Forward variance dynamics : Bergomi's model revisited

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1 Preliminaries

Most of what is presented here is taken from [1] and [2].

A variance swap with maturity T is a contract which pays out the realized variance of the logarithmic total returns up to T less a strike called the variance swap rate V_0^T , determined in such a way that the contract has zero value today.

The annualized realized variance of a stock price process S for the period $[0, T]$ with business days $0 = t_0 < ... < t_n = T$ is usually defined as

$$RV^{0,T} := \frac{d}{n} \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 .$$

The constant d denotes the number of trading days per year and is usually fixed to 252 so that $\frac{d}{n} \approx \frac{1}{T}$. We assume the market is arbitrage-free and prices of traded instruments are represented as conditional expectations with respect to an equivalent pricing measure $\mathbb{Q}$. A standard result gives that as $\sup_{i=1,\dots,n} |t_i - t_{i-1}| \longrightarrow 0$, we have

$$\sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \longrightarrow \langle \log S \rangle_T \quad \text{in probability} \tag{1.1}$$

when $(S_t)_{t\geq 0}$ is a continuous semimartingale.

Denote by V_t^T, the price at time t of a variance swap with maturity $T < \infty$. It is given under $\mathbb{Q}$ by

$$V_t^T = \mathbb{E}_t^{\mathbb{Q}}\left[RV^{0,T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\langle \log S\rangle_T\right].$$

We define the forward variance curve $(\xi^T)_{T\geq 0}$ as

$$\xi_t^T \ := \partial_T V_t^T, \qquad T \geq t \geq 0.$$

Note that, if we assume that S follows a diffusion process, $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$ with a general stochastic volatility process, σ, the forward variance is given by

$$\xi_t^T = \mathbb{E}_t^{\mathbb{Q}}\left(\sigma_T^2\right).$$

It can be seen as the forward instantaneous variance for date T, observed at t. In particular

$$\xi_t^t = \sigma_t^2, \quad \forall t \geq 0.$$

The current price of a variance swap, V_t^T, is given in terms of the forward variances as

$$V_t^T = \langle \log S\rangle_t + \int_t^T \xi_t^u du$$

The models used in practice are based on diffusion dynamics where forward variance curves are given as a functional of a finite-dimensional Markov-process:

$$\xi_t^T = G(T; t, Z_t), \tag{1.2}$$

where the function G and the m-dimensional Markov-process Z satisfy some consistency condition, which essentially ensures that for every fixed maturity $T > 0$, the forward variance $(\xi_t^T)_{t\leq T}$ is a martingale.

2 The model

Assume that a set of settlement dates is given

$$T_0 < T_1 < ... < T_n < \dots$$

and referred to as the tenor structure (we use especially the tenor structure of the VIX futures, but it can be generalized to any tenor structure). Consider an underlying asset whose price S is modeled as a stochastic process $(S_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ represents the history of the market.

We first specify the dynamics of the forward variance using a log normal specification which allows analytical pricing of European-type VIX derivatives. The dynamics of the forward variance under $\mathbb{Q}$ is assumed to be

$$\xi_t^T = \xi_0^T e^{\omega_T x_t^{T_i} - \frac{\omega_T^2}{2}\mathbb{E}(x_t^{T_i})^2}, \quad t \leq T \quad \text{and} \quad T \in]T_{i-1}, T_i] \tag{2.1}$$

where x_t^T is defined as

$$x_t^T := \sum_n \theta_n \, e^{-\kappa_n(T-t)} \int_0^t e^{-\kappa_n(t-s)} dW_s^n \tag{2.2}$$

The Brownian motions W^n are correlated with correlation coefficients $\rho_{i,j}$. The initial values of forward ξ_0^T are inputs of the model, deduced from the curve of variance-swap prices.

The number of factors introduced in this dynamics is the number of degrees of freedom that will be available to calibrate S&P's smiles, therefore a single-factor model would not be precise enough. On the other hand, the computation time in a Monte-Carlo method increases proportionally to the number of factors, therefore a two-factor model offers a good quality/time ratio. Anyway, all the following formulas do not depend on the number of factors.

The parameters of the dynamics (2.2), κ_i, θ_i and $\rho_{i,j}$, are chosen and will not be calibrated to market data, because they are not directly involved in the pricing of volatility derivatives. We follow here the approach of Bergomi [?], where he proposes some parameter sets which can be chosen. For example, in the case of 2 factors, Bergomi proposed to set $\frac{1}{\kappa_1}$ in the order of a few months, which corresponds to $\kappa_1 \approx 8$, $\frac{1}{\kappa_2}$ in the order of a few years: $\kappa_2 \approx 0.3$. The curve ω is a deterministic function of T. It is a scale factor for the volatility of ξ and it allows to control the term structure of the volatility of volatility by calibrating VIX futures and options.

2.1 The VIX Index

The VIX Futures maturing at time T quotes the expected volatility for the next 30 days. So VIX^2 represents 30-day S&P 500 variance swap rate, it is given under the risk neutral measure by

$$VIX_T = \sqrt{\mathbb{E}_T^{\mathbb{Q}}\left[\frac{1}{\delta} RV^{T,T+\delta}\right]}, \tag{2.3}$$

where $\delta = \frac{30}{365}$ and $RV^{T,T+\delta} = RV^{0,T+\delta} - RV^{0,T}$. In terms of the forward variance curve, the VIX is given by

$$VIX_T = \sqrt{\frac{1}{\delta}\int_T^{T+\delta} \xi_T^u du} = \sqrt{\frac{1}{\delta}\int_T^{T+\delta} \xi_0^u e^{\omega_u x_T^{\tau_u} - \frac{\omega_u^2}{2}\mathbb{E}(x_T^{\tau_u})^2} du},$$

where $\tau_u = \sum_{i\geq 1} T_i \mathbb{1}_{u\in]T_{i-1},T_i]}$. Note that $T_{i+1} - T_i = \delta$

We are only interested in maturities on which VIX is traded. This corresponds to the special (typically useful) case where $T = T_i$ for some $i = 1, \ldots,\ n$. Denote by $VIX_i := VIX_{T_i}$. Assumed that ω is Borel measurable and locally bounded. We have

$$VIX_i = \sqrt{\frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u e^{\omega_u x_{T_i}^{T_{i+1}} - \frac{\omega_u^2}{2}\mathbb{E}(x_{T_i}^{T_{i+1}})^2} du} \equiv \sqrt{g_i(Z)}, \tag{2.4}$$

where Z has the standard normal distribution and the function g_i is defined as

$$g_i(z) = \frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u e^{z\ \bar{\omega}_i(u) - \frac{\bar{\omega}_i^2(u)}{2}} du \tag{2.5}$$

and

$$\bar{\omega}_i(u)\ :=\ \omega_u\ \sqrt{\mathbb{E}(x_{T_i}^{T_{i+1}})^2}. \tag{2.6}$$

Using (2.4), we can evaluate any given European-like claim on VIX_i, with pay-off function f, as

$$\mathbb{E}^Q f(VIX_i) = \int_{\mathbb{R}} f(\sqrt{g_i(x)}) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx.$$

We can also express the prices of VIX options in terms of Calls and Puts on VIX2, by using the following useful representation, which is valid for any twice-differentiable function G and for any $k \in \mathbb{R}^+$

$$G(X^2) = G(k) + G'(k)(X^2 - k) + \int_k^\infty G''(K)(X^2 - K)_+ dK + \int_0^k G''(K)(K - X^2)_+ dK$$

This gives in particular the price of Calls and Put on VIX in terms of Calls and Puts on VIX^2, by extending this formula to the functions $x \longmapsto (\sqrt{x} - k)_+$ and $x \longmapsto (k - \sqrt{x})_+$

$$\mathbb{E}(VIX_i - k)_+ = \frac{1}{2k}\mathbb{E}({VIX_i}^2 - k^2)_+ - \int_{k^2}^\infty \frac{1}{4K\sqrt{K}}\mathbb{E}({VIX_i}^2 - K)_+ dK \tag{2.7}$$

$$\mathbb{E}(k - VIX_i)_+ = \frac{1}{2k}\mathbb{E}(k^2 - {VIX_i}^2)_+ + \int_0^{k^2} \frac{1}{4K\sqrt{K}}\mathbb{E}(K - {VIX_i}^2)_+ dK \tag{2.8}$$

and by call-put parity, one can express the VIX future price in terms of calls and puts on VIX^2. For every $k > 0$, we have

$$\mathbb{E}(VIX_i) = \frac{k + \mathbb{E}{VIX_i}^2}{2\sqrt{k}} - \int_0^k \frac{\mathbb{E}(K - {VIX_i}^2)_+ dK}{4K\sqrt{K}} - \int_k^\infty \frac{\mathbb{E}({VIX_i}^2 - K)_+ dK}{4K\sqrt{K}} \tag{2.9}$$

In particular, for $k = \mathbb{E}VIX_i^2$, we have

$$\mathbb{E}(VIX_i) = \sqrt{\mathbb{E}VIX_i^2} - \int_0^{\mathbb{E}VIX_i^2} \frac{\mathbb{E}(K - {VIX_i}^2)_+ dK}{4K\sqrt{K}} - \int_{\mathbb{E}VIX_i^2}^\infty \frac{\mathbb{E}({VIX_i}^2 - K)_+ dK}{4K\sqrt{K}} \tag{2.10}$$

Note that $\mathbb{E}VIX_i^2 = \frac{1}{\delta}\int_{T_i}^{T_{i+1}\xi_0^u du}$.

