Pricing Fixed and Floating Asian Options in a Discretely Monitored Framework*

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Abstract

We present a recursive approach to price discretely monitored arithmetic Asian options when the underlying asset evolves according to a generic Lévy process. Our algorithm is based on a general price recursion for Asian contracts, which can be considered to price both Fixed and Floating options.

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1 Introduction

This paper deals with a recursive approach for pricing discretely monitored arithmetic Asian options under Lévy processes. Asian options have become very popular instruments for hedging transactions whose costs are related to the average price of the underlying asset, being much cheaper than the corresponding options on the underlying asset and less subject to price manipulations near settlement.

An extensive literature deals with the pricing problem under continuous monitoring. A review can be found in Boyle and Potapchik [4]. In the discrete monitoring case, where the arithmetic mean is updated only at prefixed points in time, the pricing of Asian options is not an easy task and even in the Black-Scholes setting no analytical solution is available.
Recent contributions to Asian option pricing in the exponential Lévy setting are Albrecher [1] and Albrecher and Predota [2], Benhamou [3], Černý and Kyriakou [5], Fusai and Meucci [8], and Iseger and Oldenkamp [12]. Albrecher [1] and Albrecher and Predota [2] explore approximations of the arithmetic option

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price based on the moments of the average, but in general it is not clear which approximate distribution to choose and the approximation error is difficult to evaluate.

The Fast Fourier Transform (FFT) approach was introduced by Carverhill and Clewlow [6] in the Black-Scholes framework. Their forward density convolution algorithm requires a large discretization grid, resulting in slow convergence. Benhamou [3] extends the algorithm of Clewlow and Carverhill to some non-lognormal density and improves the numerical efficiency by recentering the density at each monitoring date, thus reducing the size of the grid. In the Lévy setting, Fusai and Meucci [8] solve the valuation problem by recursive Gaussian quadrature and derive a formula for the moments to check accuracy. Černý and Kyriakou [5] introduce a fast and accurate algorithm, using a backward price convolution, and provide an analytical upper bound for the pricing error due to truncations. Finally, Iseger and Oldenkamp [12] propose an algorithm based on the Laplace inversion technique. A different approach based on the so-called maturity randomization technique is also presented in [9, 10, 11].

This paper deals with a recursive formula as in [5, 8]. Our recursive approach, firstly presented in [10], works for very general Asian options.

2 The Exponential Lévy Model

The risk-neutral process for the stock price $(S_t)_{t\geq 0}$ is assumed to be described by

$$S(t) = S(0)\exp\left(\left(r - d + g\right)t + L(t)\right),$$

where r is the continuously compounded interest rate, d is the dividend yield, L_t is a Lévy process, and g is the so-called compensator chosen to ensure that the discounted price process is a martingale. The Lévy process is fully identified by its characteristic exponent $\psi(\omega) = \log \mathbb{E}\left(e^{i\omega L_1}\right)$, where $i = \sqrt{-1}$. Following [13], under the mean-correcting martingale measure, we set $g = -\psi(-i)$.

We are interested in pricing arithmetic Asian options under a discrete monitoring rule; that is, prices contributing to the arithmetic average are observed at equally spaced monitoring dates $t_0 = 0$, $t_1 = \Delta, \ldots, t_n = n\Delta, \ldots, t_N = N\Delta = T$. The log-return on a time interval of length Δ is defined by

$$Z_n \equiv \log \frac{S_n}{S_{n-1}} = (r - d + g)\Delta + L_n - L_{n-1}, \tag{1}$$

where $S_n = S(n\Delta)$ and the Lévy increments $L_n - L_{n-1} = L(n\Delta) - L((n-1)\Delta)$ are independent and identically distributed. It follows that Z_n has a characteristic function that does not depend on the monitoring time index n,

$$\phi_Z(\omega) = e^{(\psi(\omega) + i\omega(r-d+g))\Delta},$$

and its density f_Z can be obtained by numerical inversion of the characteristic function using the FFT or the fractional FFT, as explained in [7].

