On the Pricing of Variable Annuities with Guaranteed Minimum Benefits

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Abstract. Here we complete the pricing process of annuity with guaranteed minimum maturity benefit, guaranteed minimum death benefit or guaranteed minimum income benefit respectively based on the COS method. The exponential convergence rate and linear computational complexity make COS method highly efficient in the pricing process. We present the specific pricing process of the three annuity options under Heston model. And the three annuity options program under BS Model and Heston Model have been completed in C.

Key words. variable annuity pricing, COS method, heston model

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1 Introduction

Variable Annuities (VAs) are annuities different from the traditional annuities in having the equity-linked feature and with embedded guarantees. The insurers provide different forms of guarantees for VAs, these guarantees can be classified as guaranteed minimum death benefits (GMDB) and the guaranteed minimum living benefits (GMLB). For the GMLB, some with guarantees at the time of maturity, such as the called guaranteed minimum maturity benefits (GMMB), others with withdraw guarantees during the contract period, called guaranteed minimum withdraw benefits (GMWB). The definition of the above guaranteed benefits are slightly different in various literature, here we follow the definition of the above guaranteed benefits of [2].

The guaranteed minimum maturity benefit (GMMB) guarantees the policyholder a specific monetary amount at the maturity of the contract. A simple GMMB might be a guaranteed return of premium if the stock index falls over the term of the insurance (with an upside return of some proportion of the increase in the index if the index rises over the contract term). The guarantee may be fixed or subject to regular or equity-dependent increases. Note that this definition of GMMB corresponds to that of the GMAB with simple premium guarantees or with roll-up based guarantees in [1].

The guaranteed minimum death benefit (GMDB) guarantees the policyholder a specific monetary sum upon death during the term of the contract. Again, the death benefit may simply be the original premium, or may increase at a fixed rate of interest. More complicated or generous death benefit formulae are popular ways of tweaking a policy benefit at relatively low cost.

The guaranteed minimum income benefit (GMIB) ensures that the lump sum accumulated under a separate account contract may be converted to an annuity at a guaranteed rate. When the GMIB is connected with an equity-linked separate account, it has derivative features of both equities and bonds. In the United Kingdom, the guaranteed-annuity option is a form of GMIB. A GMIB is also commonly associated with variable-annuity contracts in the United States.

The guaranteed minimum accumulation benefit (GMAB), the policyholder has the option to renew the contract at the end of the original term, at a new guarantee level appropriate to the maturity value of the maturing contract. It is a form of guaranteed lapse and reentry option. Again the GMAB defined here refers to the GMAB with ratchet based guarantees.

The guaranteed minimum withdraw benefit (GMWB) gives the rights to the policyholder to withdraw from the individual account at a guaranteed rate at some fixed date till the maturity or the individual account is exhausted.

We consider the pricing of the GMMB, GMDB and GMIB and assume these products with a single premium. The surrender and withdraw before the maturity is not considered here.

2 Guaranteed Minimum Benefits

Denote the single premium is P. The premium forms the individual account of the policyholder, denote A_t as the account value at time t, then $A_0 = P$. The individual account is invested in a mutual fund, the time t value of the underlying mutual fund is S_t. The the dynamics of the account value of A_t is

$$\mathrm{d}A_t = \frac{\mathrm{d}S_t}{S_t}A_t - \varphi A_t \mathrm{d}t, \tag{2.1}$$

where φ is the rate of fee charged continuously by the insurer to hedge the provided guaranteed benefits and A_0 is the premium paid at time.

The policyholder is entitled to a guaranteed benefit based on the premium. Usually the guaranteed base is a percentage of the single premium αP, where $0 < \alpha \leq 1$ is the percentage of the premium that can be guaranteed. When $\alpha = 1$, it means the premium is fully guaranteed, otherwise it means partially guaranteed.

