# Gauss-Hermite quadrature on a cubic spline interpolation for the valuation of Guaranteed Minimum Withdrawal Benefit under Hull White model

Jiang Yu Nguwi

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## 1 Guaranteed Minimum Withdrawal Benefit

At $t = 0$, Policy Holder (PH) pays a lump sum premium P to the insurance company. Denote $\{t_i : i = 0, 1, 2, ..., N\}$ uniform grids of event time, where $t_0 < t_1 < ... < t_N$. Event time is the period where the withdrawal takes place. Note that there is no withdrawal at $t = t_0$, the first withdrawal occurs at $t = t_1$.

There are two accounts created upon initialization of the policy. The first account is sub-account, denote it as $A_t$. The second account is guaranteed account, denote it as $B_t$. The value of sub-account and guaranteed account is initially set to the lump sum premium $P$, $A_{t_0} = B_{t_0} = P$. The withdrawal frequency is set to be 1 by default (withdraw annually).

Denote the contract withdrawal rate $G$ and withdrawal amount at time $t_i$ $W_{t_i}$. PH can withdraw cash up to the guaranteed account value, subject to penalty charges, $\kappa$, if the withdrawal amount is more than $G$. The penalty charges is set to be 0.1 by default. Denote the amount PH receives from withdrawing $W_{t_i}$ at $t_i$ $f_{t_i}(W_{t_i})$. The function $f_{t_i}(W_{t_i})$ can be expressed as follows:

$$f_{t_i}(W_{t_i}) = \begin{cases} W_{t_i} & \text{,if } W_{t_i} \leq G \\ W_{t_i} - \kappa_{t_i}(W_{t_i} - G) & \text{,if } B_{t_i} \geq W_{t_i} > G. \end{cases}$$

Here, $\kappa_t$ (the penalty charge) is allowed to change over time. By default, it follows the same setting as in Table 1 of [3].

To illustrate the dynamics of sub-account and guaranteed account, denote $A_{t_i^{(+)}}$ and $B_{t_i^{(+)}}$ the value of sub-account and guaranteed account right after withdrawal and $A_{t_i^{(-)}}$ and $B_{t_i^{(-)}}$ the value of sub-account and guaranteed account right before withdrawal. The relationship between account value before and after withdrawal can be expressed as follows:

$$(A_{t_i^{(+)}}, B_{t_i^{(+)}}, t_i) = (max(A_{t_i^{(-)}} - W_{t_i}, 0), B_{t_i^{(-)}} - W_{t_i}, t_i).$$

At $t = t_N$, PH takes the maximum between the remaining balance in sub-account and the remaining balance in guaranteed account subject to penaly charges. Hence, the final payoff is

$$u(A_{t_N}, B_{t_N}, t_N) = max(A_{t_N}, f_{t_N}(B_{t_N})).$$

Two cases of withdrawal for this product were considered. The first case, the static approach, is when PH withdraw at the constant contract rate, $G$. The second case, the dynamic approach, is when PH withdraw at a rate to maximise his overall benefits. The optimisation problem at each withdrawal period is as follow:

$$W_{t_i} = \underset{w_{t_i} \in [0, B_{t_i^{(-)}}]}{\text{argmax}} \; u(max(A_{t_i^{(-)}} - w_{t_i}, 0), B_{t_i^{(-)}} - w_{t_i}, t_i) + f_{t_i}(w_{t_i}).$$

## 2 Hull White model

We consider the following model for the stock price:

$$\begin{cases} dS_t = r_t S_t dt + \omega^S S_t dZ_t^S \\ dr_t = \kappa^r(\theta_t^r - r_t)dt + \omega^r dZ_t^r, \end{cases}$$

where $Z_t^S, Z_t^r$ are Brownian motion and $d < Z_t^S, Z_t^r >= \rho_{Sr} dt$.