Now, the prices of Calls and Puts on VIX_i are given by the next proposition

Proposition 2.1. *If the function $\bar{\omega}_i$ is positive, then for any nonnegative strike K, the price of a call on VIX_i^2 with strike K is given by*

$$\mathbb{E}({VIX_i}^2 - K)_+ = \frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u N(-z_i^*(K) + \bar{\omega}_i(u)) du - KN(-z_i^*(K)) \tag{2.11}$$

and the price of a Put on ${VIX_i}^2$ with strike K is given by

$$\mathbb{E}(K - {VIX_i}^2)_+ = KN(z_i^*(K)) - \frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u N(z_i^*(K) - \bar{\omega}_i(u)) du \tag{2.12}$$

where N denotes the standard normal cumulative distribution function and z_i^* is defined as

$$z_i^*(K) = \inf\left\{z \in \mathbb{R} \;\middle|\; g_i(z) \geq K\right\} = g_i^{-1}(K)$$

Proof

We can write

$$\begin{aligned}
\mathbb{E}(VIX_i{}^2 - K)_+ &= \mathbb{E}(g_i(Z) - K)\mathbb{1}_{g_i(Z)\geq K} \\
&= \int_{z_i^*(K)}^{\infty} (g_i(x) - K)\frac{e^{-x^2/2}dx}{\sqrt{2\pi}} \\
&= \frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u \int_{z_i^*(K)}^{\infty} \frac{e^{-\frac{x^2}{2}+x\bar{\omega}_i(u)-\frac{\bar{\omega}_i^2(u)}{2}}dx}{\sqrt{2\pi}}du - KN(-z_i^*(K)) \\
&= \frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u N(-z_i^*(K) + \bar{\omega}_i(u))du - KN(-z_i^*(K)) \quad \square
\end{aligned}$$

2.2 Specifying $\bar{\omega}_\mathbf{i}$

We assume that $\bar{\omega}_i$ takes only two values within interval $]T_i, T_{i+1}]$. We will show that in addition to modeling the positive skew observed in the VIX options, we can calibrate, exactly, VIX future as well as "at least" one Put option by maturity.

Assumption 2.2. *The function $\bar{\omega}_i$ is decreasing and does not take more than two values over the time interval $]T_i, T_{i+1}]$.*

Denote by L_i the point where it changes its value. The curve $\bar{\omega}_i$ can then be parametrized as follows :

$$\bar{\omega}_i(t) = \zeta_i 1_{t\in]T_i,L_i]} + \beta_i\zeta_i 1_{t\in]L_i,T_{i+1}]}. \tag{2.13}$$

where $\beta_i \in [0, 1]$.

Under this assumption, F^{i2} takes the form

$$VIX_i{}^2 = m_i\left[(1-\gamma_i)e^{\zeta_i Z_i - \frac{\zeta_i^2}{2}} + \gamma_i\; e^{\beta_i\zeta_i Z_i - \frac{\beta_i^2\zeta_i^2}{2}}\right]$$

where $m_i := \frac{1}{\delta}\int_{T_i}^{T_{i+1}} \xi_0^u du$, $\gamma_i = \frac{\frac{1}{\delta}\int_{L_i}^{T_{i+1}} \xi_0^u du}{m_i}$ and the random variable $Z_i := \frac{1}{\sqrt{\mathbb{E}(x_{T_i}^{T_{i+1}})^2}} x_{T_i}^{T_{i+1}}$ has the standard normal distribution.

The price of any VIX future contract is then given as a function of the triplet $(\gamma_i, \beta_i, \zeta_i)$. This is given as function of calls and puts on VIX_i^2 by using proposition 2.12 and the equalities (2.7), (2.8) and (2.9). Noting that, with this special parametrization of the function $\bar{\omega}_i$, the price of call on ${VIX_i}^2$ with strike K is given by

$$\mathbb{E}({VIX_i}^2 - K)_+ = m_i\left[(1-\gamma_i)N(-z_i^*(K)+\zeta_i)+\gamma_i N(-z_i^*(K)+\beta_i\zeta_i)\right] - KN(-z_i^*(K)) \tag{2.14}$$

and the price of put on ${VIX_i}^2$ with strike K is given by

$$\mathbb{E}(K - {VIX_i}^2)_+ = KN(-z_i^*(K)) - m_i\left[(1-\gamma_i)N(z_i^*(K)-\zeta_i)+\gamma N(z_i^*(K)-\beta_i\zeta_i)\right] \tag{2.15}$$

Prices of VIX futures and options are then given by explicit formulas in terms of parameters γ_i, β_i and ζ_i.

We will study the particular case of VIX future and Put prices. First, to simplify the notation, denote

$$V_i^{\gamma,\beta,\zeta} := m_i\left[(1-\gamma)e^{\zeta Z_i - \frac{\zeta^2}{2}} + \gamma e^{\beta\zeta Z_i - \frac{\beta^2\zeta^2}{2}}\right],$$

for $(\zeta, \beta, \zeta) \in [0,1]\times[0,1]\times\mathbb{R}^+$. Also, let

$$F_{VIX}(\gamma,\beta,\zeta) = \mathbb{E}\sqrt{V_i^{\gamma,\beta,\zeta}} \quad \text{and} \quad p_k(\gamma,\beta,\zeta) = \mathbb{E}(k - \sqrt{V_i^{\gamma,\beta,\zeta}})_+,$$

the prices of VIX future and Put on VIX with strike k respectively.

The next result gives more information about the function giving the price of VIX future and put in terms of the parameters γ, β and ζ. The proof can be found in the appendix

Proposition 2.3. *The functions p_k and F_{VIX} are differentiable and their first partial derivatives are given by*

- $\partial_\zeta F_{VIX}(\gamma,\beta,\zeta) = -\int_0^\infty \frac{1}{4K\sqrt{K}}\left[m_i(1-\gamma)N'(z_i^*(K)-\zeta) + m_i\gamma\beta N'(z_i^*(K)-\beta\zeta)\right]dK,$
- $\partial_\beta F_{VIX}(\gamma,\beta,\zeta) = -m_i\omega\gamma\int_0^\infty N'(z_i^*(K)-\beta\zeta)\frac{dK}{4K\sqrt{K}},$
- $\partial_\gamma F_{VIX}(\gamma,\beta,\zeta) = m_i\int_0^\infty\left[N(z_i^*(K)-\beta\zeta) - N(z_i^*(K)-\zeta)\right]\frac{dK}{4K\sqrt{K}},$
- $\partial_\zeta p_k(\gamma,\beta,\zeta) = -m_i(1-\gamma)\int_{-\infty}^{z_i^*(k^2)-\zeta}\frac{KN'(K)dK}{2\sqrt{g_i(K+\zeta)}} - m_i\beta\gamma\int_{-\infty}^{z_i^*(k^2)-\beta\zeta}\frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}},$

- $\partial_\beta p_k(\gamma,\beta,\zeta) = -m_i\zeta\gamma \int_{-\infty}^{z_i^*(k^2)-\beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}}$,

- $\partial_\gamma p_k(\gamma,\beta,\zeta) = m_i \int_{-\infty}^{z_i^*(k^2)-\zeta} \frac{N'(K)dK}{2\sqrt{g_i(K+\zeta)}} - m_i \int_{-\infty}^{z_i^*(k^2)-\beta\omega} \frac{N'(K)dK}{2\sqrt{g_i(K+\beta\omega)}}$.

