The Heston Stochastic-Local Volatility Model: Efficient Monte Carlo Simulation

March 3, 2020

1 Introduction

The following method proposed by [1] deals with an efficient Monte Carlo scheme for simulating the stochastic volatility model of Heston enhanced by a non-parametric local volatility component.

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2 Theoretical framework

We consider the following local stochastic volatility model

$$\begin{aligned} \frac{dS_t}{S_t} &= r\,dt + \sigma(t, S_t)\psi(V_t)dW_t^x, \\ dV_t &= a_v(t, V_t)dt + b_v(t, V_t)dW_t^v, \\ dW_t^x dW_t^v &= \rho_{x,v}dt, \end{aligned}$$

where r denotes the risk-free interest rate, $\rho_{x,v}$ is the correlation between the corresponding Brownian motions, $\sigma(t, S_t)$ is the local volatility component, $\psi(V_t)$ controls the stochastic volatility, parameters $a_v(t, V_t)$ and $b_v(t, V_t)$ determine the drift and diffusion of the variance process respectively. In the following, we consider

$$\begin{aligned} \psi(V_t) &= \sqrt{V_t}, \\ a_v(t, V_t) &= \kappa(\overline{v} - V_t), \\ b_v(t, V_t) &= \gamma\sqrt{V_t}, \end{aligned}$$

where κ controls the speed of mean reversion, $\overline{v}$ controls a long-term mean and γ determines the volatility of the process V_t.

To be able to calibrate market smiles exactly, the volatility σ is given by the following equality

$$\sigma^2(t,K) = \frac{\sigma^2_{LV}(t,K)}{\mathbb{E}[\psi^2(V_t)|S_t = K]}$$

where σ_{LV} denotes the Dupire's local volatility.

3 Numerical algorithm

3.1 Computation of $\mathbb{E}[\psi^2(V_t)|S_t = K]$

Suppose that at a given time t_i, $i = 1, \cdots, N$ we have M pairs of Monte Carlo realizations $(s_{i,1}, v_{i,1}), \cdots, (s_{i,M}, v_{i,M})$. By grouping the pairs of realizations into bundles $[b_{i,1}, b_{i,2}],]b_{i,2}, b_{i,3}], \cdots,]b_{i,l}, b_{i,l+1}]$, we get

$$\begin{aligned}\mathbb{E}[\psi^2(V_{t_i})|S_{t_i} = s_{i,j}] &\sim \mathbb{E}[\psi^2(V_{t_i})|S_{t_i} \in]b_{i,k}, b_{i,k+1}]], \\ &\sim \frac{\mathbb{E}[\psi^2(V_{t_i})\mathbb{1}_{\{S_{t_i} \in]b_{i,k}, b_{i,k+1}]\}}]}{\mathbb{P}(S_{t_i} \in]b_{i,k}, b_{i,k+1}])}\end{aligned}$$

for $s_{i,j} \in]b_{i,k}, b_{i,k+1}]$.

If we build the l bins such that each bin contains the same number of Monte Carlo paths, we get

$$\begin{aligned}\mathbb{E}[\psi^2(V_{t_i})|S_{t_i} = s_{i,j}] &\sim \frac{\frac{1}{M}\sum_{j=1}^{M} \psi^2(v_{i,j})\mathbb{1}_{\{s_{i,j} \in]b_{i,k}, b_{i,k+1}]\}}}{\mathbb{P}(S_{t_i} \in]b_{i,k}, b_{i,k+1}])} \\ &\sim \frac{l}{M} \sum_{j \in \mathcal{J}_{i,k}} \psi^2(v_{i,j})\end{aligned}$$

where $\frac{1}{l}$ represents the probability to be in the kth bin and $\mathcal{J}_{i,k} := \{j | s_{i,j} \in]b_{i,k}, b_{i,k+1}]\}$. We summarize the method:

For each step t_i, $i = 1, \cdots, N$

1. Generate M pairs of observations $(s_{i,j}, \psi^2(v_{i,j}))$, $j = 1, \cdots, M$.
2. Order the elements $\overline{s}_{i,j} : \overline{s}_{i,1} \leq \cdots \leq \overline{s}_{i,M}$ and apply the same permutation on $(v_{i,1}, \cdots, v_{i,M})$.
3. Determine the boundary of the l bins $]b_{i,k}, b_{i,k+1}]$, $k = 1, \cdots, l$ in the following way

$$b_{i,1} = \overline{s}_{i,1}, \; b_{i,l+1} = \overline{s}_{i,N}, \; b_{i,k} = \overline{s}_{i,(k-1)M/l}, \; k = 2, \cdots, l.$$

4. For the kth bin approximate the conditional expectation by

$$\mathbb{E}[\psi^2(V_{t_i})|S_{t_i} \in]b_{i,k}, b_{i,k+1}]] \sim \frac{l}{M} \sum_{j \in \mathcal{J}_{i,k}} \psi^2(v_{i,j})$$

3.2 Simulation scheme

First, we recall the dynamics of the Heston SLV model expressed in terms of independent Brownian motions:

$$\frac{dS_t}{S_t} = r\,dt + \sigma(t, S_t)\sqrt{V_t}\left(\rho_{x,v} d\tilde{W}_t^v + \sqrt{1-\rho_{x,v}^2} d\tilde{W}_t^x\right),$$
$$dV_t = \kappa(\overline{v} - V_t)dt + \gamma\sqrt{V_t} d\tilde{W}_t^v$$

where $\tilde{W}^x$ and $\tilde{W}^v$ are independent Brownian motions. Following the scheme proposed in [1, Section 3.3], we discretize $[0, T]$ on a regular grid of size N, with step size $\Delta = \frac{T}{N}$. We get

$$\begin{aligned} v_{i+1,j} \sim& c(\Delta)\chi^2(d, \lambda(t_i, v_{i,j})), \\ x_{i+1,j} =& x_{i,j} + r\Delta - \frac{1}{2}\hat{\sigma}^2(t_i, x_{i,j})v_{i,j}\Delta + \frac{\rho_{x,v}}{\gamma}\hat{\sigma}(t_i, x_{i,j})(v_{i+1,j} - \kappa\overline{v}\Delta + v_{i,j}c_1) \\ &+ \rho_1\sqrt{\hat{\sigma}^2(t_i, x_{i,j})v_{i,j}\Delta}\tilde{Z}_x, \end{aligned}$$

where $x_{i,j} = \ln(s_{i,j})$, $\rho_1 = (1 - \rho_{x,v}^2)^{1/2}$, $c_1 = \kappa\Delta - 1$ and

$$c(\Delta) = \frac{\gamma^2}{4\kappa}(1 - e^{-\kappa\Delta}),\ d = \frac{4\kappa\overline{v}}{\gamma^2},\ \lambda(t, x) = \frac{4\kappa e^{-\kappa\Delta}}{\gamma^2(1 - e^{-\kappa\Delta})}x,$$

$\chi^2(d, x)$ represents a noncentral chi-squared distribution with d degrees of freedom and non-centrality parameter x, and

$$\hat{\sigma}^2(t_i, x_{i,j}) = \sigma^2(t_i, e^{x_{i,j}}) = \frac{\sigma_{LV}^2(t_i, s_{i,j})}{\mathbb{E}[V_{t_i}|S_{t_i} = s_{i,j}]}.$$

4 Numerical experiments

We test the algorithm on a Call option with payoff $e^{-rT}(S_T - K)_+$ with the following parameters, for different maturities and strikes :

r	s_0	v_0	κ	γ	$\rho_{x,v}$
0	1	0.0945	1.05	0.95	-0.315

We use $l = 20$ bins, $N = 100$ times steps and $M = 10^5$ Monte Carlo simulations and the local volatility σ_{LV} is given by

$$\sigma_{LV}(t, x) = 0.01 + 0.1e^{-x/s_0} + 0.01t.$$

K T	2	8
0.7	0.300028	0.309048
0.9	0.103869	0.154880
1.1	0.004426	0.059615
1.3	0.000046	0.019109
1.5	0.000002	0.005046

References

[1] A.W. Van Der Stoep, L.A. Grzelak, C.W. Oosterlee. The Heston Stochastic-Local Volatility Model: Efficient Monte Carlo Simulation. 2013.

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