3 Recursion for Arithmetic Asian Options

The starting point of our numerical approach is based on a recursive formulation for the Asian option price. Under the unified framework of [14], the payoff of an arithmetic Asian option depends on the following path-dependent random variable:

$$I_N \equiv \sum_{n=0}^{N} \lambda_n S_n, \tag{2}$$

where λ_n are deterministic. For example, the payoff of an Asian call option is given by

$$V(S_N, I_N; N) \equiv (I_N - cS_N)^+ ,$$

where $(\cdot)^+$ is the positive part function.
By suitable choices of $\{\lambda_n\}_{n=0}^N$ and c, we can describe a wide class of Asian options. In particular, standard cases are

$$\lambda_0 = \frac{\gamma}{N+\gamma} - \frac{K}{S_0}; \quad \lambda_n = \frac{1}{N+\gamma}, \quad n = 1, \ldots, N; \quad c = 0 \tag{3}$$

for fixed strike call options and

$$\lambda_0 = -\frac{\alpha\gamma}{N+\gamma}; \quad \lambda_n = -\frac{\alpha}{N+\gamma}, \quad n = 1, \ldots, N; \quad c = -1 \tag{4}$$

for floating strike calls, where $\gamma = 1$ if S_0 is included in the average, 0 otherwise, and α is a coefficient of partiality for the floating strike case. In our case, we set both α and γ equal to 1.

For pricing purposes, we can combine (1) and (2) and observe that:

$$I_{n+1} = I_n + \lambda_{n+1} S_n e^{Z_{n+1}}, \; n = 0, \ldots, \; N-1. \tag{5}$$

Therefore, using (5) and the standard backward pricing procedure, we obtain the following recursion for the option price:

$$\begin{aligned} V(S_N, I_N; N) &= (I_N - cS_N)^+, \\ V(S_n, I_n; n) &= e^{-r\Delta} \int_{-\infty}^{+\infty} f_Z(s)\, V(S_n e^s, I_n + \lambda_{n+1} S_n e^s; n+1)\, ds, \end{aligned}$$

for $n = N-1, \ldots, 0$.

Since the return distribution is independent of the current stock level and the payoff function is a homogeneous function of the spot price, then the price function is itself a homogeneous function of degree one. Thus, we can write

$$V(S_n, I_n; n) = S_n\, V\left(1, \frac{I_n}{S_n}; n\right).$$

If we set $x = I_n / S_n$, we can define

$$v_n(x) \equiv V(1, x; n),$$

where we have omitted the dependence of x on n. The function v satisfies the recursion

$$\begin{aligned} v_N(x) &= (x-c)^+, \\ v_n(x) &= e^{-r\Delta}\int_{-\infty}^{+\infty} f_Z(s)\, e^s v_{n+1}\big(xe^{-s}+\lambda_{n+1}\big)\, ds,\ \ n = N-1\ldots,0. \end{aligned}$$

Notice that, if $x = 0$, then $v_n(0) = e^{-d\Delta} v_{n+1}(\lambda_{n+1})$. Otherwise, by a change of variable in the integration ($y = xe^{-s} + \lambda_{n+1}$, i.e., $\log\left(x/\left(y - \lambda_{n+1}\right)\right) = s$), we obtain:

$$v_n(x) = \begin{cases} e^{-r\Delta}\int_{\lambda_{n+1}}^{+\infty} f_Z\Big(\log\Big(\frac{x}{y-\lambda_{n+1}}\Big)\Big)\frac{x}{(y-\lambda_{n+1})^2}\, v_{n+1}(y)\, dy & \text{if } x > 0, \\ -e^{-r\Delta}\int_{-\infty}^{\lambda_{n+1}} f_Z\Big(\log\Big(\frac{x}{y-\lambda_{n+1}}\Big)\Big)\frac{x}{(y-\lambda_{n+1})^2}\, v_{n+1}(y)\, dy & \text{if } x < 0, \end{cases} \tag{6}$$

for $n = N-1,\ldots,0$. The desired option price will be $S_0 v_0(\lambda_0)$. Notice that if $x \geq 0$, $c \leq 0$ and $\lambda_n > 0$, for $n = 1,\ldots,N$, then

$$v_n(x) = e^{-r(N-n)\Delta}\left[x + \sum_{i=0}^{N-n-1}\lambda_{N-i} e^{(N-n-i)(r-d)\Delta} - ce^{(N-n)(r-d)\Delta}\right], \tag{7}$$

for $n = N,\ldots,0$. See [10] for details.