In particular, we have

1. $\partial_\omega F_{VIX}$ *and* $\partial_\beta F_{VIX}$ *are negative.*
2. $\partial_\gamma F_{VIX}$*,* $\partial_\zeta p_k$ *and* $\partial_\beta p_k$ *are positive.*
3. *If* $k \leq \sqrt{m_i}$*, then* $\partial_\gamma p_k$ *is negative.*

3 Calibrating γ, β and ζ

A model cannot be used in practise without a reliable and reasonably quick calibration scheme. We therefore describe here how the model can be calibrated using the "explicit dependence" between VIX futures and options prices and the model parameters γ, β and ζ for each maturity T_i, given by proposition 2.3.

Data

Assume that we observe the Variance-Swap market prices for all maturities. We deduce the initial variance curve $(\xi_0^T)_{T\geq 0}$ from the market prices of Variance-Swap.

Let us also assume that we observe the VIX future price and a series of Put options on VIX, for each maturity T_i. Obviously, we will not pretend to be able to calibrate VIX futures and all European options on VIX, nevertheless we will show that for each maturity we can calibrate "exactly" both VIX future price and one Put by leaving free the parameter γ along some interval. This parameter, left free, will serve to calibrate other options and/or to reproduce the VIX skew.

Phase 1: Calibrating VIX future

The hedging of VIX options is typically done with trading in VIX futures contracts, we want the model to reproduce the VIX futures prices for each maturity T_i.

Let's denote by F the market price of VIX future for some maturity T_i and denote by $m = \frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}} \xi_0^u du$. Note that F and m must satisfy the following (no-arbitrage) condition :

$$F \leq \sqrt{m}.$$

The calibration problem of the VIX future is to find a triplet $(\gamma, \beta, \zeta) \in [0,1] \times [0,1] \times \mathbb{R}_+$ such that

$$F_{VIX}(\gamma, \beta, \zeta) \ = \ F.$$

In what follows, we will show that for every (γ, ζ) belonging to some subset of $[0,1] \times \mathbb{R}_+$, there exists a unique $\beta \in [0,1]$, such that $F_{VIX}(\gamma, \beta, \zeta) \ = \ F$.

By Proposition 2.3, we know that for $(\gamma, \zeta) \in [0,1] \times \mathbb{R}_+$, the function $\beta \longmapsto F_{VIX}(\gamma, \beta, \zeta)$ is continuous, decreasing over $[0,1]$. It is then bijective from $[0,1]$ to $\Big[F_{VIX}(\gamma, \beta = 1, \zeta), F_{VIX}(\gamma, \beta = 0, \zeta)\Big]$. (Note that $V^{\beta,0,\zeta}$ and $V^{\beta,1,\zeta}$ are lognormales)

Now, it becomes clear that if the pair (γ, ζ) is such that $F_{VIX}(\gamma, 1, \zeta) \leq F$ and $F_{VIX}(\gamma, 0, \zeta) \geq F$, then there exists $\beta \in [0,1]$ such that $F_{VIX}(\gamma, \beta, \zeta) = F$.

So, for $\gamma \geq 0$, consider the function $\zeta \longmapsto F_{VIX}(\gamma, 0, \zeta)$. From Proposition 2.3, we know that it is continuous and decreasing over $\mathbb{R}^+$ satisfying $F_{VIX}(\gamma, 0, \zeta = 0) = \sqrt{m\gamma}$ and $\lim_{\zeta \to \infty} F_{VIX}(\gamma, 0, \zeta) = 0$. It allows us to define the fuction

$$\bar{\zeta}_F : \gamma \in [0, \frac{F^2}{m}) \longmapsto \bar{\zeta}_F(\gamma) := F_{VIX}(\gamma, 0, .)^{-1}(F) \tag{3.1}$$

This function, $\bar{\zeta}_F$, is continuous, increasing over $[0, \frac{F^2}{m})$ and satisfies

$$\begin{cases} \bar{\zeta}_F(0) = \zeta_F, \\ \lim_{\gamma \to \frac{F^2}{m}} \bar{\zeta}_F(\gamma) = +\infty. \end{cases}$$

where

$$\zeta_F := 2\sqrt{\log(\frac{m}{F^2})} \tag{3.2}$$

Denote

$$\Omega_F = \left\{ (\gamma, \zeta) \in [0, \frac{F^2}{m}) \times \mathbb{R}^+ \middle| \ \zeta \in [\zeta_F, \bar{\zeta}_F(\gamma)] \right\}. \tag{3.3}$$

It is easy to check that for every $(\gamma, \zeta) \in \Omega_F$, we have $F_{VIX}(\gamma, 1, \zeta) \leq F$ and

$F_{VIX}(\gamma, 0, \zeta) \geq F$. This means that the mapping

$$\beta_F \ : \ (\gamma, \zeta) \in \Omega_F \ \longmapsto \ \beta_F(\gamma, \zeta) := F_{VIX}(\gamma, ., \zeta)^{-1}(F) \tag{3.4}$$

is well defined. In particular, for every $(\gamma, \zeta) \in \Omega_F$, we have $F_{VIX}(\gamma, \beta_F(\gamma, \zeta), \zeta) = F$ $\square$

Phase 2: Calibrating VIX Put

As mentioned above, we choose to calibrate one Put option for each maturity T_i. Denote by P the market price of this option and by K_0 its strike. Here we will try to find a family of pairs $(\gamma, \zeta) \in \Omega_F$, such that

$$p_{k_0}(\gamma, \beta_F(\gamma, \zeta), \zeta) = P.$$

The map $(\gamma, \zeta) \longmapsto p_{k_0}(\gamma, \beta_F(\gamma, \zeta), \zeta)$ is neither monotonic in γ, nor in ζ because of β_F, then we cannot obtain its inverse easily. To address this problem, we proceed as follows.

By using proposition 2.3 and in the same way as before we can define

$$\bar{\zeta}_P \ : \ \gamma \in [0, \frac{(k_0 - P)^2}{m}) \longmapsto \bar{\zeta}_P(\gamma) := p_{k_0}(\gamma, 0, .)^{-1}(P) \tag{3.5}$$

In particular, $\bar{\zeta}_P$ is continuous, increasing over $[0, \frac{(P-k_0)^2}{m})$ and satisfies

$$\begin{cases} \bar{\zeta}_P(0) = \zeta_P, \\ \lim_{\gamma \to \frac{(P-k_0)^2}{m}} \bar{\zeta}_P(\gamma) = +\infty. \end{cases}$$

where

$$\zeta_P = \sup \left\{ \zeta > 0; \ \mathbb{P}_{BS}(\sqrt{m} e^{-\frac{\zeta^2}{8}}, k_0, \frac{\zeta}{2}) \ \leq \ P \right\} \tag{3.6}$$

and

$$\mathbb{P}_{BS}(S, k, \sigma) = -SN\left(\frac{-\log(\frac{S}{k}) - \frac{\sigma^2}{2}}{\sigma} \right) + kN\left(\frac{-\log(\frac{S}{k}) + \frac{\sigma^2}{2}}{\sigma} \right).$$

Denote

$$\Omega_P = \left\{ (\gamma, \zeta) \in [0, \frac{(k_0 - P)^2}{m}) \times [\ \zeta_P, \ \bar{\zeta}_P(\gamma)] \right\} \bigvee \left\{ [\frac{(k_0 - P)^2}{m}, \frac{F^2}{m}) \times [\ \zeta_P, \infty) \right\}$$

We can define the map

$$\beta_P \ : \ (\gamma, \zeta) \in \Omega_P \longmapsto \ \beta_P(\gamma, \zeta) := p_{k_0}(\gamma, ., \zeta)^{-1}(P) \tag{3.7}$$

Now to solve the "double" calibration problem of the parameters γ, β and ζ to F and P, it suffices to find $(\gamma, \zeta) \in \Omega_F \cap \Omega_P$ such that

$$\beta_F(\gamma, \zeta) \ = \ \beta_P(\gamma, \zeta).$$

The proof of the next theorem can be found in the appendix.

Theorem 3.1. *Assume $\zeta_p \leq \zeta_F$ (where ζ_P is defined by (3.6) and ζ_F by (3.2)), then there exists $\gamma^* < \frac{(k_0-P)^2}{m}$ such that for every $\gamma \in [\gamma^*, \frac{F^2}{m})$, there exists $\zeta_\gamma = \zeta^*(\gamma, k_0, F, P) \geq 0$ such that $\beta_F(\gamma, \zeta_\gamma) \ = \ \beta_P(\gamma, \zeta_\gamma)$.*

Remark 3.1. With all market data that we have dealt with, the condition $\zeta_p \leq \zeta_F$ is satisfied for $k_0 = F$. For general case, we note that

$$\zeta_p \leq \zeta_F \iff \sigma_{IMP}(k_0) \leq \frac{\zeta_P}{2} \iff \sigma_{IMP}(k_0) \leq \frac{\zeta_F}{2}$$

where $\sigma_{IMP}(k) = \mathbb{P}_{BS}(F, k_0, .)^{-1} \ (P)$. Since "in practice", the implied volatility of VIX options are increasing with respect to the strike, then if k_0 is such that $\sigma_{IMP}(k_0) \leq \frac{\zeta_F}{2}$, so the condition is still satisfied if k_0 is replaced by $k \leq k_0$.