2.1 GMMB

The guaranteed benefit of GMMB at time of maturity $B(T)$ is the maximum of the account value and the guaranteed minimum value, i.e.

$$B(T) := \max(G_T^M, A_T),$$

where G_T^M is the guaranteed value at maturity T. The guaranteed minimum value is calculated differently according the the product design: simple case, roll-up base and ratchet base. The guaranteed minimum value is set as

$$G_T^M = \begin{cases} \alpha P \mathrm{e}^{iT}, & \text{with a compound roll-up rate} \quad i \geq 0, \\ \alpha P(1+iT), & \text{with a simple roll-up rate} \quad i \geq 0. \end{cases} \tag{2.2}$$

Note that for $i = 0$ the guarantees is of simple case, otherwise is the roll-up based with guaranteed roll-up rate $i > 0$.

2.2 GMDB

The guaranteed benefit of GMDB is a specific monetary sum that policyholder can get upon the death of insured during the term of the contract. In our model, we assume the death benefit is the total premiums accumulated at a fixed rate of interest. Assume the death time for insured is τ and at τ the guaranteed benefit of GMDB is the maximum of the current account value and the guaranteed minimum value, i.e.

$$B(\tau) := \max(G_\tau^D, A_\tau),$$

where G_τ^D is the guaranteed value at insured's mortality time. The guaranteed minimum value is calculated differently according the the product design: simple case, roll-up base and ratchet base. The guaranteed minimum value is set as

$$G_\tau^D = \begin{cases} \alpha P \mathrm{e}^{i\tau}, & \text{with a compound roll-up rate} \quad i \geq 0, \\ \alpha P(1+i\tau), & \text{with a simple roll-up rate} \quad i \geq 0. \end{cases} \tag{2.3}$$

Note that for $i = 0$ the guarantees is of simple case, otherwise is the roll-up based with guaranteed roll-up rate $i > 0$.

2.3 GMIB

With guaranteed minimum income benefit, the policyholder can choose to annuitize the guaranteed accumulation account at a guaranteed payment rate g per year. In some other situation, it will pay per month or per season. The payoff of the guaranteed benefit of GMIB at the date of annuitization T is

$$B(T) := \max(G_T^I g a(T), A_T),$$

where A_T is the account value at maturity and g is the minimum guaranteed payment rate prescribed by insurer at the outset of the insurance contract. $a(T)$ is the market price of the an annuity with payments of \$1 per year at maturity T. Same as the two types of variable annuity, the guaranteed minimum value is calculated differently according the the product design: simple case, roll-up base and ratchet base. The guaranteed minimum value is set as

$$G_T^I = \begin{cases} \alpha P \mathrm{e}^{iT}, & \text{with a compound roll-up rate} \quad i \geq 0, \\ \alpha P(1+iT), & \text{with a simple roll-up rate} \quad i \geq 0. \end{cases} \tag{2.4}$$

Note that for $i = 0$ the guarantees is of simple case, otherwise is the roll-up based with guaranteed roll-up rate $i > 0$.

To valuate the variable annuities with guarantees is to find a break-even value of φ such that the present value of benefit at time of maturity equals to the present value of the total of the continuously charged fee. We consider two models for the dynamics of the underlying mutual fund that the individual account of variable annuities invested in:the Heston stochastic volatility model and the Black Scholes Model with Vasicěk interest rate .

3 Models

3.1 The Heston Model

The underlying mutual fund value S is assumed to follow the Heston stochastic volatility model, which is given as

$$\begin{cases} \mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{v_t}S_t\mathrm{d}W_t^S, \\ \mathrm{d}v_t = \kappa_v(\theta_v - v_t)\mathrm{d}t + \sigma_v\sqrt{v_t}\mathrm{d}W_t^v, \end{cases} \tag{3.1}$$

where $\{S_t; t \geq 0\}$ and $\{v_t; t \geq 0\}$ are the equity price and the volatility of equity, respectively. And $\{W_t^{S,v}; t \geq 0\}$ are correlated standard Brownian motions with correlations ρ_{Sv} under risk-neutral measure $\mathbb{Q}$. Moreover, the interest rate r and the coefficients $\kappa_v, \theta_v, \sigma_v$ are constants representing the speed of reversion, mean of reversion and the volatility of the volatility, respectively. Denote by $\mathcal{F}_t$ the filtration generated by S_t and v_t. Then the dynamics of the account value A_t is

$$\mathrm{d}A_t = (r-\varphi)A_t\mathrm{d}t + \sqrt{v_t}A_t\mathrm{d}W_t^S. \tag{3.2}$$

