel{\text{def}}{=} \lim_{z\to 1_-} \lim_{t\to\infty} \pi(t,z) = \sum_{i\geq 1} \pi_i(\vec{r}). \tag{3.1}$$

Indeed, from the general theory of differential systems (see, for example, [1,6]), under Assumption **(L)** and for a given set of parameters, we know that a unique stationary regime of the system exists, but *possibly degenerate* in the sense that $\overline{\pi} < 1$. Note that this fact is not easy to prove by a probabilistic argument, since, due to the normalizing condition, system (2.2) cannot be interpreted as forward Kolmogorov's equations for a special birth and death process, where $r(t,z)/r(t,1)$ would represent the probability distribution function of the score of an arbitrary POI.

Throughout the rest of the paper, we shall speak of *ergodicity*, with a slight abuse of language: this will always refer to a *finite stationary* dynamical system satisfying the conditions

$$r(1) = \sum_{i\geq 1} r_i < \infty, \quad \overline{\pi} = 1. \tag{3.2}$$

This notion of *ergodicity* will be also referred to as a *non-herding* behavior, in agreement with the brief description given in Sect. 1. Conversely, a *herding* behavior will refer to *transient* phenomena, as introduced more precisely in Sect. 3.5.

3.1 Global conservation equations

Define the generating functions

$$r(z) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} r_i z^i, \quad \pi(z) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \pi_i(\vec{r}) z^i,$$

with the shortened notation $\pi(z) \equiv \pi(z;\vec{r})$, and

$$\frac{1-\theta(z)}{1-z} \stackrel{\text{def}}{=} \sum_{n\geq 0} \Theta_n z^n, \quad \frac{1-\varphi(z)}{1-z} \stackrel{\text{def}}{=} \sum_{n\geq 0} \Phi_n z^n. \tag{3.3}$$

Two characteristic values are of special interest, namely

- $r(1) = \sum_{i\geq 1} r_i$, the mean number of POIs per player;
- r_1, since μr_1 is the global death rate of POIs per player.

By (2.6), one sees easily that $r(z)$ satisfies the functional equation

$$\alpha\,[1-\theta(z)]\,\pi(z) + \mu[1-z^{-1}]r(z) = \lambda\varphi(z) - \mu r_1, \tag{3.4}$$

from which can be deduced the following corollary, which is the direct continuation of Proposition 2.2.

Corollary 3.1 *1. We have always* $\lim_{z\to 1}(1-z)r(z) = 0$ *and* $r_1 = \dfrac{\lambda}{\mu}$.
2. If in addition $\theta'(1).\varphi'(1) < \infty$, *then*

$$\frac{r(z)}{z} = \frac{\alpha\Theta(z)\pi(z)}{\mu} + \frac{\lambda\Phi(z)}{\mu}, \quad \forall|z| \le 1, \tag{3.5}$$

$$r(1) = \frac{\alpha\overline{\pi}\theta'(1) + \lambda\varphi'(1)}{\mu}. \tag{3.6}$$

Proof Immediate from Proposition 2.2 together with notation (3.3). □

Assumption (F) *In the rest of the paper, unless otherwise explicitly mentioned, the product* $\theta'(1).\varphi'(1) < \infty$ *will always be assumed to be finite*.

From Eq. (3.5), we get the immediate recursion, valid $\forall n \ge 0$,

$$r_{n+1} = \frac{\alpha}{\mu}\sum_{i=0}^{n-1}\pi_{n-i}\Theta_i + \frac{\lambda}{\mu}\Phi_n. \tag{3.7}$$

In the next sections, we examine in detail the effect of specific choices of the policy π on the behavior of the system. One of main goals of the study is to find out *the necessary and sufficient conditions* for the system to be *ergodic* in the sense given by the conditions (3.2).

3.2 Selection of POIs according to their scores

Hereafter, but in Sect. 3.4, we shall consider a set of policies where the selection of a POI depends solely on its score. Letting each score $i \ge 1$ be associated with a positive weight a_i, then we recover the policy $\mathbf{P}_{\vec{a}}$ announced in Assumption (**L**) of Sect. 2.2, namely

$$\pi_i(\vec{r}) = \frac{a_i r_i}{K}, \quad K \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} a_j r_j, \tag{3.8}$$

and (3.7) can be rewritten as

$$r_{n+1} = \frac{\alpha}{\mu K}\sum_{i=0}^{n-1} a_{n-i} r_{n-i}\Theta_i + \frac{\lambda}{\mu}\Phi_n. \tag{3.9}$$

In the context of *Ma Micro Planète*, it seems quite realistic to assume that the a_i's are *increasing with* i, since POIs with high scores should be more attractive. In the sequel, we shall mainly consider this situation of *herding effect*.