Remark 3.2. Thanks to the monotonicity properties of all the functions we have defined, the calculation of γ^* and ζ_γ are made by using a "special" binary search algorithm. This algorithm will be detailed in the appendix (see Remark **??**).

Phase 3: Calibrating γ

We can do without this calibration step if we only want to fit the future price and the Put price by choosing any value of γ between γ^* and $\frac{F^2}{m}$. Noting that $\frac{(k_0-P)^2}{m} \in [\gamma^*, \frac{F^2}{m})$. Otherwise, we can calibrate the VIX skew or another Put option on VIX.

By proposition 3.1, we know that by choosing any value of γ in $[\gamma^*, \frac{F^2}{m})$, we can find a couple $(\beta_\gamma, \zeta_\gamma)$ such that the model price of VIX future and Put on VIX with strike k_0 coincides with their market prices. There is therefore a possibility to calibrate γ to match another VIX future contract. Here we choose to calibrate with the aim to reproduce the skew of VIX at k_0. i.e the slope of the Put implied volatility of VIX at the point k_0.

In practice, the skew, at some point k, is measured as the difference of the implied volatilities of 95% and 105% strike. Now to compute the "skew" from the market data on VIX, we choose k_1: the nearest strike to k_0, on which the VIX put is available and we approximate the skew by the difference of the implied volatility of k_0 and k_1.

This step of calibration reduces to finding γ such that

$$\partial_k p_{k_0}(\gamma, \beta_F(\gamma, \zeta_\gamma); \zeta_\gamma) = \frac{P_1 - P}{k_1 - k_0}.$$

The calibration is thus reduced to minimizing

$$\left\{ \left[\frac{k_0^2}{m} - \left((1-\gamma)\ e^{\zeta_\gamma Z - \frac{1}{2}\zeta_\gamma^2} + \gamma\ e^{\zeta_\gamma \beta_F(\gamma,\zeta_\gamma) Z - \frac{1}{2}\zeta_\gamma^2 \beta_F(\gamma,\zeta_\gamma)^2} \right) \right]^2, \quad \gamma \in [\gamma^*, \frac{F^2}{m}) \right\}$$

Remark 3.3. By differentiating the Black-Scholes formula giving the price of Put with strike k_0 with respect to the strike, we can express $\partial_k p_{k_0}$in terms of the skew and the implied volatility of P at the point k_0 as

$$\partial_k \mathbb{E}(K - \sqrt{V_i^{\gamma, \beta_P(\gamma,\zeta_\gamma),\zeta_\gamma}})_+ \Big|_{k=k_0} = N(-d_2) + \mathcal{S}_i \times k_0 \sqrt{T_i} N'(-d_2),$$

where $d_2 = \frac{\log(\frac{F}{k_0}) - \frac{\sigma_{VIX}^2(k_0)}{2} T_i}{\sigma_{VIX}(k_0)\sqrt{T}}$ and $\sigma_{VIX}(k_0)$ is the implied volatility of the Put on VIX with strike k_0. We can then synthesize $\mathbb{E}_t \mathbb{1}_{k \geq VIX}$ by observing continuously the price of Put P, the future price of VIX and the skew.

4 The dynamics of the underlying asset

Until now, we have only addressed issues concerning the modelling of the forward variance curve. But, once the dynamics of forward variance has been specified, we obtain the (risk neutral) dynamics of the underlying asset $(S_t)_{t \geq 0}$ as

$$\frac{dS_t}{S_t} = r dt + \sqrt{\xi_t^t} dW_t^S, \tag{4.1}$$

where r is the annualized risk-free interest rate and ξ_t^t is given by (2.1) as

$$\xi_t^t = \xi_0^t \ e^{\omega_t x_t^{T_i} - \frac{\omega_t^2}{2}\mathbb{E}(x_t^{T_i})^2}, \quad t \in]T_{i-1}, T_i]$$

and for $t \leq T$, x_t^T is defined in (2.2) as

$$x_t^T = \sum_n \theta_n \ e^{-\kappa_n(T-t)} \int_0^t e^{-\kappa_n(t-s)} dW_s^n$$

The Brownian motion W^S is correlated with the factors X^n, denote by $\rho_n^S = \frac{d\langle W^S, X^n\rangle_t}{dt}$. The number of factors has been discussed in the beginning of this work, it corresponds to the number of degrees of freedom that will be available to fit many different smile shapes.

Now, thanks to lognormal form of the instantaneous variance, we can use the very robust approximations we obtain in [2] for the prices of the European options under this model. We can specify the correlations ρ_n^S to match the specified skew and to calibrate the ATM implied volatility

4.1 One Factor Case : Time Dependent Scott Model

In this section we consider a generalization of the model proposed by Scott 1987. Within this model, the dynamics of the underlying is given by the SDE

$$\begin{cases} \frac{dS_t}{S_t} = f(t, V_t) dW_t^S, \\ dV_t = -bV_t dt + \omega\sigma_t dW_t^V, \ d\langle W^S, W^V\rangle_t = \rho_t dt, \end{cases} \tag{4.2}$$

where $f^2(t,v) = m_t e^v$, m, ρ and σ are deterministic functions of time. Assume $\omega, \ \sigma \ \geq 0$. Assume also that $V_0 = 0$ (Otherwise, we replace m_t by $m_t \ e^{V_0 e^{-bt}}$ and V by $V - V_0$).

Denote $\varphi_X(t, \xi; \omega) := \mathbb{E} \ e^{i\xi X_t}$. Then we have

$$\varphi_X(t,\xi;\omega) \ \sim \ \psi(t,\xi;\omega) := e^{(-i\mu_1(t)\xi - \frac{\nu(t)}{2}\xi^2)} \left(1 + \sum_{n=1}^{6} (-i)^n \nu_n(t) \xi^n\right)$$

Then we have

$$\psi \ \equiv \ \varphi_X \quad [2]$$

Furthermore, if we denote by $p_X(t,x;\omega)=\mathbb{P}(X_t\in dx)$, then we have

$$p_X(t,x;\omega) \;\sim\; \frac{1}{\sqrt{2\pi\nu(t)}}e^{-\frac{(x+\mu_1(t))^2}{2\nu(t)}}\left(1+\sum_{n=1}^{6}(-1)^n\frac{\nu_n(t)}{\nu^{\frac{n}{2}}}H_n(\frac{x+\mu_1(t)}{\sqrt{\nu(t)}})\right) \tag{4.3}$$

where the H_n's are the Hermite polynomials.

This gives the following approximation for the prices of European options.

Denote by $C:\;(t,K;\omega):=\mathbb{E}(e^{X_t}-K)_+$. Then, we have

$$C(t,K;\omega) \;\sim\; C_{BS}(K,\nu)+\frac{K}{\sqrt{\nu}}N'(d_2)\Big(\sum_{n=0}^{4}\frac{z_n}{\nu^{\frac{n}{2}}}H_n(-d_2)\Big) \tag{4.4}$$

where $C_{BS}(K,\nu)\;=\;N(d_1)-KN(d_2)$, with $d_1\;=\;\frac{\log(\frac{1}{K})+\frac{1}{2}\nu}{\sqrt{\nu}}$, $d_2\;=\;\frac{\log(\frac{1}{K})-\frac{1}{2}\nu}{\sqrt{\nu}}$, $z_4\;=\;\nu_6$, $z_3=-\nu_5+z_4$, $z_2=\nu_4+z_3$, $z_1=-\nu_3+z_2$, $z_0=\nu_2+z_1=\nu_1$.[1]