Let x_t represents $\log(A_t)$, according to the Itô Lemma, the heston model turns to be:

$$\begin{cases} \mathrm{d}x_t = (r - \varphi - \frac{v_t}{2})\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^x, \\ \mathrm{d}v_t = \kappa_v(\theta_v - v_t)\mathrm{d}t + \sigma_v\sqrt{v_t}\mathrm{d}W_t^v, \end{cases} \tag{3.3}$$

For the Heston model, the characteristic function of is:

$$\Phi(\omega, \mathbf{x}, v_0) = \phi_{hes}(\omega; v_0)e^{i\omega\mathbf{x}}. \tag{3.4}$$

Here we use boldfaced value to distinguish vectors, where $\mathbf{x}$, represents $\log(A_0)$. It is written as the vector because the summation in COS method below is a matrix-vector product. See the COS part. v_0 denotes the volatility of the underlying at the initial time and $\phi_{hes}(\omega; v_0) := \Phi(\omega; 0, v_0)$ So the characteristic function of the log-asset price turns to be:

$$\begin{aligned} \phi_{hes}(\omega, v_0) = &\exp\left(i\omega(r-\varphi)\Delta t + \frac{v_0}{\sigma_v^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\kappa_v - i\rho\sigma_v\omega - D)\right) \\ &\exp\left(\frac{\kappa_v\theta_v}{\sigma_v^2}\left(\Delta t(\kappa_v - i\rho\sigma_v\omega - D) - 2\log(\frac{1-Ge^{D\Delta t}}{1-G})\right)\right). \end{aligned} \tag{3.5}$$

with

$$D = \sqrt{(\kappa_v - i\rho\sigma_v\omega)^2 + (\omega^2 + i\omega)\sigma_v^2}$$

and

$$G = \frac{\kappa_v - i\rho\sigma_v\omega - D}{\kappa_v - i\rho\sigma_v\omega + D}$$

3.2 The Black Scholes Model with Vasiček Interest Rate

Now we assume the underlying mutual fund value S_t follows the Black-Scholes model with the stochastic interest rate r_t following the Vasiček model, that is

$$\begin{cases} \mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_S S_t \mathrm{d}W_t^S, \\ \mathrm{d}r_t = \kappa_r(\theta_r - r_t)\mathrm{d}t + \eta_r \mathrm{d}W_t^r, \end{cases} \tag{3.6}$$

where the coefficients $\sigma_S, \kappa_r, \theta_r, \eta_r$ are constants representing the volatility of the mutual fund, the speed of reversion, mean of reversion and volatility of the interest rate respectively, W_t^S, V_t^r are Brownian motions under risk-neutral measure $\mathbb{Q}$ with correlation ρ_{Sr}. Under this model, the dynamics of the account value A_t is

$$\mathrm{d}A_t = (r_t - \varphi)A_t \mathrm{d}t + \sigma_S A_t \mathrm{d}W_t^S. \tag{3.7}$$

Let the $x_t = \log(A_t)$. According to Itô Lemma, the model changes to:

$$\begin{cases} \mathrm{d}x_t = (r_t - \varphi - \frac{\sigma_S^2}{2})\mathrm{d}t + \sigma_S \mathrm{d}W_t^S, \\ \mathrm{d}r_t = \kappa_r(\theta_r - r_t)\mathrm{d}t + \eta_r \mathrm{d}W_t^r, \end{cases} \tag{3.8}$$

For a given state vector $X(t) = [x(t), r(t)]^T$, and $\Phi := \Phi(\omega, X(t), t, T)$ with $x(t) = x; r(t) = r$, the following pricing PDE is satisfied according to the Feynman-Kac formula:

$$\begin{aligned} 0 =& \frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{2}\eta^2 \frac{\partial^2 \Phi}{\partial r^2} + \rho_{x,r}\sigma \frac{\partial^2 \Phi}{\partial x \partial r} + [r(t) - \varphi - \frac{\sigma^2}{2}]\frac{\partial \Phi}{\partial x} \\ &+ \kappa[\theta - r(t)]\frac{\partial \Phi}{\partial r} + \frac{\partial \Phi}{\partial t} - r(t)\Phi \end{aligned} \tag{3.9}$$

subject to terminal condition $\Phi(\omega, X(T), T, T) = exp(i\omega x(T))$. Because the PDE above is affine, its solution is of the following form according to the standard characteristic equation:

$$\Phi := \Phi(\omega, X(t), t, T) = exp(A(\omega, t, T) + B(\omega, t, T)x(t) + C(\omega, t, T)r(t)).$$

