Exact and high order discretization schemes for Wishart processes and their affine extensions

(Extended abstract)

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Short abstract

This work presents new weak second and third order schemes for Wishart processes without any restriction on its parameters. Moreover, we give the construction of an exact scheme for Wishart processes. To the best of our knowledge, this is the first exact simulation that works without any restriction on parameters. At the same time, we present a general recursive method, based on the idea studied by Nynomiya and Victoir, for getting the second weak order scheme for any affine processes defined on the symmetric positive cone matrices. Therefore, these results allow us to propose a second order scheme for more general affine diffusion involving affine positive matrices. Simulation examples will be given to emphasize the convergence of schemes on Wishart process. We also give an application in finance for the model presented by Gourieroux and Sufana [8] which can be seen as an extension of the Heston model in multi dimension.

This paper provides a second order discretization scheme for stochastic continuous affine processes on the cone of positive definite symmetric matrices. Moreover, it presents an exact scheme for Wishart process and high order schemes one. Indeed, these matrices have been introduced recently in several promising applications in finance, such as fixed-income models with stochastic correlation or in multi asset option pricing with a stochastic covariance matrices.

Let $\mathcal{M}_d(\mathbb{R})$ (respectively $\mathcal{S}_d(\mathbb{R})$, $\mathcal{S}_d^+(\mathbb{R}), \mathcal{S}_d^{+,*}(\mathbb{R}), \mathcal{S}_d^{-,*}(\mathbb{R})$, and $\mathcal{G}_d(\mathbb{R})$) denote the set of all d-square matrix (respectively symmetric, symmetric non negative matrix, symmetric positive definite, symmetric negative definite and non singular matrix).

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Short introduction to Wishart processes and their affine extensions in finance

Bru [2] has started the stochastic analysis concerning Wishart processes, which is defined as a solution of the following SDE

$$dX_t = (\alpha a^T a + bX_t + X_t b^T)dt + \sqrt{X_t} dW_t a + a^T dW_t^T \sqrt{X_t}. \tag{1}$$

We denote for short $WIS_d(x, \alpha, b, a)$ the law of $(X_t, t \geq 0)$, and $WIS_d(x, \alpha, b, a; t)$ the law of X_t.

In fact, she found that the SDE has unique weak solution (in the sense of law) in the case of $\alpha \geq d - 1$, $b \in \mathcal{S}_d(\mathbb{R})^{-,*}$ and $\sqrt{a^T a}$ commutes with b. Moreover, and under the same hypothesis on a and b; the SDE admits a strong solution in definite positive matrices if $\alpha \geq d + 1$. Recently, Cuchiero, et al. [3] in 2009 present a stochastic analysis for a large family of affine positive processes. The SDE associated to the continuous processes is the following :

$$dX_t = (\alpha + B(X_t))dt + \sqrt{X_t} dW_t a + a^T dW_t^T \sqrt{X_t} \tag{2}$$

They have proven that the SDE admits a unique weak solution, under certain condition on α and B. In the case of $\alpha - (d + 1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$, the SDE admits a strong solution that is defined on the definite positive matrices [9].

Affine processes have been studied intensively in finance, see [5][6]. Recently, affine positive matrices have arisen from a large and growing range of useful applications in finance. Since, their characteristic functions are obtained by computing the Riccati equations, the affine models present a tractable method for pricing European claims, at least for small dimension.

Wishart processes have been used first in finance by Gourieroux and Sufana [8], in order to price multi assets options. This model can be seen as an extension of the Heston model in multi dimension. Indeed, for a given Wishart process $(\Sigma_t)_{t\geq 0}$, they consider that the yield of asset denoted by $(Y_t)_{t\geq 0} \in \mathbb{R}^d$ can be presented as

$$dY_t = (r - \tfrac{1}{2} diag(\Sigma_t)^2)dt + \sqrt{\Sigma_t} dB_t$$

where Σ follows the SDE (1).

Da Fonseca, et al. [4] have extended the previous model to an affine one which takes into consideration the correlation between the Brownian motion derived by the yield and the one derived by the Wishart process.

The challenge of this kind of multi dimensional model is to figure out the right way of pricing as the dimension grows. There are two main European option pricing methods for affine models. The first one is based on the inversion of the Fourier transform. Since all processes $(X_t)_{t\geq 0}$ defined by the SDE (2) are affine, we obtain then that $\forall u \in \mathcal{S}_d(\mathbb{R})$ there are two functions ϕ and Φ defined by a certain Riccati ODE such that

$$\mathbb{E}[\exp(i\mathrm{Tr}(uX_t^x))] = \exp(\phi(t, u) + \mathrm{Tr}(\Phi(t, u)x))$$

This kind of pricing is very interesting especially if it does not involve a very high dimension, because it seems to be less efficient if the dimension grows up substantially.