The ν's are given by: $\nu(t)=\int_0^t m_s ds$ and

$$\begin{aligned}
\nu_1(t) &= \frac{\omega^2}{4}\int_0^t m_s\int_0^s \sigma_u^2 e^{-2b(s-u)}duds,\\
\nu_2(t) &= \nu_1+\frac{\omega^2}{4}\int_0^t m_\tau\int_0^\tau m_s\int_0^s \sigma_u^2 e^{-b(\tau+s-2u)}dudsd\tau-\frac{\omega}{2}\int_0^t m_\tau\int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}dsd\tau,\\
\nu_3(t) &= \mu_3(t),\quad \nu_4(t)=\mu_4(t)+\frac{\omega^2}{8}\Big(\int_0^t m_\tau\int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}dsd\tau\Big)^2,\\
\nu_5(t) &= \frac{\omega^2}{4}\Big(\int_0^t m_\tau\int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}dsd\tau\Big)^2,\;\; \nu_6(t)=\frac{\omega^2}{8}\Big(\int_0^t m_\tau\int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}dsd\tau\Big)^2.
\end{aligned}$$

[1]Note that $\nu_1-\nu_2+\nu_3-\nu_4+\nu_5-\nu_6=0$.

where

$$\mu_1(t) = \frac{1}{2}\int_0^t m_s ds + \frac{\omega^2}{4}\int_0^t m_s \int_0^s \sigma_u^2 e^{-2b(s-u)} duds,$$

$$\mu_2(t) = \mu_1(t) + \frac{\omega^2}{4}\int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} dudsd\tau - \frac{\omega}{2}\int_0^t m_\tau \int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}dsd\tau,$$

$$\begin{aligned}\mu_3(t) &= -\frac{\omega}{2}\int_0^t m_\tau \int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}dsd\tau + \frac{\omega^2}{4}\int_0^t m_\tau \int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}\int_s^\tau \rho_u\sigma_u\sqrt{m_u}dudsd\tau \\ &+\frac{\omega^2}{4}\int_0^t m_\tau\Big(\int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}ds\Big)^2 d\tau + \frac{\omega^2}{2}\int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)}dudsd\tau,\end{aligned}$$

$$\begin{aligned}\mu_4(t) &= \frac{\omega^2}{4}\int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)}dudsd\tau + \frac{\omega^2}{4}\int_0^t m_\tau\Big(\int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}ds\Big)^2 d\tau \\ &+\frac{\omega^2}{4}\int_0^t m_\tau \int_0^\tau \rho_s\sigma_s\sqrt{m_s}e^{-b(\tau-s)}\int_s^\tau \rho_u\sigma_u\sqrt{m_u}dudsd\tau\end{aligned}$$

4.2 Multi-factor case : Bergomi's model

In this section we consider a N-dimensional model of the form

$$\begin{cases} \frac{dS_t}{S_t} = rdt + f(t, V_t^1, \ldots, V_t^N)dW_t^S, \\ dV_t^n = (\alpha_n(t) - \kappa_n V_n)dt + \sigma_n(t)dW_t^n., \ d\langle W^S, W^n\rangle_t = \rho_n^S dt, \ n = 1, \ldots, N, \end{cases} \tag{4.5}$$

where $d\langle W^n, W^m\rangle_t = \rho_{n,m}dt = 0$, $\forall m, n \leq N$ and the function f is defined by

$$f^2(t, V_1, \ldots, V_N(t) = m_t \exp\left(\omega \sum_{n=1}^N V_t^n\right) \tag{4.6}$$

Assume for all $n, m \leq N$, $\rho_{n,m} = 0$. Then we have

$$P_X(t, x; \omega) \sim \frac{1}{\sqrt{2\pi\nu(t)}} e^{-\frac{(x+\mu_1)^2}{2\nu(t)}} \left(1 + \sum_{n=2}^6 (-1)^n \frac{\nu_n(t)}{\nu^{\frac{n}{2}}} H_n(\frac{x+\mu_1(t)}{\sqrt{\nu(t)}})\right) \tag{4.7}$$

where the H_n's are the Hermite polynomials.

In particular, if we set $C(t, k; \omega) := \mathbb{E}\left(e^{X_t} - K\right)_+$, we obtain

$$C(t, K; \omega) \sim C_{BS}(S, K, \nu) + (\sum_{n=2}^6 \nu_n)SN(d_1) + \frac{K}{\sqrt{\nu}}N'(d_2)\Big(\sum_{n=0}^4 \frac{z_n}{\nu^{\frac{n}{2}}}H_n(-d_2)\Big), \tag{4.8}$$

and $C_{BS}(S,K,\nu) = SN(d_1) - KN(d_2)$, with $d_1 = \frac{\log(\frac{S}{K})+\frac{1}{2}\nu}{\sqrt{\nu}}$, $d_2 = \frac{\log(\frac{S}{K})-\frac{1}{2}\nu}{\sqrt{\nu}}$ and for $n = 0, \ldots, 4$, $z_n = \sum_{i=n+2}^{6}(-1)^i \nu_i$.

where $\nu := 2\mu_1(T)$ and

$$
\begin{aligned}
\nu_2(T) &= \mu_2(T) - \mu_1(T), \quad \nu_3(T) = \mu_3(T), \\
\nu_4(T) &= \mu_4(T) + \frac{\omega^2}{8}\left(\sum_{n=1}^{N} \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds\right)^2, \\
\nu_5(T) &= \frac{\omega^2}{4}\left(\sum_{n=1}^{N} \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds\right)^2, \\
\nu_6(T) &= \frac{\omega^2}{8}\left(\sum_{n=1}^{N} \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds\right)^2.
\end{aligned}
$$

and the μ's are given by

$$
\begin{aligned}
\mu_1(T) &= \frac{1}{2}\int_0^T m_s ds + \frac{\omega^2}{4}\sum_{n=1}^N \int_0^T m_t \int_0^t \sigma_n^2(s) e^{-2\kappa_n(t-s)} ds + \frac{\omega}{2}\sum_{n=1}^N \int_0^T m_t \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds \\
&\quad + \frac{\omega^2}{4}\int_0^T m_t \left(\sum_{n=1}^N \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds\right)^2 dt, \\
\mu_2(T) &= \mu_1(T) - \frac{\omega}{2}\sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds + \\
&\quad \frac{\omega^2}{4}\sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \alpha_n(s) \int_s^t \sigma_n(r)\sqrt{m_r} dr ds + \\
&\quad \frac{\omega^2}{4}\sum_{n=1}^N \int_0^T m_t \int_0^t m_s \int_0^s \sigma_n^2(r,x) e^{-\kappa_n(t+s-2r)} dr ds \\
&\quad + \frac{\omega^2}{2}\sum_{n,m=1}^N \int_0^T m_t \int_0^t \rho_n^S \sigma_n(s)\sqrt{m_s} \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt \\
\mu_3(T) &= -\frac{\omega}{2}\sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds + \\
&\quad \frac{\omega^2}{4}\sum_{n=1}^N (\rho_n^S)^2 \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} \int_s^t \sigma_n(r)\sqrt{m_r} dr ds \\
&\quad + \frac{\omega^2}{2}\sum_{n=1}^N \int_0^T m_t \int_0^t m_s \int_0^s \sigma_n^2(s) e^{-\kappa_n(t+s-2u)} du ds dt \\
&\quad + \frac{\omega^2}{4}\int_0^T m_t \left(\sum_{n=1}^N \int_0^t \rho_n^S \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds\right)^2 dt \\
&\quad + \frac{\omega^2}{2}\sum_{n,m=1}^N \int_0^T m_t \int_0^t \rho_n^S \sigma_n(s)\sqrt{m_s} \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt, \\
\mu_4(T) &= \frac{\omega^2}{4}\sum_{n=1}^N \int_0^T m_t \int_0^t m_s \int_0^s \sigma_n^2(u) e^{-\kappa_n(t+s-2u)} du ds dt \\
&\quad + \frac{\omega^2}{4}\sum_{n=1}^N (\rho_n^S)^2 \int_0^T m_t \int_0^t \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} \int_s^t \sigma_n(u)\sqrt{m_u} du ds \\
&\quad + \frac{\omega^2}{4}\int_0^T m_t \left(\sum_{n=1}^N \int_0^t \rho_n^S \sigma_n(s)\sqrt{m_s} e^{-\kappa_n(t-s)} ds\right)^2 dt. \qquad (4.9)
\end{aligned}
$$

4.3 Application

We consider the dynamics of the underlying under the model proposed in section 1 :

$$\frac{dS_t}{S_t} = rdt + \sqrt{\xi_t^t} dW_t^S, \tag{4.10}$$

where r is the annualized risk-free interest rate and ξ_t^t is given by (2.1) as

$$\xi_t^t = \xi_0^t \; e^{\omega_t x_t^{T_i} - \frac{\omega_t^2}{2}\mathbb{E}(x_t^{T_i})^2}, \quad t \in]T_i, T_{i+1}]$$

and for $t \leq T$, x_t^T is defined in (2.2) as

$$x_t^T = \sum_n \theta_n \; e^{-\kappa_n(T-t)} \int_0^t e^{-\kappa_n(t-s)} dW_s^n$$

