ASYMPTOTIC AND EXACT PRICING OF OPTIONS ON VARIANCE

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Most of what is presented here is taken from [1].

The authors consider a discounted asset S = S0 exp(X) and a time-interval $[0, T]$ subdivided into n intervals of equal length with boundary points $tj = j\frac{T}{n}$ for $j = 1, ..., n$. The corresponding (annualized) realized variance of X over $[0, T]$ is then defined as

$$RV_n^X(T) = \frac{1}{T}\sum_{j=1}^{n}\log(S_{t_j}/S_{t_{j-1}})^2 = \frac{1}{T}\sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})^2 \tag{0.1}$$

1 Small-time asymptotics in exponential Lévy models

The asset price process is modeled as $S = S_0 e^X$ for a Lévy process X. The Lévy process X is characterized through its Lévy-Khintchine triplet $(b, \sigma^2, F(dx))$ with respect to the truncation function $h(x) = x$ or, equivalently, by its Lévy exponent, i.e., the function

$$\psi(u) = ub + \frac{1}{2}u^2\sigma^2 + \int(e^{ux} - 1 - ux)F(dx), \quad u \in i\mathbb{R},$$

for chich $\mathbb{E}\, e^{uX_t} = e^{t\psi(u)}$. One can decompose X as

$$X_t = bt + \sigma W_t + L_t,$$

where W is a standard Brownian motion and L is an independent centered pure-jump Lévy process.

The short time asymptotic of options in quadratic variation is given by the next result

Theorem 1.1. *Let X be a square-integrable Lévy process with Lévy-Khintchine triplet $(b, \sigma^2, F(dx))$ and suppose the payoff functions $g_T : \mathbb{R} \longrightarrow \mathbb{R},\ T \geq 0$ are continuous, uniformly bounded, and satisfy $\|g_T - g_0\|_\infty \longrightarrow 0$ as $T \longrightarrow 0$. Then*

$$\lim_{T\to 0} \mathbb{E}\left[g_T(\frac{1}{T}[X,X]_T)\right] = g_0(\sigma^2).$$

The analogue of this Theorem for options on the discrete realized variance reads as follows:

Theorem 1.2. *Let X be a square-integrable Lévy process with Lévy-Khintchine triplet $(b, \sigma^2, F(dx))$ and suppose that the payoff functions $g_{n,T} : \mathbb{R} \longrightarrow \mathbb{R},\ T \geq 0,\ n \in \mathbb{N}$ are continuous, uniformly bounded, and satisfy $\|g_{n,T} - g_{n,0}\|_\infty \longrightarrow 0$ as $T \longrightarrow 0$ for each $n \in \mathbb{N}$. Then*

$$\lim_{T\to 0} \mathbb{E}\left[g_{n,T}(RV_n^X(T))\right] = \mathbb{E}\left[g_{n,0}(Y_n)\right]$$

where Y_n has gamma distribution with shape parameter $n/2$ and scale parameter $2\sigma^2/n$.

2 Pricing of Option on Realized Variance

The Laplace transform of the quadratique variance can be written as

$$\mathbb{E}\left[\exp\left(-u\sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})^2\right)\right] = \left(\mathbb{E}\ e^{-uX^2_{\frac{T}{n}}}\right)^n \tag{2.1}$$

And according to [1], we have

$$\mathbb{E}\ e^{-uX_t^2} = \mathbb{E}\left[e^{t\psi(iZ\sqrt{2u})}\right], \tag{2.2}$$

where Z is a standard normal random variable.

To calculate prices of options on realisze variance one can use the method of [2]. Denote by $\varphi(u) = \mathbb{E}\, e^{iuRV}$, then we have

$$\mathbb{P}(RV > K) = \Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathcal{R}e \left(\frac{e^{-iu\ln(K)}\varphi(u)}{iu} \right) du$$

and the Delta of the option is given by

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathcal{R}e \left(\frac{e^{-iu\ln(K)}\varphi(u-i)}{iu\varphi(-i)} \right) du$$

Assuming no dividends and constant interest rates r, the initial option value is then determined as

$$C = \mathbb{E}\, VS\ \Pi_1 - K\ e^{-rt}\Pi_2 \tag{2.3}$$

The moldels considered is the tempered stable.

References

[1] Asymptotic and exact pricing options on variance. M.Keller-Ressel J.Muhle-Kar *Finance Stochastics Volume 17 (2013), issue 1* MARTIN KELLER-RESSEL AND JOHANNES MUHLE-KARBE , ASYMPTOTIC AND EXACT PRICING OF OPTIONS ON VARIANCE 1, 2

[2] Peter Carr and Dilip B. Madan, Option valuation using the fast Fourier transform. 3