Therefore, Monte Carlo methods propose the alternative way in high dimension. Moreover, it is the most simple approach to compute pathwise options. To perform Monte Carlo methods, one has to use schemes that take into consideration the definite positive condition on matrices. In the light of this, Euler scheme seems to be less efficient. Strictly speaking, the Euler scheme is not well defined since it does not necessary keep the positivity constraint in each time step of discretization, and the matrix square root is then not defined. One has then to modify it to get a scheme that is well-defined. This can be done as follows:

$$X_{t_{i+1}} = X_{t_{i+1}} + \Delta_i\{\alpha + B(X_{t_i})\}\Delta_i + \sqrt{X_{t_i}^+}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T\sqrt{X_{t_i}^+}$$

where $\Delta_i = t_{i+1} - t_i$ and $\forall x \in \mathcal{S}_d(\mathbb{R}),\ (x)^+ = q^T(\lambda)^+ q$ such that q is an orthogonal matrix, λ is the eigen matrix of x and $\forall (i,j) \in \{1,\ldots,d\}^2,\ ((\lambda)^+)_{i,j} = \mathbb{1}_{i=j}(\lambda_{i,j})^+$. On the one hand, the Euler scheme is then expensive in term of time computation because it involves diagonalization procedure (to compute $(.)^+$). On the other hand, as in the case of CIR process, the Euler scheme leads to poor results under the stressed financial market conditions.

Our main results

The first part of the work presents a new method to simulate exactly a Wishart process without restriction on parameters. In the literature, exact simulation methods for Wishart processes have been studied for a specific value of its parameter (namely $\alpha \in \mathbb{N}$, see [2] [11]). In the general case, we present a method based on the splitting infinitesimal operator. First of all, we give some property about the canonical Wishart process $WIS_d(x, \alpha, 0, I_d)$ which is defined as a solution for the following SDE

$$dX_t = \alpha I_d dt + \sqrt{X_t}dB_t + dB_t^T\sqrt{X_t} \tag{3}$$

where $\alpha > d - 1$.

Indeed, if we consider the infinitesimal generator associated to a canonical Wishart process $(X_t)_{t\geq 0}$, we have proven that $\forall d \in \mathbb{N}$ there is a sequence of generator $(L_i)_{\{1,\ldots,d\}}$, such that

$$L = \sum_{i=1}^{d} L_i \text{ and that } \forall (i,j) \in \{1,\ldots,d\}^2,\ L_iL_j = L_jL_i.$$

In this case, we give an interesting property about the commutativity of operator. If we consider two operators L_X and L_Y associated respectively to two independent affine processes $(X_t^x)_{t\geq 0}$ and $(Y_t^y)_{t\geq 0}$, starting respectively from x and y, such that $L_X L_Y = L_Y L_X$, then we obtain

$$X_t^{Y_t^z} \underset{Law}{=} Z_t^z$$

where $(Z_t^z)_{t\geq 0}$ is an affine process associated to the operator $L_X + L_Y$, starting from z. It is then sufficient to simulate exactly X_t and Y_t to obtain an exact scheme for the process Z_t. Formally, the result can be easily proven. Indeed, for a smooth function $f : \mathcal{S}_d^+(\mathbb{R}) \to \mathbb{R}$, we obtain

$$\begin{array}{rcl} \mathbb{E}(f(X_t^{Y_t^z})) = \mathbb{E}(\mathbb{E}(f(X_t^{Y_t^z}))|Y_t^z) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}(L_X^k f(Y_t^z)) & = & \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=0}^{\infty} \frac{t^j}{j!} L_Y^j L_X^k f(z) \\ & = & \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_X + L_Y)^k f(z) \\ & = & \mathbb{E}(f(Z_t^z)) \end{array}$$

where we have used the commutativity property of operators in the second equality (this result is proved in detail in our work for affine diffusions).

Consequently, it is sufficient to find the adequate way to simulate each process given by each operator $(L_i)_{i\in\{1,\ldots,d\}}$. By the symmetry argument, it is sufficient to study the solution of the operator L_1, because the solutions associated to operators $(L_i)_{2\leq i\leq d}$ are obtained by simple permutation between the first index and i^{th} index of the solution L_1.