If we assume $d\langle W^n, W^m \rangle_t = 0$, $\forall n \neq m$, we obtain an approximation of the prices of European options on S given by the previous formulas, by taking $\omega = 1$, $m \equiv \xi_0$ and

$$\alpha_l(t) = -\frac{\omega_t^2}{2}\theta_l^2 \; e^{-2\kappa_l(T_{i+1}-t)}, \quad \sigma_l(t) := \theta_l \omega_t e^{-\kappa_l(T_{i+1}-t)}, \; t \in]T_i, T_{i+1}], \quad l = 1, \; \ldots, N. \tag{4.11}$$

Some useful calculations

When $m \equiv m_i \quad \in]T_i, T_{i+1}]$ and $\omega \equiv \omega_i \quad \in]T_i, T_{i+1}]$ we have

$$\begin{aligned}
A_n &:= \int_0^{T_M} m_t \int_0^t \sigma_n^2(s) e^{-2\kappa_n(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} m_i \frac{e^{-2\kappa_n T_i} - e^{-2\kappa_n T_{i+1}}}{2\kappa_n} \int_0^{T_i} \sigma_n^2(s) e^{2\kappa_n s} ds + \sum_{i=0}^{M-1} m_i \int_{T_i}^{T_{i+1}} \int_{T_i}^t \sigma_n^2(s) e^{-2\kappa_n(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} (a_n^{1,i} + a_n^{2,i})
\end{aligned}$$

where

$$\begin{aligned}
a_n^{1,i} &:= m_i \frac{e^{-2\kappa_n T_i} - e^{-2\kappa_n T_{i+1}}}{2\kappa_n} \sum_{j=0}^{i-1} \omega_j^2 \theta_n^2 e^{-2\kappa_n T_{j+1}} \frac{e^{4\kappa_n T_{j+1}} - e^{4\kappa_n T_j}}{4\kappa_n} \\
a_n^{2,i} &= m_i \omega_i^2 \theta_n^2 \frac{\left(1 - e^{-2\kappa_n(T_{i+1}-T_i)}\right)^2}{8\kappa_n^2}
\end{aligned}$$

$$
\begin{aligned}
B_n &:= \int_0^{T_M} m_t \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds \\
&= \sum_{i=0}^{M-1} m_i \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \int_0^{T_i} \alpha_n(s) e^{\kappa_n s} ds + \sum_{i=0}^{M-1} m_i \int_{T_i}^{T_{i+1}} \int_{T_i}^{t} \alpha_n(s) e^{-\kappa_n(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} m_i \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \sum_{j=0}^{i-1} \int_{T_j}^{T_{j+1}} \alpha_n(s) e^{\kappa_n s} ds + \sum_{i=0}^{M-1} m_i \int_{T_i}^{T_{i+1}} \int_{T_i}^{t} \alpha_n(s) e^{-\kappa_n(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} (b_n^{1,i} + b_n^{2,i})
\end{aligned}
$$

where

$$
\begin{aligned}
b_n^{1,i} &:= m_i \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \sum_{j=0}^{i-1} \left(-\frac{\omega_j^2}{2} \theta_n^2 \right) e^{-2\kappa_n T_{j+1}} \frac{e^{3\kappa_n T_{j+1}} - e^{3\kappa_n T_j}}{3\kappa_n} \\
b_n^{2,i} &= m_i \left(-\frac{\omega_i^2}{2} \theta_n^2 \right) \frac{1 - 3e^{-2\kappa_n(T_{i+1}-T_i)} + 2e^{-3\kappa_n(T_{i+1}-T_i)}}{6\kappa_n^2}
\end{aligned}
$$

$$
\begin{aligned}
C_n &:= \int_0^{T_M} m_t \int_0^t \rho_n^S(s) \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds \\
&= \sum_{i=0}^{M-1} m_i \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \int_0^{T_i} \rho_n^S(s) \sigma_n(s) \sqrt{m_s} e^{\kappa_n s} ds + \\
&\quad \sum_{i=0}^{M-1} m_i \int_{T_i}^{T_{i+1}} \int_{T_i}^{t} \rho_n^S(s) \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} (c_n^{1,i} + c_n^{2,i})
\end{aligned}
$$

where

$$
\begin{aligned}
c_n^{1,i} &:= m_i \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \sum_{j=0}^{i-1} \rho_{n,j}^S \sqrt{m_j} \omega_j \theta_n e^{-\kappa_n T_{j+1}} \frac{e^{2\kappa_n T_{j+1}} - e^{2\kappa_n T_j}}{4\kappa_n} \\
c_n^{2,i} &= m_i \sqrt{m_i} \rho_{n,i}^S \omega_i \theta_n \frac{\left(1 - e^{-\kappa_n(T_{i+1}-T_i)}\right)^2}{2\kappa_n^2}
\end{aligned}
$$

$$
\begin{aligned}
D_{n,m} &:= \int_0^{T_M} m_t \int_0^t \rho_n^S \sigma_n(s)\sqrt{m_s} \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_t \int_0^t \rho_n^S \sigma_n(s)\sqrt{m_s} \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds \ dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_i \left(\int_0^{T_i} \cdots + \int_{T_i}^t \ldots \right) \left(\int_0^{T_i} \cdots + \int_{T_i}^t \ldots \right) dt \\
&= \sum_{i=0}^{M-1} \left(D_{n,m}^{1,i} + D_{n,m}^{2,i} + D_{n,m}^{3,i} + D_{n,m}^{4,i} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
D_{n,m}^{1,i} &= b_m^{1,i} \sum_{j=0}^{i-1} \rho_{n,j}^S \sqrt{m_j} \theta_n \omega_j \frac{1-e^{-\kappa_n(T_{j+1}-T_j)}}{\kappa_n} \\
D_{n,m}^{2,i} &= b_m^{2,i} \sum_{j=0}^{i-1} \rho_{n,j}^S \sqrt{m_j} \theta_n \omega_j \frac{1-e^{-\kappa_n(T_{j+1}-T_j)}}{\kappa_n} \\
D_{n,m}^{3,i} &= \rho_{n,i}^S \theta_n \omega_i m_i^{\frac{3}{2}} \left(\frac{e^{-\kappa_m T_{i+1}} - e^{-\kappa_n \Delta_i - \kappa_m T_i}}{\kappa_n(\kappa_n - \kappa_m)} - \frac{e^{-\kappa_n T_{i+1}} - e^{-\kappa_n \Delta_i - \kappa_n T_{i+1}}}{\kappa_n} \right) \times \\
& \sum_{j=0}^{i-1} \left(-\frac{\omega_j^2}{2} \theta_m^2 \right) e^{-2\kappa_m T_{j+1}} \frac{e^{3\kappa_m T_{j+1}} - e^{3\kappa_m T_j}}{3\kappa_m} \\
D_{n,m}^{4,i} &= \frac{\rho_{n,i}^S \theta_n \theta_m^2}{2\kappa_n \kappa_m} \left(-\frac{\omega_i^3}{2}\right) m_i^{\frac{3}{2}} \times \\
& \left(\frac{1-e^{-(\kappa_n+\kappa_m)\Delta_i}}{\kappa_n+\kappa_m} - \frac{e^{-2\kappa_n \Delta_i} - \frac{\kappa_n}{\kappa_m} e^{-(\kappa_n+\kappa_m)\Delta_i}}{\kappa_n - \kappa_m} - \frac{e^{-\kappa_n \Delta_i}}{\kappa_m} - \Delta_i e^{\kappa_m T_i - (\kappa_n+\kappa_m)\Delta_i} \right)
\end{aligned}
$$

$$
\begin{aligned}
E_n &:= \int_0^{T_M} m_t \int_0^t m_s \int_0^s \sigma_n^2(u) e^{-\kappa_n(t+s-2u)} du ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_t \int_0^t m_s \int_0^s \sigma_{n,i}^2(u) e^{-\kappa_n(t+s-2u)} du ds dt
\end{aligned}
$$