Since the a_i's are assumed to form an increasing sequence, there exists $\lim_{i\to\infty} a_i$. When this limit is not finite, the following simple interesting proposition holds.

Proposition 3.2 *If* $\lim_{i\to\infty} a_i = \infty$*, then the system is non-ergodic.*

Proof The relation (3.9) leads to the simple bound

$$r_{n+1} \geq \frac{\alpha}{\mu K} a_n r_n \Theta_0 = \frac{\alpha}{\mu K} a_n r_n,$$

so that, when the a_i's are unbounded, system (3.2) admits no solution with $K < \infty$.

Nonetheless, it turns out there is in this case exactly one admissible solution to (3.2), namely

$$r_{n+1} = \frac{\lambda}{\mu}\Phi_n, \quad \text{with} \quad K = \infty,$$

which implies also (see (3.1)) $\overline{\pi} = 0$. Here, it is worth checking that $\pi(\vec{r})$ does not satisfy a uniform Lipschitz condition, since

$$\lim_{j\to\infty} \frac{a_j}{\sum_{i\geq 1} a_i r_i} = \infty.$$

□

Consequently the only interesting case is when $\{a_i, i \geq 1\}$ is a *non-decreasing sequence* tending to a *finite limit*, which ad libitum can be taken equal to 1, since the a_i's are defined up to a constant.

3.3 When scores of selected POIs are exactly increased by one

As it will appear in Sects. 5.1 and 5.2, a complete solution to the non-linear Eq. (3.9) for a general $\theta(z)$ seems to be technically almost untractable. In this section, in order to get ideas about the general behavior of the system, we analyze in detail the simple (but nonetheless not elementary!) case $\theta(z) = z$, for which Eq. (3.9) becomes the following first order recursive sequence

$$r_{n+1} = \frac{\alpha}{\mu K} a_n r_n + \frac{\lambda}{\mu}\Phi_n. \tag{3.10}$$

Simple algebra carried out on this last equation leads to the formula

$$r_{n+1} = \frac{\lambda}{\mu} A_n \sum_{j=0}^{n} \frac{\Phi_j}{A_j}\left(\frac{\alpha}{\mu K}\right)^{n-j}, \quad n \geq 0, \tag{3.11}$$

where

$$A_n \stackrel{\text{def}}{=} \prod_{i=1}^{n} a_i.$$

The constant K can then be recovered as

$$\begin{aligned}K &= \sum_{i\geq 1} a_i r_i = \frac{\lambda}{\mu}\sum_{i=0}^{\infty} a_{i+1}A_i \sum_{j=0}^{i}\frac{\Phi_j}{A_j}\left(\frac{\alpha}{\mu K}\right)^{i-j}\\ &= \frac{\lambda}{\mu}\sum_{j=0}^{\infty}\frac{\Phi_j}{A_j\left(\frac{\alpha}{\mu K}\right)^j}\sum_{i=j}^{\infty} A_{i+1}\left(\frac{\alpha}{\mu K}\right)^i\\ &= \frac{\lambda K}{\alpha}\sum_{j=0}^{\infty}\frac{\Phi_j}{A_j\left(\frac{\alpha}{\mu K}\right)^j}\sum_{i=j+1}^{\infty} A_i\left(\frac{\alpha}{\mu K}\right)^i,\end{aligned}$$

which leads to the following implicit equation

$$\frac{\alpha}{\lambda} = F\left(\frac{\alpha}{\mu K}\right), \tag{3.12}$$

allowing us to determine K, after having set

$$\begin{cases} F(z) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} u_k z^k,\\ u_k = \sum_{j=0}^{\infty} \frac{A_{j+k}\Phi_j}{A_j}, \quad \forall k \geq 1. \end{cases} \tag{3.13}$$