Here, we only consider a particular case of the function $\theta_t^r$, flat curve. By definition of flat curve, $\theta_t^r = \bar{r_0} + \frac{{\omega^r}^2}{2{\kappa^r}^2}(1 - exp(-2{\kappa^r}^2 t))$.

By default, the initial values of asset $S_0$ is 1 and $r_0$ is set to be 0.05. On the other hand the parameters are by default set to be $\bar{r_0} = 0.05$, $\rho_{Sr} = 0.5$, $\kappa^r = 1$, $\omega^r = 0.2$, $\omega^S = 0.2$. By default, the maturity is five years and the policy type is static ($G = 20$).

## 3 Gauss-Hermite quadrature on a cubic spline interpolation

We refer to [1] for complete description of the method. GHQC method works by estimating integration problem by Gauss-Hermite quadrature. The integration problem is as follows:

$$\begin{aligned} u(A_{t_n^{(+)}}, B_{t_n^{(+)}}, r_{t_n}) &= E_{t_n^{(+)}}[e^{-\int_{t_n}^{t_{n+1}} r(\tau)d\tau} u(A_{t_{n+1}^{(-)}}, B_{t_{n+1}^{(-)}}, r_{t_{n+1}}) | A_{t_n^{(+)}}, B_{t_n^{(+)}}, r_{t_n}] \\ &= E_{t_n^{(+)}}[e^{-\int_{t_n}^{t_{n+1}} r(\tau)d\tau}] \int\int p(A', r' | A_{t_n^{(+)}}, r_{t_n}) u(A', B_{t_{n+1}^{(-)}}, r') dA' dr', \end{aligned}$$

where $p(A', r' | A_{t_n^{(+)}}, r_{t_n})$ is the joint conditional distribution of $A$ and $r$. During withdrawal period, there will be a jump in the value of contract. The jump condition is:

$$u(A_{t_n^{(+)}}, B_{t_n^{(+)}}, r_{t_n}) = \begin{cases} u(max(A_{t_n^{(-)}} - \kappa, 0), B_{t_n^{(-)}} - \kappa, t_n) + f_{t_n}(\kappa) & \text{, if static;} \\ \max\limits_{w_{t_n} \in [0, B_{t_n^{(-)}}]} u(max(A_{t_n^{(-)}} - w_{t_n}, 0), B_{t_n^{(-)}} - w_{t_n}, t_n) + f_{t_n}(w_{t_n}) & \text{, if dynamic.} \end{cases} \tag{1}$$

To evaluate $E_{t_n^{(+)}}[e^{-\int_{t_n}^{t_{n+1}} r(\tau)d\tau}]$, closed form solution is provided in [1]. However, we point out that in [1], Vasicek model is used. Hence, some formulas are different in the case of Hull White model.

$$\begin{aligned} E_{t_n^{(+)}}[e^{-\int_{t_n}^{t_{n+1}} r(\tau)d\tau}] &= e^{A_{t_n,t_{n+1}} - r(t_n)B_{t_n,t_{n+1}}}, \\ B_{t_n,t_{n+1}} &= \frac{1}{\kappa^r}(1 - e^{-\kappa^r(t_{n+1}-t_n)}), \\ A_{t_n,t_{n+1}} &= (\bar{r_0} - \frac{\omega^{r2}}{2\kappa^{r2}})(B_{t_n,t_{n+1}} + t_n - t_{n+1}) - \frac{\omega^{r2}}{4\kappa^r} B^2_{t_n,t_{n+1}} \\ &+ \frac{\omega^{r2}}{4\kappa^{r3}}(2 - 2\kappa^r(t_{n+1} - t_n) - 2e^{-\kappa^r(T-t)} - 2e^{-\kappa^r(T+t)} + e^{-2\kappa^r t} + e^{-2\kappa^r T}). \end{aligned}$$