Denote by

$$\begin{aligned}
E_{n,i} &:= \int_{T_i}^{T_{i+1}} m_i \int_0^t m_s \int_0^s \sigma_{n,i}^2(u) e^{-\kappa_n(t+s-2u)} du ds dt \\
&= m_i \sum_{j=0}^{i-1} \int_{T_i}^{T_{i+1}} e^{-\kappa_n t} \int_{T_j}^{T_{j+1}} m_j e^{-\kappa_n s} \int_0^s \sigma_{n,i}^2(u) e^{2\kappa_n u} du ds dt \\
&\quad + m_i \int_{T_i}^{T_{i+1}} e^{-\kappa_n t} \int_{T_i}^{t} m_i e^{-\kappa_n s} \int_0^s \sigma_{n,i}^2(u) e^{2\kappa_n u} du ds dt \\
&= m_i \sum_{j=0}^{i-1} \int_{T_i}^{T_{i+1}} e^{-\kappa_n t} \int_{T_j}^{T_{j+1}} m_j e^{-\kappa_n s} \left(\int_0^{T_j} + \int_{T_j}^{s} \right) ds dt \\
&\quad + m_i \int_{T_i}^{T_{i+1}} e^{-\kappa_n t} \int_{T_i}^{t} m_i e^{-\kappa_n s} \left(\int_0^{T_i} + \int_{T_i}^{s} \right) ds dt \\
&= m_i \Big(\sum_{j=0}^{i-1} E_n^{i,j} + \tilde{E}_n^i \Big)
\end{aligned}$$

where

$$\begin{aligned}
E_n^{i,j} &= \int_{T_i}^{T_{i+1}} e^{-\kappa_n t} \int_{T_j}^{T_{j+1}} m_j e^{-\kappa_n s} \left(\int_0^{T_j} + \int_{T_j}^{s} \right) ds dt \\
&= m_j \frac{e^{-\kappa_n T_j} - e^{-\kappa_n T_{j+1}}}{\kappa_n} \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \sum_{l=0}^{j-1} \theta_n^2 \omega_l^2 e^{-2\kappa_n T_{l+1}} \frac{e^{4\kappa_n T_{l+1}} - e^{4\kappa_n T_l}}{4\kappa_n} + \\
&\quad m_j \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \; \theta_n^2 \omega_j^2 \frac{e^{-2\kappa_n T_{j+1}}}{4\kappa_n} \left(\frac{e^{3\kappa_n T_{j+1}} - e^{3\kappa_n T_j}}{3\kappa_n} - e^{4\kappa_n T_j} \frac{e^{-\kappa_n T_j} - e^{-\kappa_n T_{j+1}}}{\kappa_n} \right)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{E}_n^i &= \int_{T_i}^{T_{i+1}} e^{-\kappa_n t} \int_{T_i}^{t} m_i e^{-\kappa_n s} \left(\int_0^{T_i} + \int_{T_i}^{s} \right) ds dt \\
&= m_i \left(\frac{e^{-2\kappa_n T_i} - e^{-2\kappa_n T_{i+1}}}{2\kappa_n} - e^{-\kappa_n T_i} \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \right) \sum_{l=0}^{i-1} \theta_n^2 \omega_l^2 e^{-2\kappa_n T_{l+1}} \frac{e^{4\kappa_n T_{l+1}} - e^{4\kappa_n T_l}}{4\kappa_n} \\
&\quad + m_i \theta_n^2 \omega_i^2 \frac{e^{-2\kappa_n T_{i+1}}}{4\kappa_n} \left(\frac{e^{3\kappa_n T_{i+1}} - e^{3\kappa_n T_i}}{9\kappa_n^2} - \frac{4e^{3\kappa_n T_i}}{3\kappa_n} \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} + \frac{e^{4\kappa_n T_i}}{\kappa_n} \frac{e^{-2\kappa_n T_i} - e^{-2\kappa_n T_{i+1}}}{2\kappa_n} \right)
\end{aligned}$$

$$
\begin{aligned}
E_n &:= \int_0^{T_M} m_t \int_0^t m_s \int_0^s \sigma_n^2(u) e^{-\kappa_n(t+s-2u)} du ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_M} m_t \int_{T_i}^t m_s \int_{T_i}^s \sigma_{n,i}^2(u) e^{-\kappa_n(t+s-2u)} du ds dt
\end{aligned}
$$

where

$$
\sigma_{n,i}(t) := \theta_n(\omega_i e^{-\kappa_n T_{i+1}} - \omega_{i-1} e^{-\kappa_n T_i}) e^{\kappa_n t},\ t \geq 0, \quad n = 1,\ \ldots, N.
$$

Then

$$
E_n \;:=\; \sum_{i=0}^{M-1} \theta_n^2(\omega_i e^{-\kappa_n T_{i+1}} - \omega_{i-1} e^{-\kappa_n T_i})^2 \sum_{j=i}^{M-1} \int_{T_j}^{T_{j+1}} m_j e^{-\kappa_n t} \int_{T_i}^t m_s e^{-\kappa_n s} \int_{T_i}^s e^{4\kappa_n u} du ds dt
$$

Denote by

$$
E_n^{i,j} \;:=\; \int_{T_j}^{T_{j+1}} m_j e^{-\kappa_n t} \int_{T_i}^t m_s e^{-\kappa_n s} \int_{T_i}^s e^{4\kappa_n u} du ds dt
$$

Then

$$
\begin{aligned}
E_n^{i,j} &:= \int_{T_j}^{T_{j+1}} m_j e^{-\kappa_n t} \left(\int_{T_i}^{T_j} m_s e^{-\kappa_n s} \int_{T_i}^s e^{4\kappa_n u} du ds + \int_{T_j}^t m_s e^{-\kappa_n s} \int_{T_i}^s e^{4\kappa_n u} du ds \right) dt \\
&= m_j \sum_{l=i}^{j-1} m_l \left(\frac{e^{3\kappa_n T_{l+1}} - e^{3\kappa_n T_l}}{12\kappa_n^2} - e^{4\kappa_n T_i} \frac{e^{-\kappa_n T_l} - e^{-\kappa_n T_{l+1}}}{4\kappa_n^2} \right) \times \frac{e^{-\kappa_n T_j} - e^{-\kappa_n T_{j+1}}}{\kappa_n} + \\
&\quad m_j^2 \left(\frac{e^{2\kappa_n T_{j+1}} - e^{2\kappa_n T_j}}{24\kappa_n^3} - (e^{3\kappa_n T_j} + 3e^{\kappa_n(4T_i - T_j)}) \frac{e^{-\kappa_n T_j} - e^{-\kappa_n T_{j+1}}}{12\kappa_n^3} + e^{4\kappa_n T_i} \frac{e^{-2\kappa_n T_j} - e^{-2\kappa_n T_{j+1}}}{8\kappa_n^3} \right)
\end{aligned}
$$

$$
\begin{aligned}
F_{n,m} &:= \int_0^{T_M} m_t \int_0^t \rho_n^S \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds \int_0^t \rho_m^S \sigma_m(s) \sqrt{m_s} e^{-\kappa_m(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_t \int_0^t \rho_n^S \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds \int_0^t \rho_m^S \sigma_m(s) \sqrt{m_s} e^{-\kappa_m(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_i \left(\int_0^{T_i} \cdots + \int_{T_i}^t \cdots \right) \left(\int_0^{T_i} \cdots + \int_{T_i}^t \cdots \right) dt \\
&= \sum_{i=0}^{M-1} \left(F_{n,m}^{1,i} + F_{n,m}^{2,i} + F_{n,m}^{3,i} + F_{n,m}^{4,i} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
F^{1,i}_{n,m} &= m_i \frac{e^{-(\kappa_n+\kappa_m)T_i} - e^{-(\kappa_n+\kappa_m)T_{i+1}}}{\kappa_n+\kappa_m}\ f^i_n\ f^i_m \\
F^{2,i}_{n,m} &= \tilde{f}^i_{n,m}\ f^i_m \\
F^{3,i}_{n,m} &= \tilde{f}^i_{m,n}\ f^i_n \\
F^{4,i}_{n,m} &= \frac{m_i^2\omega_j^2\theta_n\theta_m\rho^S_{n,i}\rho^S_{m,i}}{4\kappa_n\kappa_m}\left(\frac{1-e^{-2(\kappa_n+\kappa_m)\Delta_i}}{\kappa_n+\kappa_m} + \frac{e^{-2\kappa_n\Delta_i} - e^{-2\kappa_m\Delta_i}}{\kappa_n-\kappa_m}\right)
\end{aligned}
$$