So, the problem of showing the existence of a *strictly positive and finite constant* K is exactly tantamount to finding a strictly positive real number x, such that

$$\frac{\alpha}{\lambda} = F(x) = \sum_{k=1}^{\infty} u_k x^k.$$

Hence, in order to get effective ergodicity conditions, one has to check the lower and upper bounds of $F(x)$, which is an increasing function of x, for $x \geq 0$. We note first that $F(x)$ is finite for small enough $|x|$ if, and only if,

$$u_1 = \sum_{j=0}^{\infty} a_{j+1}\Phi_j < \infty. \tag{3.14}$$

But, ex hypothesis, $A_{k+1}/A_k = a_k$ increases to 1 from below, so that u_k is a monotone decreasing sequence and inequality (3.14) is plainly equivalent to

$$\varphi'(1) = \sum_{j\geq 0}\Phi_j < \infty. \tag{3.15}$$

As for the upper bound, we have the immediate inequality

$$u_k \leq \varphi'(1), \quad \forall k \geq 1,$$

and hence the radius of convergence of $F(z)$ is exactly equal to 1. Now, in order to decide whether the consistency relation (3.12) admits a solution, the key point will be to analyze the quantity

$$M = \lim_{x \to 1_-} F(x).$$

There are two possibilities.

(i) If M is infinite, then for any value of λ, α, μ, it is possible to find a unique admissible finite K, and we shall say that the system is *ergodic*.
(ii) Conversely, if $M < \infty$, then it will act as a limiting value for α/λ, and hence a non-trivial *ergodicity condition* takes place.

The situation is summarized in the following brief statement.

Theorem 3.3 *Assume that* $\lim_{i \to \infty} a_i \nearrow 1$.

(i) The system is never ergodic when $\varphi'(1) = \infty$.
(ii) When $\varphi'(1) < \infty$, *the system is ergodic if, and only if,*

$$\frac{\alpha}{\lambda} \leq \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{A_{j+k} \Phi_j}{A_j}, \tag{3.16}$$

where the r.h.s. of the inequation above may be infinite.

It is interesting to note a few not so intuitive facts related to (3.16): first, this condition does not depend on the value of μ, as long as $\mu > 0$ obviously. Second, the ergodicity range increases when the initial score increases in distribution—that is, when the Φ_i's increase.

To propose a somehow concrete classification, let us concentrate for the remainder of this section on the reasonably general set of sequences satisfying, for some strictly positive constants γ and ν,

- $a_1 > 0$,
- the a_i's form an increasing sequence,
- $a_i = 1 - \gamma i^{-\nu} + \mathcal{O}\big(i^{-\nu-1}\big)$.

The following proposition describes the *phase transition* phenomenon that occurs when the parameters vary.

Proposition 3.4 *Assume the sequence* a_i *satisfies the above conditions, and that* $\theta(z) = z$. *Then,*

- *if* $\nu > 1$, *or* $\nu = 1$ *and* $\gamma \leq 1$, *the system is always ergodic;*
- *if* $\nu < 1$, *or* $\nu = 1$ *and* $\gamma > 1$, *there is a finite ergodicity bound* M *if, and only if,* $\sum_{j>0} j^{\nu+1} \varphi_j < \infty$.

Proof The first step is to obtain a convenient estimate for A_n when n is large. Using the property $0 < a_1 < a_i < 1$, valid for all i, one gets

$$\log A_n = \sum_{i=1}^{n} \log a_i = -\gamma \sum_{i=i}^{n} i^{-\nu} + \mathcal{O}(1).$$

The finiteness of M depends strongly on the properties of the series $\sum A_n$, which can be deduced from the following estimates.

- If $\nu > 1$, then the Dirichlet series $\sum_{i \geq 1} i^{-\nu}$ converges to some positive number, so that the series $\sum A_n$ diverges, and then clearly $M = +\infty$.
- If $\nu = 1$, then $A_n \approx C_1 n^{-\gamma}$. When $\gamma \leq 1$, $\sum A_n$ still diverges and $M = +\infty$; conversely, when $\gamma > 1$,

$$\frac{1}{A_j} \sum_{i=j+1}^{\infty} A_i \approx C_2 j,$$

  and M is finite whenever the second moment $\sum_{j>0} j^2 \varphi_j$ of the distribution $\{\varphi_k, k \geq 1\}$ exists.
- If $\nu < 1$, then $A_n = \mathcal{O}(e^{-\gamma n^{1-\nu}})$. Using classical techniques (Riemann sums and integration by parts), one gets

$$\sum_{i=j+1}^{\infty} e^{-\gamma i^{1-\nu}} \approx \int_{j}^{\infty} e^{-\gamma x^{1-\nu}} \mathrm{d}x \approx \frac{j^{\nu}}{\gamma(1-\nu)} e^{-\gamma j^{1-\nu}},$$

  and hence M is finite whenever $\sum_{j>0} j^{\nu+1} \varphi_j$ is.

