Pricing European options with wavelet and the characteristic function : implementation in PREMIA

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Abstract

Applying the wavelet approximation method proposed in [OO13], we price the European option in Heston model. This method improved the accuracy problem of COS method for call option (COS method has to rely on the put call parity to compute the price of call option) and long maturity contract. And this method can be extended to the extreme heavy tail model like CGMY model.

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1 Option pricing with wavelet

Consider the risk-neutral valuation formula of an European option

$$v(x,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(v(y,T)|x) = \mathrm{e}^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)\mathrm{d}y, \tag{1.1}$$

where $v(x,t)$ denotes the option value at time t with the state variable value x for the underlying asset at time t, T is the maturity, t is the initial date, $\mathbb{E}^{\mathbb{Q}}$ the expectation operator under the risk-neutral measure $\mathbb{Q}$, x and y are state variables of the logarithm of the underlying asset at time t and T, respectively, $f(y|x)$ is the conditional probability density of y given x and r is the risk neutral interest rate.

Since the density function $f(y|x)$ is expected to decay to zero as y goes to infinity, the infinity integration range is truncated to $[a,b] \subset \mathbb{R}$, the the risk-neutral price of the European option is

$$v^c(x,t) = \mathrm{e}^{-r(T-t)}\int_a^b v(y,T)f(y|x)\mathrm{d}x. \tag{1.2}$$

According to the theory of multiresolution analysis (MRA), the conditional probability density truncated in a bounded interval $[a, b]$,

$$f^c(y|x) := \begin{cases} f(y|x), & \text{if} \quad y \in [a, b], \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

can be generated by the scaling function ϕ^j with a unique sequence of coefficients $\{c^j_{m,k}\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$

$$f^c_{m,j}(y|x) = \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k}(x)\phi^j_{m,k}\left((j+1)\cdot\frac{y-a}{b-a}\right), \quad j \geq 0, \tag{1.4}$$

where m is the scale of approximation and we consider only the scaling function with order $j = 0, 1$. For $j = 0$ we have the scaling function of Haar wavelet system, while for $j = 1$, we have the linear B-splines.

Then by applying the scaling function to the conditional probability density function, the risk-neutral price of an European option can be computed by

$$v^c_{m,j}(x, t) = \mathrm{e}^{-r(T-t)} \int_a^b v(y, T) f^c_{m,j}(y|x)\mathrm{d}y = \mathrm{e}^{-r(T-t)} \sum_{k=0}^{(j+1)\cdot(2^m-1)} c^j_{m,k}(x) \cdot V^{j,\alpha}_{m,k}, \tag{1.5}$$

where

$$V^{j,\alpha}_{m,k} := \int_a^b v(y, T)\phi^j_{m,k}\left((j+1)\cdot\frac{y-a}{b-a}\right)\mathrm{d}y, \tag{1.6}$$

with α is the option index, $\alpha = 1$ for call and $\alpha = -1$ for put.

2 Heston model and its characteristic function

We consider the Heston model for the underlying process, its dynamics is given by the following stochastic differential equations,

$$\begin{cases} \mathrm{d}x_t &= \left(\mu - \frac{1}{2}u_t\right)\mathrm{d}t + \sqrt{u_t}\mathrm{d}W_{1t}, \\ \mathrm{d}u_t &= \lambda\left(\overline{u} - u_t\right)\mathrm{d}t + \eta\sqrt{u_t}\mathrm{d}W_{2t}, \end{cases} \tag{2.1}$$

where x_t denotes the log-asset price variable and u_t the variance of the asset price process. Parameters $\lambda \geq 0, \overline{u} \geq 0$ and $\eta \geq 0$ are called the speed of mean reversion, the mean level of variance, and the volatility of volatility, respectively. Furthermore, the Brownian motions W_{1t} and W_{2t} are assumed to be correlated with correlation coefficient ρ. Under the risk neutral measure, $\mu = r - d$, where r is the risk-free interest rate and d is the dividend rate.

