Spatially coupled quantum LDPC codes

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***Abstract*— We propose here a new construction of spatially coupled quantum LDPC codes using a small amount of entangled qubit pairs shared between the encoder and the decoder which improves quite significantly all other constructions of quantum LDPC codes or turbo-codes with the same rate.**

I. Introduction

Quantum codes suitable for iterative decoding. Turbo-codes [1] and LDPC codes [2] and their variants are one of the most satisfying answers to the problem of devising codes promised by Shannon's theorem. They display outstanding performances for a large class of error models with a decoding algorithm of reasonable complexity. Generalizing these codes to the quantum setting seems a promising way to efficiently approach the quantum capacity, and quantum generalizations of LDPC codes have indeed been proposed in [3].

However, it has turned out that the design of high performance quantum LDPC codes is much more complicated than in the classical setting. In particular, most constructions suggested in the literature [4], [3], [5], [6], [7], [8], [9], [10], [11] suffer from having either a bounded minimum distance or a vanishing rate. There are only a few exceptions, namely [12], [13], [14], [15]. However in all these constructions, unlike in the classical setting, there are issues with the decoder: 4-cycles in their Tanner graph if decoding is performed over $\mathbb{F}_4$, code degeneracy which impairs the decoder [16].

On the other hand, generalizing turbo-codes to the quantum setting has first been achieved in [17]. However this construction had rather poor performance under iterative decoding. In [18] it was shown that it was possible to come up with quantum turbo-codes with good performance under iterative decoding. However, the families of codes constructed in this article have bounded minimum distance and the performance of these codes degrades for large blocklength. It was even proved there that it is not possible to obtain quantum serial turbo-codes with unbounded minimum distance and with an iterative decoding algorithm which converges. This is due to the fact that it can be proved that quantum convolutional encoders which are at the same time non-catastrophic and recursive do not exist [18].

Spatially coupled quantum LDPC codes. Spatially coupled LDPC have been introduced in [19] (they were named terminated convolutional LDPC codes there). They might be viewed in the following way, take several several instances of a certain LDPC code family, arrange them in a row and then mix the edges of the codes randomly among neighboring layers. Moreover fix the bits of the first and last layers to zero. It has soon been found out that iterative decoding behaves much better for this code than for the original LDPC code. A breakthrough occurred when it was proved that for the binary erasure channel, the noise threshold (that is the maximal probability of erasure which can be sustained by a code of infinite length) under iterative decoding of the spatially coupled ensemble coincides with the noise threshold under Maximum A Priori (MAP) decoding of the underlying LDPC code (which consists in taking the optimal decision for each bit) [20]. This has some dramatic consequences.

- The MAP threshold can be significantly better than the iterative decoding threshold and this especially when the degrees of the LDPC code get large.
- The MAP threshold can be already quite close to capacity for small regular degrees (for instance for a (4,8) regular LDPC code) and converges to capacity quickly as the degrees increase. This allows to use for instance regular degrees in the construction and we do not need to choose well optimized LDPC code ensembles with a large amount of degree 2 nodes which give codes of minimum distance at most logarithmic in the code length [21], [22]. LDPC codes with linear minimum distance in the blocklength can be chosen in the spatially coupled construction. This simplifies considerably the design of long codes with very low error floor and excellent performance under iterative decoding.
- This nice behavior does not only hold for the erasure channel, it actually holds for all binary input memoryless output-symmetric channels (BIMS) [23] and this *universally*: the authors construct there a single spatially coupled ensemble which attains a desired probability of error after decoding at a certain desired gap from capacity for all BIMS channels.

All these nice features of classical spatially coupled LDPC codes suggest to study whether they have a quantum analogue. The fact that spatially coupled LDPC codes may afford to have large degrees and still perform well under iterative decoding would be quite interesting in the quantum setting, since by the very nature of the quantum construction of stabilizer codes the rows of the parity-check matrix of the quantum code have to belong to the code which is decoded by the iterative decoder. This implies that we should have rather large row weights to avoid severe error-floor phenomena and/or oscillatory behavior of iterative decoding which degrades significantly its performance [16]. A first step in this direction was achieved in [24] where a certain family of quantum spatially coupled LDPC codes was suggested. They showed a family of (quantum) rate $\frac{1}{2}$ codes which correct for a length of $181,000$ qubits a depolarizing error of 0.03 for a probability of error after decoding of about 10^{-4}. This improves the underlying LDPC code construction since codes of this class which are 2 times longer only have this error probability after decoding for a depolarizing error probability of about 0.026. This should be put in perspective with the hashing bound capacity of the depolarizing channel (which is a lower bound for the true capacity of this channel) which says that there are quantum codes of rate $\frac{1}{2}$ which can operate successfully up to a depolarizing error probability of 0.075. The authors stayed in the classical stabilizer formalism and this complicates matters significantly, since there is no way with this construction alone to have a satisfactory quantum analogue of bits fixed to zero.