This concludes the proof of the proposition, the results of which will be illustrated in Sect. 4.5. □

3.4 Selecting POIs in terms of their cumulative score distribution

All models considered so far are based on exogenous parameters a_i, supposed to be known and independent of $\vec{r}(t)$. Actually, it turns out that most of the computations remain valid when these parameters are a *functional of the state* $\vec{r}(t)$. We tackle hereafter such a model, for which ergodicity conditions can be explicitly obtained.

The selection policies family we are interested in will be expressed as a functional of the cumulative score distribution. Let

$$P_i \stackrel{\text{def}}{=} \frac{1}{r(1)} \sum_{j=1}^{i} r_j$$

be the proportion of POIs with score less of equal to i, and let f be a positive convex function on [0, 1] such that $f(0) = 0$, $f(1) = 1$ and $f'(1) < \infty$. Then the probability of selecting a POI with a score less or equal to i is chosen to be $f(P_i)$ or, equivalently,

$$\pi_i(\vec{r}) = f(P_i) - f(P_{i-1}). \tag{3.17}$$

An example of such a policy is given by the following simple algorithm: $c > 1$ POIs are selected uniformly, and the one with the largest score will be visited. This model is reminiscent of the queueing model of [10], which proves to be very efficient in thermodynamical limit. The probability of selecting a POI with score equal to i is then given by (3.17) when $f(x) = x^c$.

From a formal point of view, we remain in the framework of Sect. 3.2, as it can be seen just by setting

$$a_i = \frac{r(1)}{f'(1)r_i}\big[f(P_i) - f(P_{i-1})\big], \quad K = \frac{r(1)}{f'(1)}, \tag{3.18}$$

and remarking that the $a_i's$ are no more exogenous constants as before. To check they form an increasing positive sequence bounded by 1, it suffices to note that, since f is a convex function of x,

$$\frac{r(1)}{r_i}\big[f(P_i) - f(P_{i-1})\big] \le f'(P_i) \le \frac{r(1)}{r_i}\big[f(P_{i+1}) - f(P_i)\big] \le f'(1).$$

While the equations are similar, the situation is nonetheless very different from that encountered in Sect. 3.2. The model is more difficult to solve, as the π_i's do not have a simple expression in terms of the score i, so that a formal solution like (3.11) is not really usable; however, they can be computed by recurrence, since the normalization constant $r(1)$ is given by (3.6). As a consequence, the ergodicity condition is easier to obtain and, interestingly enough, appears to be *insensitive* to the distributions θ and φ.

Theorem 3.5 *Assume that the POIs are visited according to policy* (3.17)*, with f strictly convex, and therefore $1 < f'(1) < \infty$. Then, according to the definition given in Sect. 3, the system is ergodic if, and only if,*

$$\frac{\alpha\theta'(1)}{\lambda\varphi'(1)} \le \frac{1}{f'(1) - 1}.$$

Proof The recurrence relation (3.7) takes the form

$$P_{n+1} - P_n = \frac{\alpha}{\mu r(1)} \sum_{i=0}^{n-1} \big(f(P_{n-i}) - f(P_{n-i-1})\big)\Theta_i + \frac{\lambda}{\mu r(1)}\Phi_n,$$

which by a direct summation yields

$$P_{n+1} = \frac{\alpha}{\mu r(1)} \sum_{i=0}^{n-1} f(P_{n-i})\Theta_i + \frac{\lambda}{\mu r(1)} \sum_{i=1}^{n} \Phi_i, \quad \forall n \geq 0. \tag{3.19}$$

The form (3.19) allows for a direct graphical analysis of the convergence problem, starting from the following simple facts. Using (3.6) with $\overline{\pi} = 1$, it is straightforward to check that the non-decreasing sequence defined by (3.19) is bounded by 1. Therefore, it always converges to a finite value

$$P_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} P_n \leq 1,$$

that satisfies, by a direct application of the celebrated Toeplitz lemma,

$$P_\infty = \frac{\alpha}{\mu r(1)}\theta'(1) f(P_\infty) + \frac{\lambda}{\mu r(1)}\varphi'(1).$$