Here let $A := A(\omega, t, T), B := B(\omega, t, T), C := C(\omega, t, T)$, we find the following partial derivatives:

$$\frac{\partial \Phi}{\partial t} = \Phi\frac{\partial A}{\partial t} + x(t)\frac{\partial B}{\partial t} + r(t)\frac{\partial C}{\partial t} \tag{3.10}$$

$$\frac{\partial \Phi}{\partial x} = B\Phi, \quad \frac{\partial^2 \Phi}{\partial x^2} = B^2\Phi, \quad \frac{\partial^2 \Phi}{\partial x \partial r} = BC\Phi \tag{3.11}$$

$$\frac{\partial \Phi}{\partial r} = C\Phi, \quad \frac{\partial^2 \Phi}{\partial r^2} = C^2\Phi \tag{3.12}$$

According to the derivatives, we can rewrite the PDE:

$$\begin{aligned} 0 =& \frac{1}{2}\sigma^2 B^2 + \frac{1}{2}\eta^2 C^2 + \rho_{x,r}\sigma\eta BC + [r(t) - \varphi - \frac{\sigma^2}{2}]B + \kappa[\theta - r(t)]C \\ &+ \frac{\partial A}{\partial t} + x(t)\frac{\partial B}{\partial t} + r(t)\frac{\partial C}{\partial t} - r(t) \end{aligned} \tag{3.13}$$

We find the following ODEs by collecting the terms for x(t), r(t):

$$\frac{\partial B}{\partial t} = 0 \tag{3.14}$$

$$\frac{\partial C}{\partial t} = -B + \kappa C + 1 \tag{3.15}$$

$$\frac{\partial A}{\partial t} = -\kappa\theta C - \frac{1}{2}\eta^2 C^2 - \rho_{x,r}\sigma\eta BC - \frac{1}{2}\sigma^2 B^2 + (\varphi + \frac{1}{2}\sigma^2)B \tag{3.16}$$

With the boundary condition $A(u,0) = 0, \quad B(u,0) = iu, \quad C(u,0) = 0$. The solution of the ODEs is given by:

$$B(u,\Delta t) = i\omega \tag{3.17}$$

$$C(u,\Delta t) = (i\omega - 1)\kappa^{-1}(1 - e^{\kappa\Delta t}) \tag{3.18}$$

$$A(u,\Delta t) = \kappa^{-1}(Le^{2\kappa\Delta t} + Ne^{\kappa\Delta t} - L - N) + M\Delta t \tag{3.19}$$

The expressions of L, M, N are as followings:

$$L = -\frac{1}{4}\eta^2(i\omega - 1)^2\kappa^{-2}$$

$$N = \eta^2(i\omega - 1)^2\kappa^{-2} + (i\omega - 1)\kappa^{-1}(\rho\sigma\eta i\omega + \kappa\theta)$$

$$M = \frac{1}{2}[i\omega\sigma^2 + \sigma^2\omega^2 - \eta^2(i\omega - 1)^2\kappa^{-2}] + i\omega\varphi - (i\omega - 1)\kappa^{-1}(\rho\sigma\eta i\omega + \kappa\theta)$$

By the analysis above, we find the characteristic function for the BS model with vasicêk interest rate:

$$\Phi = exp(\kappa^{-1}(Le^{2\kappa\Delta t} + Ne^{\kappa\Delta t} - L - N) + M\Delta t + i\omega x + (i\omega - 1)\kappa^{-1}(1 - e^{\kappa\Delta t})r) \tag{3.20}$$