We choose here another route which assumes some additional resource consisting of shared entangled qubits between the transmitter and the receiver and which are noiseless on the receiver side [25]. This is called entanglement-assisted quantum error correction. In particular, the orthogonality constraints are less stringent than in the stabilizer formalism. It can also be viewed as stabilizer codes where some qubits (namely the halves of the maximally entangled qubits which are on the receiver side) are noise free. This is exactly what is needed to have an equivalent in the quantum world of information bits set to zero. Entanglement is used here in order to have qubits participating to the quantum code without having to sending them and therefore allowing them to be error free.

Our construction can now be described as follows. We start by giving a spatially coupled version of a construction of quantum LDPC codes suggested by [6] based on a couple of orthogonal (classical) LDPC codes obtained from low density generator matrix (LDGM) codes. This gives a stabilizer code of rate $\frac{1}{4}$ and a few first layers and a few last layers of the spatially coupled construction are error-free because these outermost layers are formed by the qubits of the receiver side which are not sent and are therefore noiseless. We use therefore only a very moderate amount of this resource in our construction. Despite this fact, we obtain a tremendous performance improvement over other families of codes of rate $\frac{1}{4}$. In our case, the probability of error after decoding drops down sharply after $p = 0.102$. This is not a real threshold since these codes are LDGM codes and have therefore a constant minimum distance, but no deterioration of the "threshold" could be observed experimentally when the length increases and this even for the quite large lengths (up to 76800) which were considered. This should be compared to the hashing bound capacity for codes of rate $\frac{1}{4}$ which corresponds to a depolarizing noise of $p_c \approx 0.1269$. Moreover our construction belongs to the family of CSS codes [26], [27] and from the way our codes are decoded, we decode namely two binary codes of rate $\frac{3}{8}$ affected by a binary symmetric channel of crossover probability $p' = \frac{2p}{3}$, we can not expect that this kind of strategy would be able to operate successfully for depolarizing noise above $p_0 \approx 0.1087$. Finally, we also demonstrate that our scheme is able to tolerate some moderate error noise on the qubits of the outermost layers without suffering severe performance loss. This is in strong contrast with catalytic error correction [28] or other coding strategies making use entangled qubits such as quantum polar coding [29] which can not not tolerate any amount of noise on these qubits.

II. Entanglement assisted stabilizer codes

We review in this section the entanglement assisted stabilizer code formalism [25]. The style of presentation we adopt here is to suit a readership familiar with classical codes but not with quantum information theory. Let us recall that the trace hermitian inner product between $E = (E_i)_{1 \leq i \leq n}$ and $F = (F_i)_{1 \leq i \leq n}$ in $\mathbb{F}_4^n$ is given by:

$$E \star F \triangleq \sum \operatorname{Tr} E_i \bar{F}_i, \tag{1}$$

where $\bar{u} = u^2$ and $\operatorname{Tr} u = u + \bar{u}$ for u in $\mathbb{F}_4$.

An entanglement assisted stabilizer code of type $[[n, k; c]]$ using c entangled qubit pairs shared between the transmitter and the receiver is defined by

Definition 1 (entanglement assisted stabilizer code): An entanglement assisted stabilizer code of type $[[n, k; c]]$ is specified by a matrix $\mathbf{H}$ over $\mathbb{F}_4$ of size $(n+c-k) \times (n+c)$ and a choice of a set I of c columns of $\mathbf{H}$ such that:

(i) The rows of $\mathbf{H}$ are independent over $\mathbb{F}_2$ and orthogonal with respect to the trace hermitian inner product,

(ii) If we let $\mathbf{H}'$ be the submatrix of $\mathbf{H}$ formed by erasing the aforementioned c columns in $\mathbf{H}$ and if we denote by $\mathbf{H}'_i$ the i-th row of $\mathbf{H}'$, then the matrix $\mathbf{M}' \triangleq (\mathbf{H}'_i \star \mathbf{H}'_j)_{\substack{1 \leq i \leq n+c-k \\ 1 \leq j \leq n+c-k}}$ has rank c.