It is actually possible for the sequence to converge to some $P_\infty < 1$ but, $f(.)$ being a convex function, this can only happen when the slope at the point 1 is greater than 1. This concludes the theorem, since (3.6) implies the equivalence

$$\frac{\alpha\theta'(1)}{\mu r(1)} f'(1) \leq 1 \quad \Longleftrightarrow \quad \frac{\alpha\theta'(1)}{\lambda\varphi'(1)} \leq \frac{1}{f'(1) - 1}.$$

□

3.5 About transience: herding phenomena

Hereafter, we do not pretend to give exhaustive rigorous proofs of all our claims, which can rather be viewed as *very likely true* conjectures, and have been verified by several numerical simulation runs.

Our concern is to extend Theorem 3.3, when there is no solution to (3.16), i.e., when its right-hand side member is finite and α is too large. In fact, it appears that the system has still a stationary regime, but a *finite number of POIs have an infinite score*. This means that, as $t \to \infty$, the selection policy $\pi(\vec{t})$ becomes *defective*, namely [see Eq. (3.1)] when

$$\overline{\pi} < 1.$$

This implies exactly

$$\pi_i(\vec{r}) = \frac{a_i r_i}{\sum_{j\geq 1} a_j r_j + \delta},$$

where δ is a positive constant representing the proportion of POIs going to infinity. Obviously, $\delta = 0$ when the system is *normally* ergodic, i.e., no escape of mass to infinity. By contrast, $\delta > 0$ corresponds to a phase transition with condensation. Then the consistency Eq. (3.12) must be modified and becomes

$$1 = \frac{\lambda}{\alpha} F\left(\frac{\alpha}{\mu K}\right) + \frac{\delta}{K}, \tag{3.20}$$

which has *always* a solution (K, δ) by monotonicity with respect to K, remembering that $F(1) < \infty$, while $F(1+\varepsilon) = \infty, \forall \varepsilon > 0$, and $\alpha > \lambda F(1)$. Equation (3.20) is not sufficient to determine the two unknowns K and δ. To get a second equation, we resort to the following heuristic *stochastic least action principle*, which in our opinion should be provable in a very general context.

Ansatz 3.6 The least action principle. *When a multicomponent irreducible Markovian system ceases to be ergodic, the first component(s) becoming infinite with positive probability can be identified by* continuity *with respect to the set of parameters $\mathcal{P}$ defining the system.*

This definition, which might look obscure to the reader, can be rephrased by saying that the phase transition takes place as soon as one touches some regions (hyperplanes or surfaces) in $\mathcal{P}$. This is for instance the case for the famous Jackson–Kelly stochastic queueing networks.

Applying this principle leads to say that we are looking for the minimal $\delta > 0$ ensuring the system

$$\vec{r} = \lim_{t\to\infty} \vec{r}(t)$$

is no more ergodic, so that *some POIs go to infinity*. This implies necessarily

$$\frac{\alpha}{\mu K} = 1. \tag{3.21}$$

Then (3.20) and (3.21) yield at once

$$\begin{cases} \delta = \dfrac{\alpha - \lambda F(1)}{\mu}, \\ \overline{\pi} = \dfrac{K-\delta}{K} = \dfrac{\lambda F(1)}{\alpha} < 1, \\ r(1) = \dfrac{\lambda}{\mu}[F(1) + \varphi'(1)]. \end{cases}$$

A possible way to decide analytically when $\overline{\pi} < 1$ would be to design a converging iterative scheme to solve system (2.2) or (2.6) (see, for example, the polling network analyzed in [2]): this appears to be another interesting, but practically intricate problem.

4 Some simple examples and limit cases

For the sake of completeness, we quote hereafter simple peculiar cases, which hopefully shed light on some aspects of the behavior of the system.

4.1 No creation of POIs, i.e., $\lambda = 0$

In this case, the system is empty at steady state for any Markovian policy. The argument is elementary, since the vector state $\vec{0}$ is absorbing due to the $M/M/\infty$ character of the score decreasing process.