The characteristic function of log-asset price x_T given $x_t = x$ for model (2.1) is

$$\begin{aligned}\psi(w) &= \exp(-iwx) \\ &\cdot \exp\left[-iw\mu(T-t) + \frac{u_0}{\eta^2}\left(\frac{1-\mathrm{e}^{-D(T-t)}}{1-G\mathrm{e}^{-D(T-t)}}\right)(\lambda + i\rho\eta w - D)\right] \\ &\cdot \exp\left\{\frac{\lambda\overline{u}}{\eta^2}\left[(\lambda+i\rho\eta w - D)(T-t) - 2\log\left(\frac{1-G\mathrm{e}^{-D(T-t)}}{1-G}\right)\right]\right\},\end{aligned} \tag{2.2}$$

with

$$D = \sqrt{(\lambda + i\rho\eta w)^2 + (w^2 - iw)\eta^2} \quad \text{and} \quad G = \frac{\lambda + i\rho\eta w - D}{\lambda + i\rho w + D}. \tag{2.3}$$

The truncated bounds a and b of the Heston model (2.1) for (1.2) is the same as for the COS method:

$$[a, b] := \left[x + c_1 - L\sqrt{c_2}, x + c_1 + L\sqrt{c_2}\right], \quad \text{with} \quad x = x_t - \log K,$$

where L is a constant taking value large enough ($L = 12$ for Heston model in our example) so that the truncated error satisfies the computation precise requirement, and c_n denotes the nth cumulant of $x_T - \log K$, which was provided in Appendix of [FO09] with

$$\begin{aligned}c_1 &= \mu T + \left(1 - \mathrm{e}^{-\lambda T}\right)\frac{\overline{u} - u_0}{2\lambda} - \frac{1}{2}\overline{u}T, \\ c_2 &= \frac{1}{8\lambda^3}\Big[\eta T\lambda \mathrm{e}^{-\lambda T}(u_0 - \overline{u})(8\lambda\rho - 4\eta) + \lambda\rho\eta(1-\mathrm{e}^{-\lambda T})(16\overline{u} - 8u_0) + 8\lambda^2(u_0 - \overline{u})(1 - \mathrm{e}^{-\lambda T}) \\ &\quad + 2\overline{u}\lambda T(-4\lambda\rho\eta + \eta^2 + 4\lambda^2) + \eta^2((\overline{u} - 2u_0)\mathrm{e}^{-2\lambda T} + \overline{u}(6\mathrm{e}^{-\lambda T} - 7) + 2u_0)\Big].\end{aligned}$$

Next we will derive the coefficients $V^{j,\alpha}_{m,k}$ and $c^j_{m,k}$ in (1.5).

3 Coefficients $V^{j,\alpha}_{m,k}$

Here we consider the approximation by Haar wavelet, i.e. $j = 0$. For the approximation by B-splines, i.e. $j = 1$, please refer to Section 3.3.2 of [OO13].

For an European option with strike K and payoff function at time of maturity T as $[\alpha(S_T - K)]^+$, the option value at time T is

$$v(y, T) = [\alpha K(\mathrm{e}^y - 1)]^+,$$

with $\alpha = 1$ for call option and $\alpha = -1$ for put option. By applying into (1.2) the above option value and the Haar wavelet for $\phi^j_{m,k}(j = 0)$

$$\phi^0_{m,k}(x) = \begin{cases} 2^{m/2}, & \frac{k}{2^m} \le \frac{y-a}{b-a} < \frac{k+1}{2^m}, \\ 0, & \text{otherwise,} \end{cases}$$

we have for $k = 0, 1, \cdots, 2^m - 1$ and $a < 0 < b$,

$$\begin{aligned} V^{0,1}_{m,k} &:= \int_a^b K(\mathrm{e}^y - 1)^+ \phi^0_{m,k}\left(\frac{y-a}{b-a}\right)\mathrm{d}y = \int_0^b K(\mathrm{e}^y - 1)\phi^0_{m,k}\left(\frac{y-z}{b-a}\right)\mathrm{d}y \\ &= \begin{cases} 2^{m/2} K\left(\mathrm{e}^{\gamma_k} - \mathrm{e}^{\delta_k} + \delta_k - \gamma_k\right), & \gamma_k > 0, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{3.1}$$

where the upper script $0, 1$ in $V^{0,1}_{m,k}$ denotes $j = 0$ for Haar wavelet and $\alpha = 1$ for call option and $\triangle_m := \frac{b-a}{2^m}$, $\beta_k := a + k\triangle_m, \gamma_k := \beta_k + \triangle_m, \delta_k := \max(0, \beta_k)$.