$\mathbf{H}$ is called a parity-check matrix for the stabilizer code. When $c = 0$, the code is a called a *stabilizer code*.

Remarks

1) This definition is not completely standard, generally such a code is specified by the subgroup generated by the rows $\mathbf{H}'_i$ which satisfy Condition (ii), but it is better suited to our construction (we start with a construction of stabilizer codes) and specify the c columns later on.
2) An equivalent way of expressing Condition (ii) which can be proved by symplectic geometry arguments (see for instance [30, Lemma 2]) is the fact that the group generated by the $\mathbf{H}'_i$'s can be generated by $2c + (n - k - c)$ elements of $\mathbb{F}_4^n$ $u_1, u_2, \dots, u_c, v_1, \dots, v_c, w_1, \dots, w_{n-k-c}$ which are all orthogonal to each other (meaning that $u_i \star u_j = 0$ for instance) with the exception of $u_i \star v_i = 1$ for all i in $\{1, \dots, c\}$.
3) Such a choice of matrix amounts to a particular syndrome measurement.

Error model and information available for decoding. We consider Pauli channels and in this case the errors which occur are elements of $\mathbb{F}_4^{n+c}$. A very important channel error of this kind is the depolarizing channel model. It is given by the following definition.

Definition 2 (Depolarizing channel): The depolarizing channel on n qubits of error probability p picks up an element $E \in \mathbb{F}_4^n$ by choosing randomly the coordinates E_i of E independently of each other according to $\mathbf{P}(E_i = 0) = 1 - p, \mathbf{P}(E_i = 1) = \mathbf{P}(E_i = \omega) = \mathbf{P}(E_i = \bar{\omega}) = \frac{p}{3}$.

The entanglement assisted setting consists in encoding k information qubits together with $n-k-c$ ancilla qubits and c maximally entangled pairs of qubits (making up for a total of $n+c$ qubits), the transmitter and the receiver holding each one qubit of these pairs, and encoding takes place only on the transmitter side (the c qubits held by the receiver do not participate in the encoding). The n qubits held by the transmitter are sent through a noisy channel whereas the c qubits on the receiver side are noiseless. We assume that the noisy channel is a Pauli channel meaning that the error which occurs is an element $(E_i)_{1 \leq i \leq n+c}$ of $\mathbb{F}_4^{n+c}$ with $E_i = 0$ for the c positions i which correspond to the c qubits on the receiver side. If the model is a depolarizing channel model the n remaining coordinates of E are distributed as explained in Definition 2. The c columns of $\mathbf{H}$ which are involved in the definition of the entanglement assisted stabilizer code correspond to these positions for which $E_i = 0$, that is the positions belonging to I.

At this point the receiver measures $n+c-k$ qubits and obtains the following syndrome

Definition 3 (error syndrome): The error syndrome associated to an error $E = (E_i)_{1 \leq i \leq n+c} \in \mathbb{F}_4^{n+c}$ with respect to a parity-check matrix $\mathbf{H}$ with rows $H_1, \dots, H_{n+c-k}$ is the binary vector

$$s(E) \triangleq (E \star H_i)_{1 \leq i \leq n+c-k}.$$

By assumption on the error model, $E_i = 0$ if $i \in I$, therefore if we let E' be the element of $\mathbb{F}_4^n$ obtained by throwing away the c positions which are associated to the c qubits on the receiver side, that is $E' = (E_i)_{i \notin I}$, we have

$$s(E) = (E' \star H'_i)_{1 \leq i \leq n+c-k}.$$

Minimum distance. Apart from the fact that the rows of the parity-check matrix have to satisfy the aforementioned orthogonality conditions there is another fundamental difference with the classical setting. It can be checked that not all errors change the quantum state belonging to an entanglement assisted stabilizer code. More precisely

Fact 1: Let $\mathscr{G}$ be the group generated by the rows of $\mathbf{H}$ and let $\mathscr{S}$ be the subgroup of $\mathscr{G}$ of elements E which are such that $E_i = 0$ on all positions belonging to I. $\mathscr{S}$ is called the stabilizer group of the code. The set of errors which leaves the (continuous) entanglement assisted stabilizer code invariant is given by its stabilizer group.