3.1 Put-Call Parity

In the previous section we show a recursion to price Asian call option (C) with arithmetic average and discrete monitoring dates. The corresponding put price (P) can be computed by the following put-call parity condition: since

$$(I_N - cS_N)^+ - (cS_N - I_N)^+ = I_N - cS_N$$

then it holds

$$\begin{aligned} C(S_0, I_0; 0) - P(S_0, I_0; 0) &= e^{-rN\Delta}\mathbb{E}_0(I_N - cS_N) \\ &= S_0\left(\sum_{n=0}^{N} e^{-r\Delta(N-n)}\lambda_n - c\right). \end{aligned}$$

Thus, given the price of a fixed strike call option, we can compute easily the price of the corresponding put option using the put-call parity above.

4 Numerical Discretization

This section aims to describe how the recursive algorithm can be efficiently implemented. For ease of exposition, we consider only the fixed (3) and floating case (4).

4.1 Floating Strike Asian Option

For a floating strike call option, being $\lambda_n = \lambda := -\alpha/(N+\gamma) < 0$, for $n = 1, \ldots, N$, we always have $x < 0$; thus the general recursion (6) can be written as:

$$\begin{aligned} v_N(x) &= (x+1)^+, \\ v_n(x) &= -e^{-r\Delta} \int_{-\infty}^{\lambda} f_Z\left(\log\left(\frac{x}{y-\lambda}\right)\right) \frac{x}{(y-\lambda)^2} v_{n+1}(y)\, dy, \end{aligned} \tag{8}$$

for $n = N-1, \ldots, 0$.

In order to compute the recursive integrals above, we have to

1. truncate the domain to a finite one $\Omega_T = (T, \lambda)$. Numerical experiments show that the best choice is to consider a bound T such that $|T|$ decreases with respect to the number of monitoring dates N. In our numerical experiments, we set $T = -\frac{3}{2} - \frac{30}{N}$, see [10] for details.
2. discretize the integral in (8) by applying an appropriate quadrature formula. In our numerical experiments we consider a Gauss-Legendre quadrature rule.

If the chosen quadrature rule provides nodes x_i and weights w_i, $i = 1, \ldots, m$, (8) is approximated by

$$v_n(x_i) = -e^{-r\Delta} \sum_{j=1}^{m} w_j f_Z\left(\log\left(\frac{x_i}{x_j-\lambda}\right)\right) \frac{x_i}{(x_j-\lambda)^2} v_{n+1}(x_j),\ i = 1, \ldots, m,$$

which can be rewritten in matrix form as

$$\mathbf{v}_n = \mathbf{K}\mathbf{D}\mathbf{v}_{n+1}, \quad n = N-1, \ldots, 0, \tag{9}$$

where

- $(\mathbf{v}_n)_i = v_n(x_i)$, $i = 1, \ldots, m$;
- $\mathbf{K}$ is the square matrix with elements

  $$K_{ij} = -e^{-r\Delta} f_Z\left(\log\left(\tfrac{x_i}{x_j-\lambda}\right)\right) \tfrac{x_i}{(x_j-\lambda)^2},$$

  $i, j = 1, \ldots, m$;
- $\mathbf{D}$ is the diagonal matrix containing the weights w_i, $i = 1, \ldots, m$.