Similarly we derive the coefficient for an European put option with strike K and payoff function $(K - S_T)^+$,

$$\begin{aligned} V^{0,-1}_{m,k} &:= \int_a^b K(1 - \mathrm{e}^y)^+ \phi^0_{m,k}\left(\frac{y-a}{b-a}\right)\mathrm{d}y = \int_a^0 K(1 - \mathrm{e}^y)\phi^0_{m,k}\left(\frac{y-z}{b-a}\right)\mathrm{d}y \\ &= \begin{cases} 2^{m/2} K\left(\mathrm{e}^{\beta_k} - \mathrm{e}^{\xi_k} + \xi_k - \beta_k\right), & \beta_k < 0, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{3.2}$$

with $\xi_k = \min(0, \gamma_k)$.

If $a < b < 0$, then $V^{0,1}_{m,k} = 0$, $V^{0,-1}_{m,k} = 2^{m/2}K\left(\mathrm{e}^{\beta_k} - \mathrm{e}^{\gamma_k} + \gamma_k - \beta_k\right)$, for $k = 0, \cdots, 2^m - 1$. And conversely, if $0 < a < b$, then $V^{0,-1}_{m,k} = 0$, $V^{0,1}_{m,k} = 2^{m/2}K\left(\mathrm{e}^{\gamma_k} - \mathrm{e}^{\beta_k} + \beta_k - \gamma_k\right)$, for $k = 0, \cdots, 2^m - 1$.

4 Coefficients $c^j_{m,k}$

We consider the approximation in a fixed interval $[a, b]$ and provide the computation formula here, for the derivation of the formula please refer to Section 3.1 of [OO13].

The coefficients $c^j_{m,k}, k = 1, \cdots, (j + 1) \cdot (2^m - 1)$ can be approximated as

$$c^j_{m,k} \approx \frac{1}{Md^k}\left(Q_{m,j}(d) + (-1)^k Q_{m,j}(-d) + 2\sum_{s=1}^{M-1} \mathcal{R}(Q_{m,j}(d\mathrm{e}^{ih_s}))\cos(kh_s)\right), \tag{4.1}$$

where $M = 2^m, d > 0$ is the radius and taking value of 0.9995 in our example, $h = \frac{\pi}{M}$ and $h_s = sh$ for all $s = 0, \cdots, M$.

$$Q_{m,j}(z) := \frac{2^{\frac{m}{2}}(j+1)z^{-\frac{2^m(j+1)a}{b-a}}\hat{f}\left(\frac{2^m(j+1)}{b-a}i \cdot \log(z)\right)(\log(z))^{j+1}}{(b-a)(z-1)^{j+1}}, \tag{4.2}$$

$\hat{f}(w)$ is the Fourier transform of $f(y|x)$, i.e. the characteristic function of the conditional probability density function

$$\hat{f}(w) = \int_{-\infty}^{+\infty} \mathrm{e}^{-iwy} f(y|x)\mathrm{d}y.$$

For Heston model (2.1), the characteristic function $\hat{f}(w)$ is given as $\psi(w)$ in (2.2).
For the approximation in $\mathcal{R}$, please refer to Section 3.2 of [OO13].

References

[OO13] Ortiz-Gracia, L., Oosterlee, C.W., Robust pricing of European options with wavelets and the characteristic function. *SIAM J. SCI. COMPUT.*, 35(5), 1055-1084, 2013.

[FO09] Fang, F., Oosterlee, C.W., A novel pricing method for European options based on Fourier-Cosine series expansions. *SIAM J. SCI. COMPUT.*, 35(5), 1055-1084, 2009. 1, 3, 4, 5

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