4.2 The lower bound $\alpha = 0$

Here no POI will see any increase of score upon being visited, and hence one gets a clear lower bound for the model. Solving (3.5) leads to

$$r_i = \frac{\lambda}{\mu} \sum_{k \geq i} \varphi_k, \quad r(1) = \frac{\lambda \varphi'(1)}{\mu}, \quad r'(1) = \frac{\lambda \big[\varphi''(1) + 2\varphi'(1)\big]}{2\mu}.$$

Consequently, when $\alpha \neq 0$, we have the following bound, valid for any point selection scheme (i.e. policy) π,

$$r'(1) \geq \frac{\lambda \big[\varphi''(1) + 2\varphi'(1)\big]}{2\mu},$$

and therefore the mean score $r'(1)/r(1)$ of a POI can be finite only when $\varphi''(1)$ is finite. This leads to some unintuitive behavior, since it was shown in Proposition 3.4 that having $\varphi''(1) = \infty$ can at the same time make the system always ergodic.

4.3 Score independent visits

When the POIs are visited uniformly regardless of their score, which amounts to the choice

$$\pi_i(\vec{r}) = \frac{r_i}{\sum_{i \geq 1} r_i},$$

we have $\pi(z) = r(z)/r(1)$. Then Eq. (3.5) allows us to derive immediately the corresponding generating function, which takes the compact explicit form

$$r(z) = \frac{\lambda r(1)(\varphi(z) - 1)}{\alpha(1 - \theta(z)) + \mu r(1)(1 - z^{-1})},$$

where, for $\lambda \neq 0$,

$$r(1) = \frac{\alpha\theta'(1) + \lambda\varphi'(1)}{\mu},$$

and one can check the denominator of $r(z)$ does not vanish for $|z| < 1$. Moreover, $r(1)$ is clearly finite and given by Eq. (3.6). So, we have obtained a unique function $r(z)$ analytic in the unit disc and continuous for $|z| = 1$. Consequently the system admits of a unique limit invariant measure $\{r_i, i \geq 1\}$ *for any value of the parameters*. Intuitively, this is due to the fact that the decrease rate of the scores is somehow proportional to the number of points, like in a $M/M/\infty$ queue. But, when the selection process favors heavily enough the POIs having a big score, proper ergodicity conditions appear, as in the models analyzed in Sect. 3.2.

4.4 An unstable model with a score dependent policy

A policy that may seem attractive at first sight is to give to each POI a weight proportional to its score

$$\pi_i(\vec{r}) = \frac{ir_i}{\sum_{i\geq 1} ir_i},$$

so that $\pi(z) = zr'(z)/r'(1)$ and (3.5) becomes a differential equation

$$\alpha\,[1-\theta(z)]\,\frac{zr'(z)}{r'(1)} + \mu[1-z^{-1}]r(z) = \lambda[\varphi(z)-1].$$

However, the weight $a_i = i$ given to the score i is unbounded, and from Proposition 3.2 we conclude that no choice of parameters can lead to a stable system under this policy. More precisely, *there exists at least one POI having an infinite score*.

4.5 A case study of Proposition 3.4

Assuming the conditions of Proposition 3.4 hold, let us take in addition $\varphi(z) = z$ and, for some $\gamma > 0$,

$$a_i = \left(\frac{i}{i+1}\right)^{\gamma}.$$

This implies $A_k = 1/(k+1)^{\gamma}$ and therefore (3.11–3.12) become

$$r_{k+1} = \frac{\lambda}{\mu}\frac{1}{(k+1)^{\gamma}}\left(\frac{\mu K}{\alpha}\right)^{k},$$

$$\frac{\alpha}{\lambda} = F(x) = \sum_{k=1}^{\infty} A_k x^k = \Phi(x,\gamma,1) - 1,$$

where Φ stands for the classical Lerch function [5, formula 9.550]

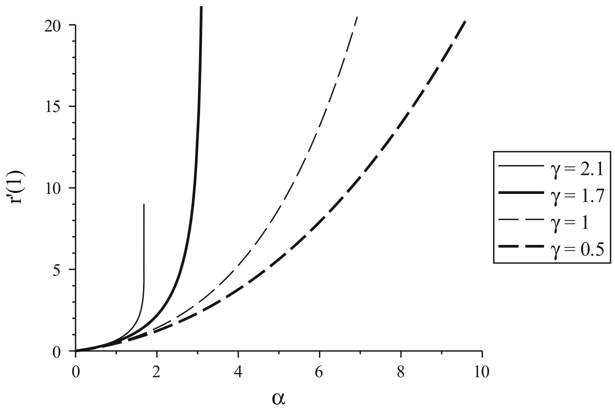

Fig. 1 Expected score of POIs per player $r'(1)$ as a function of the visit rate α. The parameters $\lambda = \mu = 3$ remain fixed

$$\Phi(x, \gamma, v) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{x^k}{(k+v)^{\gamma}}.$$