For a classical linear code the minimum distance of the code is equal to the minimum weight of an nonzero error of zero syndrome. The minimum distance of an entanglement assisted stabilizer code is defined by

Definition 4 (minimum distance): The minimum distance of a stabilizer code is the minimum Hamming weight of an error E with zero syndrome which does not belong to the stabilizer group and which is such that $E_i = 0$ for $i \in I$.

Entanglement assisted CSS codes. The codes that we are going to construct here belong to a subclass of entanglement assisted stabilizer codes, which is called the class of entanglement assisted CSS codes. They consist of codes which are defined by a parity-check matrix whose rows contain either only 1's and 0's or only ω's and 0's. In this case we may partition the rows of $\mathbf{H}$ as

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_1 \\ \omega \mathbf{H}_\omega \end{pmatrix}$$

where $\mathbf{H}_1$ and $\mathbf{H}_\omega$ are binary matrices. The orthogonality constraint on the rows of $\mathbf{H}$ translates into an orthogonality constraint between the rows of $\mathbf{H}_1$ and $\mathbf{H}_\omega$: the rows of $\mathbf{H}_1$ have to be orthogonal to the rows of $\mathbf{H}_\omega$, or what is the same, if we let $\mathcal{C}_1$ be the code with parity-check matrix $\mathbf{H}_1$ and $\mathcal{C}_\omega$ be the code with parity-check matrix $\mathbf{H}_\omega$, then we should have

$$\mathcal{C}_\omega^\perp \subset \mathcal{C}_1.$$

Condition (ii) has also a simple expression in terms of $\mathbf{H}_1$ and $\mathbf{H}_\omega$. Let I_1 be the set of indices of the rows of $\mathbf{H}$ which belong to $\mathbf{H}_1$ and I_ω be the set of indices of the rows of $\mathbf{H}$ which belong to $\mathbf{H}_\omega$. If we let $\mathbf{H}'_1$ and $\mathbf{H}'_\omega$ be the submatrices of $\mathbf{H}_1$ and $\mathbf{H}_\omega$ formed by the columns which are not in I and if we denote by $\mathbf{H}'_1(i)$ the row of index i in $\mathbf{H}_1$ and $\mathbf{H}'_\omega(i)$ the row of index i in $\mathbf{H}'_\omega$, then Condition (ii) is equivalent to the fact that the matrix $\mathbf{M}' \triangleq (< \mathbf{H}'_1(i), \mathbf{H}'_\omega(j) >)_{i \in I_1, j \in I_\omega}$ has rank c where $< \mathbf{x}, \mathbf{y} >= \sum_i x_i y_i$ is the standard inner product between $\mathbf{x} = (x_i)_i$ and $\mathbf{y} = (y_i)_i$ which are vectors in $\mathbb{F}_2^n$. There is a simple way to check Condition (ii) which is given by the following proposition.

Proposition 1: Let $\mathbf{H}_1^{''}$ and $\mathbf{H}_\omega^{''}$ be the submatrices of $\mathbf{H}_1$ formed by the c columns which belong to I. A necessary and sufficient for Condition (ii) to hold is that $\mathbf{H}_1^{''}$ and $\mathbf{H}_\omega^{''}$ are both of rank c.

Moreover a suboptimal decoding of the quantum code can be performed by decoding $\mathcal{C}_1$ and $\mathcal{C}_\omega$, since the error E can be written in a unique way as $E = E^1 + \omega E^\omega$ with E^1 and E^ω in $\mathbb{F}_2^{n+c}$ and by noticing that the only entries of $s(E^1)$ which are non zero correspond to the rows of $\mathbf{H}$ which belong to $\mathbf{H}_\omega$. Take only these rows for the syndrome s^1 of E^1, and we obtain

$$s^1(E^1) = (\omega \mathbf{H}_\omega(i) \star E^1)_{i \in I_\omega} = (< \mathbf{H}_\omega(i), E^1 >)_{i \in I_\omega}.$$

Similarly we have

$$s^\omega(\omega E^\omega) = (\mathbf{H}_1(i) \star \omega E^\omega)_{i \in I_1} = (< \mathbf{H}_1(i), E^\omega >)_{i \in I_1}.$$

A depolarizing channel model of probability of error p translates into a binary symmetric channel error model of probability of error $\frac{2p}{3}$ for E^1 and E^ω.