4 Pricing Process with COS Method

After the introduction above about the three variable annuities and two different models about underlying assets, we now analyse the specified pricing procedure. The characteristic function $\Phi(\omega)$ and the possibility density function are a Fourier pair. Here we use the COS method to complete the pricing process. The convergence rate of COS method is exponential and the computational complexity is linear. It has been proved to be a efficient way in pricing of some underlying dynamics, such as the Heston model.[3]

4.1 Pricing Process of GMMB under Heston Model

The present value of GMMB $g(\varphi)$ at time of maturity or the present value of the expected liability should equals to the present value of the total of the continuously charged fee $f(\varphi)$.

$$g(\varphi) = e^{-rT}\mathbf{E}^{\mathbb{Q}}[(G_T^M - A_T)_+|\mathcal{F}_0] = e^{-rT}\int_a^b (G_T^M - e^y)f(y)\mathrm{d}y. \tag{4.1}$$

with $y = \log(A_t)$

Here we use the COS method to get the COS formula with the cosine series coefficientU_k of $g(\varphi)$.If the guaranteed minimum value is set with a compound roll-up rate with the characteristic function of Heston model above, we find $g(\varphi)$ turns to be

$$g(\varphi) = e^{-rT}{\sum_{k=0}^{N-1}}' Re\left\{\phi_{hes}\left(\frac{k\pi}{b-a}; v_0\right)e^{ik\pi\frac{y-a}{b-a}}\right\}U_k. \tag{4.2}$$

with

$$U_k = \frac{2}{b-a}\int_a^b (G_T^M - e^y)_+ \cos(k\pi\frac{y-a}{b-a})\mathrm{d}y$$

The cosine series coefficients, $\chi(k)$, of $h(x_t) = G_T^M = \alpha P e^{iT}$ on $[c,d] \subset [a,b]$:

$$\chi_k(c,d) := \int_c^d \alpha P e^{iT} \cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \tag{4.3}$$

and the cosine series coefficients, ψ_k of $h(x_t) = e^{x_t}$ on $[c,d] \subset [a,b]$:

$$\psi_k(c,d) := \int_c^d e^y \cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \tag{4.4}$$

Basic calculus shows that

$$\chi_k(c,d) = \alpha P e^{iT} \begin{cases} \left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi} & k \neq 0 \\ (d-c) & k = 0 \end{cases} \tag{4.5}$$

and

$$\begin{aligned} \psi_k(c,d) = \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}&\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - \cos\left(k\pi\frac{c-a}{b-a}\right)e^c\right. \\ &\left.+\frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right] \end{aligned} \tag{4.6}$$

We can get the present value of the GMMB $g(\varphi)$ by the derivation above. As for the sum of the continuously charged fee by the insurance company $f(\varphi)$ is derived by the following:

$$\begin{aligned} f(\varphi) &= \mathbf{E}^{\mathbb{Q}}(\int_0^T \varphi A_t e^{-rt}\mathrm{d}t|\mathcal{F}_0) \\ &= \mathbf{E}^{\mathbb{Q}}(\sum\nolimits_{n=0}^{M-1} \varphi \int_{n\Delta t}^{(n+1)\Delta t} A_t e^{-rt}\mathrm{d}t|\mathcal{F}_0) \\ &= \sum\nolimits_{n=0}^{M-1} \mathbf{E}^{\mathbb{Q}}(\varphi A_{n\Delta t} e^{-rn\Delta t}\Delta t|\mathcal{F}_0) \\ &= \sum\nolimits_{n=0}^{M-1} \Delta t e^{-rn\Delta t}\varphi \mathbf{E}^{\mathbb{Q}}(e^{i\omega x_{n\Delta t}}|\mathcal{F}_0)_{\omega=-i} \end{aligned} \tag{4.7}$$

Here we make $\omega = -i$, $x_{n\Delta t} = \log(A_{n\Delta t})$. And we take the compound interest rate for example above.

Valuating the GMMB contract is to determine the expense feeφ such that the present value of the expected benefit $g(\varphi)$ is equivalent to present value of the total paid fee

$$g(\varphi) = f(\varphi) \tag{4.8}$$

Since σ_v, θ_v,κ_v,ρ,r,i,α,P,T are constant, we could get the φ by Newton Recursion Method with the program in C language. Because GMMB is a kind of put option, here we find the truncation range $[c,d] = [a,0]$.