When $\varphi(z) = z$, the computation of the expected score of POIs per player $r'(1)$ is easy:

$$r'(1) = \sum_{k=1}^{\infty} k r_k = \sum_{k=1}^{\infty} \frac{\lambda}{\mu} \sum_{k=1}^{\infty} k A_k \left(\frac{\alpha}{\mu K}\right)^k = \frac{\lambda}{\mu} \frac{\alpha}{\mu K} F'\left(\frac{\alpha}{\mu K}\right).$$

The above formulas allow us to represent the expected score per player as a function of α in a parametric way, by letting x vary from 0 to 1 (Fig. 1). As expected from the theoretical results obtained above, 3 different situations prevail:

- $\gamma \leq 1$: the system is always ergodic (dashed curves);
- $1 < \gamma \leq 2$: there is an explicit ergodicity condition on α, and $r'(1)$ tends to infinity at the boundary;
- $\gamma > 2$: there is an explicit ergodicity condition on α, but $r'(1)$ remains bounded.

5 Miscellaneous outcomes and open problems

The next three paragraphs present briefly some ideas and mathematical questions concerning the general model.

5.1 General approach via an integral equation

Consider the situation of Sect. 3.2, and let $a(z) \stackrel{\text{def}}{=} \sum_{j \geq 1} a_j z^j$. Denote the respective convergence radii of $r(\cdot)$ and $a(\cdot)$ by ρ_r and ρ_a. Then, by Hadamard's multiplication theorem [8] and definition (3.8), we can write

$$\pi(z) = \frac{1}{2i\pi K} \int_{\mathcal{L}} r(w) a\left(\frac{z}{w}\right) \frac{\mathrm{d}w}{w},$$

where $\mathcal{L}$ stands for a circle of radius ρ, centered at the origin, such that $|z|/\rho_a < \rho < \rho_r$. This yields the non-linear integral equation

$$\frac{\mu r(z)}{z} = \frac{\alpha[1-\theta(z)]}{2i\pi K(1-z)} \int_{\mathcal{L}} r(w) a\left(\frac{z}{w}\right) \frac{\mathrm{d}w}{w} + \frac{\lambda(1-\varphi(z))}{1-z}. \tag{5.1}$$

Methods of solution of (5.1) (e.g. via boundary value problem or Fredholm integral equation) depend heavily on the explicit form of $a(\cdot)$. Alas, tractable explicit expressions can hardly be derived (nor expected!), even when $a(\cdot)$ is a polynomial.

5.2 General approach via a matrix form solution

The problem becomes more intricate when $\theta(z)$ is an arbitrary probability generating function with $\varphi'(1) < \infty$. Then the following approach, based on the recursion (3.9), involves infinite matrices and seems to be computationally quite acceptable. We sketch hereafter its main lines.

Clearly, Eq. (3.9) can be rewritten in the vector form

$$\vec{U}_{n+1} = M_n\left(\frac{\alpha}{\mu K}\right)\vec{U}_n + \frac{\lambda}{\mu}\vec{V}_{n+1}, \tag{5.2}$$

where

- $\vec{U}_n$ is the n-column vector $[r_n r_{n-1} \ldots r_1]^T$, where ^{T} denotes the transpose operation;
- $\vec{V}_n$ is the n-column vector $[\Phi_{n-1}, 0, \ldots, 0]^T$;
- $M_n(x)$ is a $(n+1) \times n$ matrix for all $x > 0$. Its first row is the vector $x\vec{B}_n$, where $\vec{B}_n$ denotes the row vector

  $$[a_n\Theta_0, a_{n-1}\Theta_1, \ldots, a_1\Theta_{n-1}],$$

  and the remaining internal matrix is exactly the $n \times n$ identity matrix denoted by I_n.