4.2 Fixed Strike Asian Option

For a fixed strike call option, being $\lambda_n = \lambda := 1/(N+\gamma) > 0$, for $n = 1, \dots, N$, considering Equation (7), the general price recursion (6) becomes:

$$\begin{aligned} v_N(x) &= (x)^+ , \\ v_n(x) &= \begin{cases} e^{-r(N-n)\Delta}\left[x + \lambda e^{(r-d)\Delta}\frac{1-e^{(N-n)(r-d)\Delta}}{1-e^{(r-d)\Delta}}\right] & \text{if } x \geq 0, \\ -e^{-r\Delta}\int_{-\infty}^{\lambda} f_Z\left(\log\left(\frac{x}{y-\lambda}\right)\right)\frac{x}{(y-\lambda)^2}\, v_{n+1}(y)\,dy & \text{if } x < 0, \end{cases} \end{aligned} \tag{10}$$

for $n = N-1, \dots, 0$. Since the above formula can be rewritten as

$$\begin{aligned} v_n(x) &= -e^{-r\Delta}\Big(\int_{-\infty}^{0} f_Z\left(\log\left(\frac{x}{y-\lambda}\right)\right)\frac{x}{(y-\lambda)^2}\, v_{n+1}(y)\,dy \\ &\quad + \int_{0}^{\lambda} f_Z\left(\log\left(\frac{x}{y-\lambda}\right)\right)\frac{x}{(y-\lambda)^2}\, v_{n+1}(y)\,dy\Big) \quad \text{if } x < 0, \end{aligned}$$

we implement the recursive formula truncating the domain as above and considering nodes x_i and weights w_i, $i = 1, \dots, m$, for the interval $(T, 0)$, and x_{m+i} and weights w_{m+i}, $i = 1, \dots, m$, for the integral on $(0, \lambda)$. Thus the recursive formula can be rewritten in matrix form as

$$\mathbf{v}_n = \mathbf{K}\mathbf{D}\mathbf{w}_{n+1}, \quad n = N-1, \dots, 0,, \tag{11}$$

where

- $(\mathbf{v}_n)_i = v_n(x_i)$, $i = 1, \dots, m$;
- $(\mathbf{w}_n)_i = v_n(x_i)$, $i = 1, \dots, 2m$;
- $\mathbf{K}$ is a rectangular matrix with elements

  $$K_{ij} = -e^{-r\Delta} f_Z\left(\log\left(\frac{x_i}{x_j-\lambda}\right)\right)\frac{x_i}{(x_j-\lambda)^2},$$

  $i = 1, \dots, m$, $j = 1, \dots, 2m$;
- $\mathbf{D}$ is the diagonal matrix containing the weights w_i, $i = 1, \dots, 2m$.

4.3 Implementation

The pricing algorithm based on the recursive procedures (9) and (11) is:

1. compute the density f_Z by numerical inversion of the characteristic function using the fractional FFT;
2. compute the quadrature nodes and weights, and thus the diagonal matrix $\mathbf{D}$;

3. compute the matrix $\mathbf{K}$ and the vector $\mathbf{v}_N$ ($\mathbf{w}_N$) for the floating (fixed) case;

4. for $n = N-1, \cdots, 0$

   - compute $\mathbf{z} := \mathbf{D}\mathbf{v}_{n+1}$ ($\mathbf{z} := \mathbf{D}\mathbf{w}_{n+1}$) - the computational cost is $O(m)$ operations;
   - compute $\mathbf{v}_n = \mathbf{K}\mathbf{z}$ - the computational cost is $O(m^2)$ operations;
   - for the fixed strike case, using $\mathbf{v}_n$ and (7), compute $\mathbf{w}_n$;

5. given $\mathbf{v}_0$ or $\mathbf{w}_0$, compute the option price $S_0 v_0(\lambda_0)$ by using an interpolation technique.

The computational cost of the procedure is $O(m^2)$, due to the matrix vector multiplication $\mathbf{K}\mathbf{z}$. The density function f_Z is computed on an interval (l, u) such that the probability for Z to fall out of this interval is 10^{-8}. Out of this interval, f_Z is set equal to 0. Thus we can speed up the algorithm storing only the non-zero elements of the matrix $\mathbf{K}$.

4.4 Numerical Validation

In this section we validate our pricing algorithm. In Table 2 we assume that the underlying asset evolves according to the Double Exponential (DE) Lévy process introduced by Kou and we price fixed strike Asian options; in Table 3 we deal with both the CGMY and the Normal-Inverse Gaussian (NIG) process, considering the same kind of derivative contracts. Finally, in Table 4 we price floating strike Asian options considering the Jump Diffusion (JD) process proposed by Merton. See Table 1 for the characteristic functions of the considered Lévy processes. We recall that the two parametrizations of the characteristic exponent of the NIG process are connected by the relation $\alpha = \sigma^{-1}\sqrt{k^{-1} + \theta^2\sigma^{-2}}$, $\beta = \theta\sigma^{-2}$ and $\delta = \sigma k^{-0.5}$.