We propose the following to determine the interval of integration [a,b] within the COS method.

$$[a,b] = [c_1 - L\sqrt{c_2}, c1 + L\sqrt{c_2}] \quad L = 20 \tag{4.9}$$

where the c_n denotes the n-th cumulant of $\ln(A_T/G_T^M)$. The cumulants for the Heston model are as below

$$
\begin{aligned}
c_1 =& (r-\varphi)T + (1-e^{-\kappa T})\frac{\theta - v_0}{2\kappa} - \frac{1}{2}\theta T \\
c_2 =& \frac{1}{8\kappa^3}(\sigma T\kappa e^{-\kappa T}(v_0-\theta)(8\kappa\rho - 4\sigma) + \kappa\rho\sigma(1-e^{-\kappa T})(16\theta - 8v_0) + 2\theta\kappa T(-4\kappa\rho\sigma + \sigma^2 + 4\sigma^2) \\
&+ \sigma^2((\theta - 2v_0)e^{-2\kappa T} + \theta(6e^{-\kappa T} - 7) + 2v_0) + 8\kappa^2(v_0-\theta)(1-e^{-\theta T}))
\end{aligned}
$$

4.2 Pricing Process of GMDB under Heston Model

The present value of GMDB $g(\varphi)$ at insured's mortality time or the present value of the expected liability should equals to the present value of the total of the continuously charged fee $f(\varphi)$, it means

$$
g(\varphi) = \mathbf{E}^{\mathbb{Q}}[e^{-r\tau}(G_\tau^D - A_\tau)_+|\mathcal{F}_0] = \mathbf{E}^{\mathbb{Q}}(\int_0^{T\wedge\tau} \varphi A_t e^{-rt}\mathrm{d}t|\mathcal{F}_0) = f(\varphi). \tag{4.10}
$$

where τ is the future lifetime of the policyholder.

Assume that the mortality risk can be diversified, then we can rewrite the left hand side of the equation by taking conditional expectation on τ

$$
\int_0^T \mathbf{E}^{\mathbb{Q}}[e^{-rt}(G_t^D - A_t)_+|\mathcal{F}_0]{}_tp_x\mu_{x+t}\mathrm{d}t \tag{4.11}
$$

where${}_tp_x$ and μ_{x+t} represent the survival probability and force of mortality. There are some supposition about force of mortality. Here we choose the constant force of mortality supposition and we can get

$$
{}_tp_x\mu_{x+t} = \mu e^{-\mu t}
$$

As for the part $\mathbf{E}^{\mathbb{Q}}[e^{-rt}(G_t^D - A_t)_+|\mathcal{F}_0]$, it has slight different from GMMB. The different parts are that we get the integral upon t, so the integral range $[a, b]$ will change with variable t. We can program it by the definition of integral in C and the characteristic function about x_t will be the tiny time period Δt instead of the whole period T.

And the right hand side of the equation will be

$$
\begin{aligned}
f(\varphi) &= \mathbf{E}^{\mathbb{Q}}(\int_0^T \varphi A_t e^{-rt}{}_tp_x\mu_{x+t}\mathrm{d}t|\mathcal{F}_0) \\
&= \mathbf{E}^{\mathbb{Q}}(\sum\nolimits_{n=0}^{N-1}\varphi\int_{n\Delta t}^{(n+1)\Delta t} A_t e^{-rt}\mu e^{-\mu t}\mathrm{d}t|\mathcal{F}_0) \\
&= \sum\nolimits_{n=0}^{N-1}\mathbf{E}^{\mathbb{Q}}(\varphi A_{n\Delta t}e^{-rn\Delta t}\mu e^{-\mu\Delta t}\Delta t|\mathcal{F}_0) \\
&= \sum\nolimits_{n=0}^{N-1}\Delta t e^{-rn\Delta t}\varphi\mu e^{-\mu\Delta t}\mathbf{E}^{\mathbb{Q}}(e^{i\omega x_{n\Delta t}}|\mathcal{F}_0)_{\omega=-i}
\end{aligned} \tag{4.12}
$$

By the analysis above, we can get the insurance fee φ in C by the Newton Recursion Method or Secant Method.