Let us consider $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$ and $c \in \mathcal{G}_n$ such that $(x)_{2\leq i,j\leq d} = cc^T$, with $d-1$ is the rank of $(x)_{2\leq i,j\leq d}$. We obtain then that the process $(X_t^x)_{t\geq 0}$ starting from x and defined as follows is a solution of the operator L_1 :

$$X_t^x = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} U_t + \sum_{i=1}^{d-1} (Y_t^i)^2 & Y_t^T \\ Y_t & I_{d-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c^T \end{pmatrix}$$

such that $\forall i \in \{1, \ldots, n\}$

$$\begin{array}{rcll} dU_t & = & (\alpha - (d-1))dt + 2\sqrt{U_t} dW_t^1 & ; \quad U_0 = x_{\{1,1\}} - \sum_{i=1}^{d-1} (Y_0^i)^2 \geq 0 \\ dY_t^i & = & dW_t^{i+1} & ; \quad Y_0 = (c)^{-1}(x_{\{1,i\}})_{2\leq i\leq d} \end{array}$$

$(W_t^i)_{1\leq i\leq d}$ is a vector of standard Brownian motions

When $x \in \mathcal{S}_d^+(\mathbb{R})$ but $x \notin \mathcal{S}_d^{+,*}(\mathbb{R})$, we give a numerical method that generalize the one presented above. It is based on permutation matrices and Cholesky decomposition modified Golub [7] (called the Outer product). The last part of this section, we show that there is an equality in the sense of law between the canonical Wishart process and the full parameter one based on the Laplace transform. Indeed,

$$WIS_d(y, \alpha, b, a; t) \underset{Law}{=} a^T A_t^{\tilde{b}} WIS_d(x, \alpha, 0, I_d; t) A_t^{\tilde{b}} a \tag{4}$$

Where $x = (A_t^{\tilde{b}})^{-1} \exp(t\tilde{b})(a^{-1})^T y a^{-1} \exp(t\tilde{b}^T)(A_t^{\tilde{b}})^{-1}$, $\tilde{b} = (a^{-1})^T b(a^{-1})$, $A_t^b = \sqrt{\frac{\int_0^t \exp(sb)\exp(sb^T)ds}{t}}$, such that $a \in \mathcal{G}_d$. We stress that the last result gives the simulation of Wishart process without any restriction on its parameters.

The second part of this work is about high order discretization of a continuous positive symmetric processes. The first step was to find the canonical affine positive matrices by which we can generate all other processes. After a geometric argument, we prove that for all positive affine continuous processes $(Y_t)_{t\geq 0}$ driving by the SDE (2), there is a matrix $\beta \in \mathcal{G}_d(\mathbb{R})$ such that $Y_t = \beta X_t \beta^T$, and X_t is defined as a solution of the following SDE

$$dX_t = \alpha + B(X_t)dt + \sqrt{X_t}dB_t I_d^n + I_d^n dB_t^T \sqrt{X_t} \tag{5}$$

Where α and I_d^n are diagonal matrices such that $\min_{1\leq i\leq d} \alpha_{i,i} \geq d-1$ and $\forall i \in \{1,\ldots,d\},\ (I_d^n)_{i,i} = \nVdash_{i\leq n}$.

We can prove easily that the operator associated to the SDE (5) can be written as $\sum_{i=1}^n L_i + L_B = \tilde{L}_n + L_B$, such that that L_B is associated to an ODE first order. It is then sufficient to simulate two independent processes associated to L_B and $\tilde{L}_n$ and simulate a a second order scheme for the global process by the usual method e.g Ninomiya-Victoir scheme [10], defined as

$$U\hat{X}_t^{\hat{Y}_t^z} + (1-U)\hat{Y}_t^{\hat{X}_t^z}$$

where U is an independent Bernoulli variable with parameter $\frac{1}{2}$, and $\hat{X}_t^x$, $\hat{Y}_t^y$ are two independent second weak order schemes associated to L_B and $\tilde{L}_n$ respectively, starting from x and y respectively, and both are defined on $\mathcal{S}_d^+(\mathbb{R})$.

Since it is simple to find the adequate scheme associated to the ODE operator L_B, it is then sufficient to study the solution concerning the operator $\tilde{L}_n$. In this perspective, we use the framework introduced previously by Alfonsi [1], and after some technical results, we give a weak high order scheme for the operator $\tilde{L}_n$, and we present then the second order scheme for any affine process defined on the positive matrices $\mathcal{S}_d^+(\mathbb{R})$. In the case of Wishart process, we can reach higher order scheme by using the equality in law (4). Therefore, we propose a third order scheme for Wishart process based on the third order scheme for the CIR process developed in [1].

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