III. Our construction

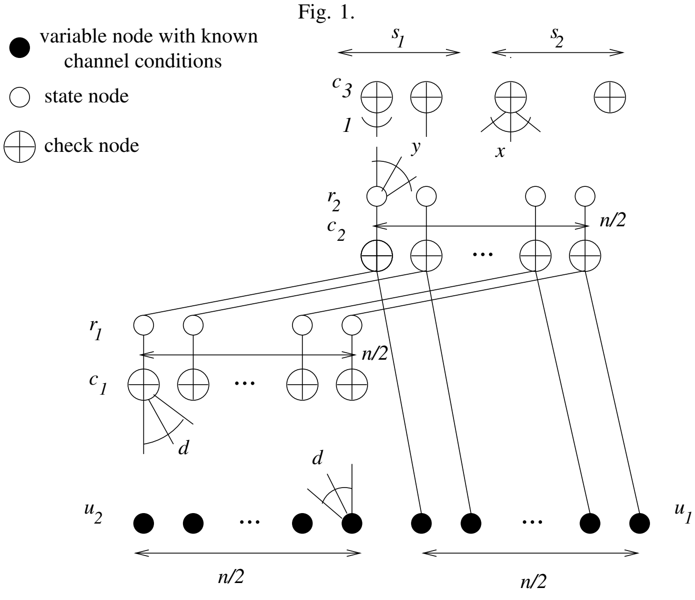
Basically the idea of our construction is to exploit an idea due to [31], [6] which begins with the observation that the dual of a low density generator matrix code is a low density parity-check code. This can be exploited to yield a CSS code at the expense of a constant minimum distance. However, if the weights of the rows of the low density generator matrix are chosen to be large enough, this is not necessarily a problem.

Our observation is now just that the dual of a spatially coupled low density generator matrix code is (essentially) a spatially coupled low density parity-check code.

A. Overview of the construction of [6]

Let us present the construction of a low density generator matrix code given in [6]. A Tanner graph used for decoding this code is depicted in Figure 1. The length of the code is n and half of the variable nodes are of degree 1 (they correspond to u_1) in the Tanner graph, while the other half (which corresponds to u_2) is of some constant degree d. There is a first set of check nodes, corresponding to c_1, all of degree $d+1$ which form a bipartite subgraph of degree d with the variable nodes of degree d. There is a matching of these check nodes with $n/2$ state nodes (corresponding to r_1 in the figure) and there are two matchings between the $n/2$ check nodes of the second level (corresponding to c_2) and the variable nodes of u_1 and r_1 respectively. Then there is a last matching between the $n/2$ check nodes of c_2 and the $n/2$ state nodes of r_2. Finally the subgraph of the Tanner graph formed by the state nodes of r_2 and the last level of check nodes corresponding to c_3 has three type of nodes:
- s_1 check nodes of degree 1 (this implies that the associated state node of r_2 should be equal to 0,
- s_2 check nodes of some constant degree x,
- all the state nodes of r_2 are of some constant degree y in the subgraph.

Fig. 1.



The purpose of the check nodes of degree 1 of the last level is to ensure that iterative decoding does not get stuck at the initial stage (it corresponds to some kind of doping of the last level of state nodes corresponding to r_2). If we denote this code by $\mathcal{C}_1$, then it is proved in [6] that there exists another code $\mathcal{C}_\omega$ with the same Tanner graph structure which satisfies $\mathcal{C}_\omega^\perp \subset \mathcal{C}_1$. This is basically a consequence of the fact that $\mathcal{C}_1$ is an LDGM code whose dual is an LDPC code and of the particular form of the low density generator matrix of $\mathcal{C}_1$. The LDGM structure implies however that these codes have constant minimum distance, the point is here that the weight of the rows of the low density generator matrix can be chosen to be large enough so that this does not deteriorate iterative decoding performances.