Table 1: Characteristic exponents of some parametric Lévy processes

Model	$\psi(\omega)$
CGMY	$C\Gamma(-Y)\left((M - i\omega)^Y - M^Y + (G + i\omega)^Y - G^Y\right)$
DE	$-\frac{1}{2}\sigma^2\omega^2 + \lambda\left(\frac{(1-p)\eta_2}{\eta_2 + i\omega} + \frac{p\eta_1}{\eta_1 + i\omega} - 1\right)$
NIG	$-\delta\left(\sqrt{\alpha^2 - (\beta + i\omega)^2} - \sqrt{\alpha^2 - \beta^2}\right)$
NIG	$\frac{1}{k} - \frac{1}{k}\sqrt{1 - 2\theta k\omega - k\sigma^2\omega^2}$
JD	$-\frac{1}{2}\sigma^2\omega^2 + \lambda\left(e^{i\omega\mu - \frac{1}{2}\omega^2\gamma^2} - 1\right)$

First of all, we consider as a benchmark the MC method. We price a fixed strike call option with strike $K = 100$, 250 monitoring dates, and maturity $T = 1$. We assume that the risk-free interest rate is equal to 3.67% and that the underlying asset does not pay any dividend ($d = 0$), $S(0) = 100$. All the numerical experiments have been performed with a personal computer equipped with Windows 7 and an Intel Core i7 Q720 1600 MHz processor with 6 GB of RAM. All the computed prices fall into the confidence intervals.

Table 2: Fixed strike Asian Call options: prices and CPU time (in seconds).

Model	Parameters	m	Price	CPUtime	CI
DE	$\sigma = 0.120381$	1000	5.34888	6.8	4.837-5.301
	$\lambda = 0.330966$	2000	5.07276	10.3	
	$p = 0.2071$	3000	5.07061	21.8	
	$\eta_1 = 9.65997$, $\eta_2 = 3.13868$	4000	5.07025	39.9	
		5000	5.07017	54.8	

We now consider as a benchmark the results reported in [5], pricing a fixed strike call option with strike $K = 100$, 50 monitoring dates, and maturity $T = 1$. We assume that the risk-free interest rate is equal to 4% and that the underlying asset does not pay any dividend ($d = 0$), $S(0) = 100$. The prices are reported in Table 3.

Table 3: Fixed strike Asian Call options: prices. A comparison with [5].

Model	Parameters	m	Price	Benchmark
CGMY	C=0.6509	1000	7.34768	7.3474
	G=5.853	2000	7.34628	
	M=18.27	3000	7.34692	
	Y=0.8	4000	7.34714	
		5000	7.34723	
		6000	7.34731	
NIG	$\delta = 0.7543$	1000	7.33883	7.3426
	$\alpha = 12.3407$	2000	7.34150	
	$\beta = -5.8831$	3000	7.34214	
		4000	7.34236	
		5000	7.34247	
		6000	7.34254	

Finally we deal with a floating strike Asian option, pricing a contract with maturity $T = 1$ and with 100 monitoring dates. We assume that the underlying asset evolves according to the Jump Diffusion model proposed by Merton. The option prices are reported in Table 4, while as benchmark price we consider the one proposed in [9].

Table 4: Floating strike Asian Call options: prices. A comparison with [9].

Model	Parameters	m	Price	Benchmark
JD	$\sigma = 0.126349$	1000	3.91208	5.1701
	$\lambda = 0.174814$	2000	5.17883	
	$\mu = -0.390078$	3000	5.17025	
	$\gamma = 0.338796$	4000	5.17032	
		5000	5.17025	
		6000	5.17026	

From Tables 3 and 4 we notice that setting $m = 4000$ is enough to obtain three decimal places accuracy.

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