Then, we would need to calculate distribution of ln A and r. Denote $lnA_{t_n^{(+)}} = x^*$ and $r_{t_n} = r^*$. The conditional joint distribution of $lnA_{t_{n+1}^{(-)}}$ and $r_{t_{n+1}}$ given $x*$ and $r*$ is bivariate Normal density distribution. Its mean, variance and covariance are as follow:

$$
\begin{aligned}
\mu_r(r^*) &= r^* e^{-\kappa^r \Delta n} + (\bar{r_0} - \frac{\omega^{r2}}{\kappa^{r2}}) b_n + \frac{\omega^{r2}}{2\kappa^{r2}} a_n + \frac{\omega^{r2}}{2\kappa^{r2}} b_n^2; \\
\tau_r^2 &= \frac{\omega^{r2}}{2\kappa^r} a_n; \\
\mu_x(x^*, r^*) &= x^* + \frac{b_n}{\kappa^r}(r^* + \frac{b_n \omega^{r2}}{2\kappa^{r2}}) + (\bar{r_0} - \frac{\omega^{r2}}{\kappa^{r2}})(\Delta_n - \frac{b_n}{\kappa^r}) - \frac{\rho_{Ar}\omega^S \omega^r}{\kappa^{r2}}(\kappa^r \Delta_n - b_n) - (\alpha + \frac{1}{2}\omega^{S2})\Delta_n \\
&\quad - \frac{\omega^{r2}}{4\kappa^{r3}}(2(1 - \kappa^r \Delta_n) - 4e^{-\kappa^r \Delta_n} + e^{-2\kappa^r \Delta_n} + 1); \\
\tau_x^2 &= \omega^{S2}\Delta_n + \frac{\omega^{r2}}{2\kappa^{r3}}(2\kappa^r \Delta_n - 4b_n + a_n) + \frac{2\rho_{Sr}\omega^S \omega^r (\kappa^r \Delta_n - b_n)}{\kappa^{r2}}; \\
\rho_{xr} &= \frac{1}{\tau_x \tau_r}(\frac{\rho_{Sr}\omega^S \omega^r b_n}{\kappa^r} + \frac{\omega^{r2}}{2\kappa^{r2}}(2b_n - a_n)).
\end{aligned}
$$

where $b_n = 1 - e^{-\kappa^r \Delta_n}$, $a_n = 1 - e^{-2\kappa^r \Delta_n}$ and $\Delta_n = t_{n+1} - tn$. Then, the conditional joint distribution of $Y_1 = (lnA_{t_{n+1}^{(-)}} - \mu_x)/\tau_x$ and $Y_2 = (r_{t_{n+1}} - \mu_r)/\tau_r$ is standard bivariate Normal density distribution.

Now, denote $a = \frac{1}{2}(\sqrt{1+\rho_{xr}} + \sqrt{1-\rho_{xr}})$ and $b = \frac{1}{2}(\sqrt{1+\rho_{xr}} - \sqrt{1-\rho_{xr}})$.As proposed by Shevchenko & Luo, by a change of variable $Y_1 = \sqrt{2}(aZ_1 + bZ_2)$ and $Y_2 = \sqrt{2}(bZ_1 + aZ_2)$, the integration problem becomes

$$
u(A_{t_n^{(+)}}, B_{t_n^{(+)}}, r_{t_n}) = e^{A_{t_n,t_{n+1}} - r(t_n)B_{t_n,t_{n+1}}} \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-Z_1^2 - Z_2^2} u^{(Z)}(Z_1, B_{t_{n+1}^{(-)}}, Z_2) dZ_1 dZ_2.
$$

The key idea is to basically solve the above integration problem using Gauss Hermite quadrature.

To approximate the integration problem, Gauss Hermite quadrature is applied to every grid point $(A_m, r_k), m = 0, 1, ..., M$, and $k = 0, 1, ..., K$. Then, $X_m = ln(A_m)$ and $r_k$ are discretized according to [2] with parameters $S_{left} = 0.8S_0$, $S_{right} = 1.2S_0$, $\xi_{max} = 20S_0$, $\xi_{min} = \frac{S_0}{10000}$, $d_1 = \frac{S_0}{20}$, $R_{max} = 10r_0$, $c = r_0$, $d_3 = \frac{R_{max}}{400}$.