4.3 Pricing Process of GMIB under Heston Model

From above introduction about GMIB, we know there are two more parameters in GMIB, which are g and $a(T)$. The present value of GMIB $g(\varphi)$ at maturity or the present

Table 1: **Estimate Value of g for a 30-year Term Certain Annuity**

Interest Rate	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Annuity Value	25.81	22.40	19.60	17.29	15.37	13.76	12.41	11.26	10.27	9.43
Estimate Value of g	0.039	0.045	0.051	0.058	0.065	0.073	0.081	0.089	0.097	0.106

value of the expected liability should equals to the present value of the total of the continuously charged fee $f(\varphi)$, it means

$$g(\varphi) = \mathbf{E}^{\mathbb{Q}}[e^{-rT}(G_T^I g a(T) - A_T)_+ | \mathcal{F}_0] = \mathbf{E}^{\mathbb{Q}}(\int_0^T \varphi A_t e^{-rt} \mathrm{d}t | \mathcal{F}_0) = f(\varphi). \tag{4.13}$$

The two important points for GMIB are the value of $a(T)$ and g. About the value of a(T), which represents the market price of the term certain annuity, the interest rate is constant and fixed under $HestonModel$, that means the value of $a(T)$ at the maturity

$$a(T) = \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) \tag{4.14}$$

About the value of g, we refer to the method [4]. If we choose the 30-year term certain annuity, the fair price of g will be approximately the inverse of the valued of a 30-year term certain annuity. We can get the following table by setting different interest rates.

From the table above, we can find the value of g is between 4%-10% with the changeable interest rate. Actually the interest rate hardly exceed 10% in realistic world based on the date in European Financial Market. Besides, the insurance company chooses the value g in a conservative way for funding the guarantees. And considering their conservative strategy, the value of g will smaller that the fair price of estimate g. In our paper we consider the value of g between 6.5%-7.5% is the conservative value from insurance company's perspective.

By the analysis above, we can get the estimate value of g and $a(T)$. And the present value of expected liability burdened by the insurer and the present value of total charge are similar as GMMB.

5 Program in C

We will implement the calculation of the valuation of the GMMB, GMDB, GMIB under Heston model and the Black-Scholes model respectively. The main function for the three cases are specified as follows.

5.1 GMMB

For the GMMB under Heston model:
double ap_fouriercosine_gmmb_heston(double A0, double v0, double r, double divid, double sigma, double rho, double kappa,double alpha, double theta, double maturity, double premium, double rollup_rate, int N, double *ptprice)

For the GMMB under Black-Scholes model:
double ap_fouriercosine_gmmb_heston(double A0, double r, double divid, double

sigma,double alpha, double maturity, double premium, double rollup_rate, int N, double *ptprice)

alpha: percentage of premium guaranteed;
T: maturity of GMMB.

5.2 GMDB

For the GMDB under Heston model:
double ap_fouriercosine_gmdb_heston(double d,double A0, double v0, double r, double divid, double sigma, double rho, double kappa,double alpha, double theta, double maturity, double premium, double rollup_rate, int N, double *ptprice);
For the GMDB under Black-Scholes model:
double ap_fouriercosine_gmdb_bs(double d,double A0, double r, double divid, double sigma,double alpha, double maturity, double premium, double rollup_rate, int N, double *ptprice)

alpha: percentage of premium guaranteed;
T: maturity of GMDB.

5.3 GMIB

For the GMIB under Heston model:
double ap_fouriercosine_gmib_heston(double y,double g,double A0, double v0, double r, double divid, double sigma, double rho, double kappa,double alpha, double theta, double maturity, double premium, double rollup_rate, int N, double *ptprice)
For the GMIB under Black-Scholes model:
double ap_fouriercosine_gmib_heston(double y ,double g,double A0, double v0, double r, double divid, double sigma, double rho, double kappa,double alpha, double theta, double maturity, double premium, double rollup_rate, int N, double *ptprice)

g: minimum guaranteed payment rate;
y: market price of term certain annuity(a(T)).

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