By direct algebra, the solution takes the form

$$\vec{U}_{n+1} = \frac{\lambda}{\mu} \sum_{i=1}^{n+1} Q_{n,i}\left(\frac{\alpha}{\mu K}\right)\vec{V}_i, \quad n \geq 0, \tag{5.3}$$

where

$$\begin{cases} Q_{n,i}(x) = M_n(x) M_{n-1}(x) \ldots M_i(x), & i \leq n, \\ Q_{n,n+1}(x) \stackrel{\text{def}}{=} I_{n+1}. \end{cases}$$

The next step is to carry out the scalar product of (5.3) with the row vector

$$\vec{A}_{n+1} \stackrel{\text{def}}{=} [a_{n+1}, \ldots, a_1],$$

and then to let $n \to \infty$. Keeping in mind that $K = \sum_{i \geq 1} a_i r_i$, we get the following final relationship, which is formally similar to the one given by (3.12),

$$\frac{\alpha}{\lambda} = \frac{\alpha}{\mu K} \sum_{n=0}^{\infty} \sum_{i=1}^{n+1} \vec{A}_{n+1} \widetilde{Q}_{n,i}\left(\frac{\alpha}{\mu K}\right) \vec{V}_i. \tag{5.4}$$

From the definition of M_n, each term of the double sum in (5.4) is clearly a *decreasing* scalar function of K. Hence, ergodicity conditions could be obtained along the same lines as in Theorem 3.3, by analyzing the series in the right-hand side member of (5.4).

5.3 The case $\theta'(1).\varphi'(1) = \infty$

When either $\theta'(1)$ or $\varphi'(1)$ are infinite, we are a priori facing a *transient* phenomenon as $r(1) = \infty$. However, the system can have an interesting behavior provided that $K = \sum_{i \geq 1} a_i r_i < \infty$, which yields necessarily

$$\liminf_{i \to \infty} a_i = 0,$$

but this would correspond to a *non-herding* system, which is an other story, although most of the mathematical arguments presented in Sect. 3 could be readily applied.

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Appendix: Generators, cores and weak convergence

Here we quote the material necessary for the proof of Theorem 2.2. The results are borrowed from [3].

Theorem 6.1 *If O is the generator of a strongly continuous semigroup $\{T(t)\}$ on $\mathcal{L}$, then its domain $\mathcal{D}(O)$ is dense in $\mathcal{L}$ and O is closed.*

Definition 6.2 *[3, p. 17] Let O be a closed linear operator with domain $\mathcal{D}(O)$. A subspace S of $\mathcal{D}(O)$ is said to be a* core *for O if the closure of the restriction of O to S is equal to O, i.e., if $\overline{O_{|S}} = O$.*

The next proposition is an important criterion to characterize a core.

Theorem 6.3 *[3, p. 17] Let O be the generator of a strongly continuous contraction semigroup $\{T(t)\}$ on $\mathcal{L}$. Let $\mathcal{D}_0$ and $\mathcal{D}$ be dense subspaces of $\mathcal{L}$ with $\mathcal{D}_0 \subset \mathcal{D} \subset \mathcal{D}(O)$. If $T(t) : D_0 \to \mathcal{D}$ for all $t \geq 0$, then $\mathcal{D}$ is a core for O.*

The proof of Theorem 2.1 relies heavily on the next general proposition.

Theorem 6.4 *[3, Theorem 6.1, p. 28] In addition to* $\mathcal{L}$*, let* $\mathcal{L}_k$, $k \geq 1$*, be a sequence of Banach spaces,* $\Pi_k : \mathcal{L} \to \mathcal{L}_k$ *be a bounded linear transformation, subject to the constraint* $\sup_k \|\Pi_k\| < \infty$*. Let also* $\{T_k(t)\}$ *and* $\{T(t)\}$ *be strongly continuous contraction semigroups on* $\mathcal{L}_k$ *and* $\mathcal{L}$ *with generators* O_k *and* O*. We write* $f_k \to f$ *to mean exactly*

$$f \in \mathcal{L}, \quad f_k \in \mathcal{L}_k \quad \text{for } k \geq 1, \quad \text{and} \lim_{k \to \infty} \|f_k - \Pi_k f\| = 0.$$

Then, if D *is a core for* O*, the following are equivalent:*

(a) For each $f \in \mathcal{L}$, $T_k(t)\Pi_k f \to T(t)f$ *for all* $t \geq 0$*, uniformly on bounded intervals.*
(b) For each $f \in \mathcal{L}$, $T_k(t)\Pi_k f \to T(t)f$ *for all* $t \geq 0$.
(c) For each $f \in D$*, there exists* $f_k \in \mathcal{D}(O_k)$ *for each* $k \geq 1$*, such that* $f_k \to f$ *and* $O_k f_k \to Of$.

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