Note that we define $\xi_{min} = \frac{S_0}{10000}$ instead of 0 because if we set it to be 0, we have trouble defining $ln(S)$.

We then have the following approximation (details please see [1]):

$$
u(A_m, B_{t_n^{(+)}}, r_k) \approx e^{A_{t_n,t_{n+1}} - r(t_n)B_{t_n,t_{n+1}}} \frac{1}{\pi} \sum_{i=1,j=1}^{q_1,q_2} \lambda_i^{(q_1)} \lambda_j^{(q_2)} u(A_{ijkm}, B_{t_{n+1}^{(-)}}, r_{ijk}), \tag{2}
$$

$$
A_{ijkm} = exp(\sqrt{2}\tau_x(a\xi_i^{(q_1)} + b\xi_j^{(q_2)}) + \mu_x(X_m, r_k)), \tag{3}
$$

$$
r_{ijk} = \sqrt{2}\tau_r(b\xi_i^{(q_1)} + a\xi_j^{(q_2)}) + \mu_r(r_k). \tag{4}
$$

We provide two possible simulations. The coarser one sets $m_1 = 50$, $m_3 = 30$, and the order of Hermite polynomial for $S$ and $r$ are 5 and 3 respectively; the coarser one sets $m_1 = 150$, $m_3 = 50$, and the order of Hermite polynomial for $S$ and $r$ are 9 and 5 respectively. By default, it uses the finer simulation.

# 4 Implementation

The implementation of the method can be summarised as follows:

- **Step 1** Uniformly discretize guaranteed account space $B_j, j = 0, 1, ..., J$. Discretize sub-account space and interest rate space $(A_m, r_k), m = 0, 1, ..., M$, and $k = 0, 1, ..., K$ according to the above.

- **Step 2** For each grid point, initialize the value of the contract $u(A_{t_N^{(-)}}, B_{t_N^{(-)}}, r_{t_N}) = max(A_{t_N^{(-)}}, (1-\kappa)B_{t_N^{(-)}} + \kappa * min(G, B_{t_N^{(-)}}))$.

- **Step 3** For each given $\bar{B}$, use bicubic spline in $A$ and $r$ to obtain continuous function of $u(A, \bar{B}, r)$. Then, the value of $u(A_{t_{N-1}^{(+)}}, B_{t_{N-1}^{(+)}}, r_{t_{N-1}})$ is evaluated by 2.

- **Step 4** Jump condition 1 is applied to evaluate $u(A_{t_{N-1}^{(-)}}, B_{t_{N-1}^{(-)}}, r_{t_{N-1}})$. If evaluating for dynamic case, one-dimensional cubic spline is applied in $B$ to choose the optimal withdrawal amount.

- **Step 5** Repeat step 3 and step 4 for $t = t_{N-1}, t_{N-2}, ..., t_0$.

- **Step 6** Obtain $u(A_{t_0}, B_{t_0}, r_{t_0})$ and compare the value to $A_{t_0}$. Repeat step 1 to step 5 until we find the value of $\alpha$ such that $u(A_{t_0}, B_{t_0}, r_{t_0}) = A_{t_0}$.This $\alpha$ is the fair fee that the insurance company should charge on PH. Secant method is used to find $\alpha$.

## References


[1] Shevchenko, P. V., & Luo, X. (2017). Valuation of variable annuities with guaranteed minimum withdrawal benefit under stochastic interest rate. Insurance: Mathematics and Economics, 76, 104-117. 2, 3

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[3] Chen, Z., Vetzal, K., & Forsyth, P. A. (2008). The effect of modelling parameters on the value of GMWB guarantees. Insurance: Mathematics and Economics, 43(1), 165-173. 1