B. The associated spatially coupled construction

There is a spatially coupled version of these codes which can be described as follows. Take $L+2\delta$ such codes, number them with $0, 1, \dots, L+2\delta-1$ and consider the associated Tanner graphs (and say that the t-th Tanner graph corresponds to level t). For each $i \in \{-\delta, \dots, \delta\}$, and $t \in \{0, 1, \dots, L+2\delta-1\}$, we swap a fraction $\frac{1}{2\delta+1}$ of the edges which link a variable node with a check node at level t with an edge which links a variable node to a check node at level $t+i \mod (L+2\delta)$ such that the variable node at level t is now adjacent to the check node at level $t+i \mod (L+2\delta)$ and vice versa. We do not swap the edges which link the state nodes to variable nodes on the other hand. The variable nodes and the check nodes which are of degree 1 have their corresponding edge which stays at the same level. The variable node positions at positions $\{0, \dots, 2\delta-1\}$ will correspond to the set I of the entanglement assisted code (and we also check that Condition of Proposition 1 holds). In other words, these variable nodes are set to zero when we perform iterative decoding. We used the window decoder described in [32] to reduce its complexity.

We have performed the computations for the following parameters

TABLE I

	L	δ	d	x	y	s_1	s_2
type 1	20	2	20	9	3	$\frac{15n}{64}$	$\frac{9n}{64}$
type 2	20	2	25	9	3	$\frac{15n}{64}$	$\frac{9n}{64}$

IV. RESULTS

The x-axis and the y-axis of the following curves give respectively the depolarizing error probability and the probability of error after decoding.

As shown in Fig. 2, the spatially coupled codes clearly outperform significantly the previously known LDPC code constructions as well as the quantum turbo-code constructions. The type 1 codes (SC1A of length $N = 19200$, SC 1B, $N = 38400$ and SC 1C $N = 76800$) are slightly better than the type 2 codes (SC 2A: $N = 19200$, SC 2B: $N = 38400$) as illustrated by Fig. 3.

These codes can also tolerate some moderate amount of depolarizing noise on the 2δ levels which are fixed to 0 (they belong to I) as shown by Fig. 4 which shows various depolarizing noise levels p on these positions for a type 1 code of length 38400.

V. ACKNOWLEDGEMENT

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Fig. 2.

Comparison with other codes of rate $\frac{1}{4}$.

- SC1C: spatially coupled code of type 1 and length 76800
- Garcia-Liu: [6]
- Codes of length $n = 785700$, $n = 242500$ from [15]
- turbo codes: [33]
- MacKay: [3]
- Lou-Garcia: [31]
- Camara-Ollivier-Tillich: [5]

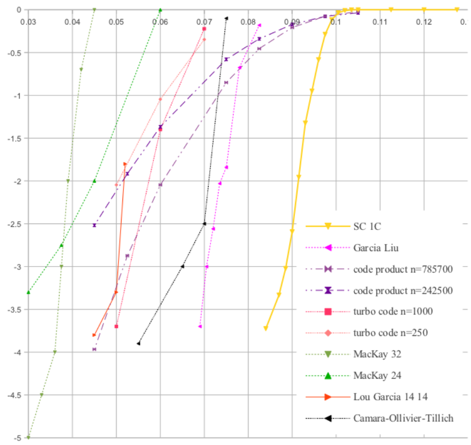


Fig. 3.

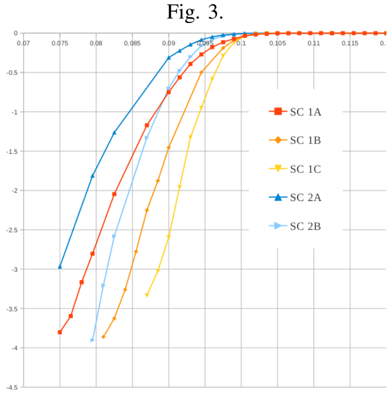
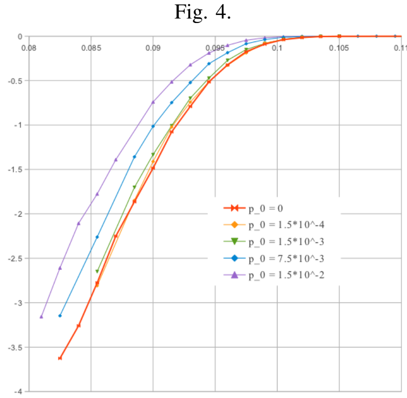


Fig. 4.



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