with

$$
\begin{aligned}
f^i_n &:= \sum_{j=0}^{i-1} \sqrt{m_j}\omega_j\rho^S_{n,j}\theta_n e^{-\kappa_n T_{j+1}} \frac{e^{2\kappa_n T_{j+1}} - e^{2\kappa_n T_j}}{2\kappa_n} \\
\tilde{f}^i_{n,m} &:= m_i\sqrt{m_i}\rho^S_{n,i}\omega_i\theta_n \frac{e^{-\kappa_n T_{i+1}}}{2\kappa_n}\left(\frac{e^{(\kappa_n-\kappa_m)T_{i+1}}}{\kappa_n-k_m} - \frac{2\kappa_n e^{(\kappa_n-\kappa_m)T_i}}{\kappa_n^2-\kappa_m^2} + \frac{e^{2\kappa_n T_i-(\kappa_n+\kappa_m)T_{i+1}}}{\kappa_n+\kappa_m}\right)
\end{aligned}
$$

$$
\begin{aligned}
G_n &:= \int_0^{T_M} m_t \int_0^t \rho^S_n\sigma_n(s)\sqrt{m_s}e^{-\kappa_n(t-s)}ds\left(\int_0^t \rho^S_n\sigma_n(s)\sqrt{m_s}ds - \int_0^s \rho^S_n\sigma_n(s)\sqrt{m_s}ds\right)dt \\
&= \sum_{i=0}^{M-1}\int_{T_i}^{T_{i+1}} m_t \int_0^t \rho^S_n\sigma_n(s)\sqrt{m_s}e^{-\kappa_n(t-s)}ds\left(\int_0^t \sigma_n(s)\sqrt{m_s}ds - \int_0^s \rho^S_n\sigma_n(s)\sqrt{m_s}ds\right)dt \\
&= \sum_{i=0}^{M-1}\int_{T_i}^{T_{i+1}} m_i\left(\int_0^{T_i}\cdots + \int_{T_i}^t \cdots\right)\left(\int_0^{T_i}\cdots + \int_{T_i}^t\cdots\right)dt - \\
&\quad \sum_{i=0}^{M-1}\sum_{j=0}^{i-1}\int_{T_i}^{T_{i+1}} m_i \int_{T_j}^{T_{j+1}} \rho^S_n\sigma_n(s)\sqrt{m_s}e^{-\kappa_n(t-s)}ds\int_0^s \rho^S_n\sigma_n(s)\sqrt{m_s}dsdt - \\
&\quad \sum_{i=0}^{M-1}\int_{T_i}^{T_{i+1}} m_i \int_{T_i}^t \rho^S_n\sigma_n(s)\sqrt{m_s}e^{-\kappa_n(t-s)}ds\int_0^s \rho^S_n\sigma_n(s)\sqrt{m_s}dsdt \\
&= \sum_{i=0}^{M-1}\left(G^{1,i}_n + G^{2,i}_n + G^{3,i}_n + G^{4,i}_n - \tilde{G}^{1,i}_n - \tilde{G}^{2,i}_n\right)
\end{aligned}
$$

where

$$\begin{aligned}
G_n^{1,i} &= m_i \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \; g_{1,n}^i \; g_{2,n}^i \\
G_n^{2,i} &= m_i \; g_{1,n}^i \; g_{3,n}^i \\
G_n^{3,i} &= m_i \; g_{2,n}^i \; g_{4,n}^i \\
G_n^{4,i} &= \frac{m_i^2 \omega_i^2 \theta_n^2 (\rho_{n,i}^S)^2}{2\kappa_n^2} \left(\frac{1 - 2e^{-\kappa_n \Delta_i} + 3e^{-2\kappa_n \Delta_i} - 2e^{-3\kappa_n \Delta_i}}{2\kappa_n} - \Delta_i e^{-2\kappa_n \Delta_i} \right) \\
\tilde{G}_n^{1,i} &= m_i \sum_{j=0}^{i-1} \sqrt{m_j} \rho_{n,j}^S \omega_i \theta_n \; \tilde{g}_{1,n}^{i,j} \\
\tilde{G}_n^{2,i} &= m_i \; g_{4,n}^i \sum_{j=0}^{i-1} g_{2,n}^j + m_i^2 (\rho_{n,i}^S)^2 \theta_n^2 \omega_i^2 \; \frac{\left(1 - e^{-\kappa_n \Delta}\right)^3}{6\kappa_n^3}
\end{aligned}$$

with

$$\begin{aligned}
g_{1,n}^i &:= \sum_{j=0}^{i-1} \sqrt{m_j} \rho_{n,j}^S \omega_j \theta_n e^{-\kappa_n T_{j+1}} \frac{e^{2\kappa_n T_{j+1}} - e^{2\kappa_n T_j}}{2\kappa_n} \\
g_{2,n}^i &:= \sum_{j=0}^{i-1} \sqrt{m_j} \rho_{n,j}^S \omega_j \theta_n e^{-\kappa_n T_{j+1}} \frac{e^{\kappa_n T_{j+1}} - e^{\kappa_n T_j}}{\kappa_n} \\
g_{3,n}^i &:= \sqrt{m_i} \rho_{n,i}^S \omega_i \theta_n \frac{e^{-\kappa_n T_{i+1}}}{\kappa_n} \left(\Delta_i - \frac{1 - e^{-\kappa_n \Delta_i}}{\kappa_n} \right) \\
g_{4,n}^i &:= \sqrt{m_i} \rho_{n,i}^S \omega_i \theta_n \left(\frac{1 - e^{-\kappa_n \Delta_i}}{\kappa_n} \right)^2 \\
\tilde{g}_n^{i,j} &= g_{2,n}^j \; e^{-\kappa_n T_{j+1}} \; \frac{e^{-\kappa_n T_i} - e^{-\kappa_n T_{i+1}}}{\kappa_n} \; \frac{e^{2\kappa_n T_{j+1}} - e^{2\kappa_n T_j}}{2\kappa_n} + \\
& \quad \frac{\rho_{n,i}^S \theta_n \omega_i \sqrt{m_i}}{\kappa_n} \; e^{-\kappa_n T_{i+1}} \; \frac{e^{2\kappa_n T_{i+1}} - e^{2\kappa_n T_i}}{2\kappa_n} \left(\Delta_i - \frac{1 - e^{-\kappa_n \Delta_i}}{\kappa_n} \right)
\end{aligned}$$

$$
\begin{aligned}
H_{n,m} &:= \int_0^{T_M} m_t \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_t \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt \\
&= \sum_{i=0}^{M-1} \int_{T_i}^{T_{i+1}} m_i \left(\int_0^{T_i} \cdots + \int_{T_i}^t \cdots \right) \left(\int_0^{T_i} \cdots + \int_{T_i}^t \cdots \right) dt \\
&= \sum_{i=0}^{M-1} \left(H_{n,m}^{1,i} + H_{n,m}^{2,i} + H_{n,m}^{3,i} + H_{n,m}^{4,i} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
H_{n,m}^{1,i} &= m_i \frac{e^{-2(\kappa_n+\kappa_m)T_i} - e^{-2(\kappa_n+\kappa_m)T_{i+1}}}{2(\kappa_n+\kappa_m)} \; h_n^i \; h_m^i \\
H_{n,m}^{2,i} &= \tilde{h}_{n,m}^i \; h_m^i \\
H_{n,m}^{3,i} &= \tilde{h}_{m,n}^i \; h_n^i \\
H_{n,m}^{4,i} &= \frac{m_i \omega_i^4 \theta_n^2 \theta_m^2}{36 \kappa_n \kappa_m} \Big(\frac{1 - e^{-2(\kappa_n+\kappa_m)\Delta_i}}{2(\kappa_n+\kappa_m)} - \frac{e^{-3\kappa_m \Delta_i} - e^{-2(\kappa_n+\kappa_m)\Delta_i}}{2\kappa_n - \kappa_m} - \frac{e^{-3\kappa_n \Delta_i} - e^{-2(\kappa_n+\kappa_m)\Delta_i}}{2\kappa_m - \kappa_n} \\
&\qquad + \frac{e^{-2(\kappa_n+\kappa_m)\Delta_i} - e^{-3(\kappa_n+\kappa_m)\Delta_i}}{\kappa_m + \kappa_n} \Big)
\end{aligned}
$$

with

$$
\begin{aligned}
h_n^i &:= -\frac{1}{2} \sum_{j=0}^{i-1} \omega_j^2 \theta_n^2 e^{-2\kappa_n T_{j+1}} \frac{e^{3\kappa_n T_{j+1}} - e^{3\kappa_n T_j}}{3\kappa_n} \\
\tilde{h}_{n,m}^i &:= -m_i \frac{\omega_i^2}{2} \theta_n^2 \left(\frac{e^{-\kappa_m T_{i+1}}}{3\kappa_n(2\kappa_n - \kappa_m)} - \frac{e^{-2\kappa_n \Delta_i - \kappa_m T_i}}{(2\kappa_n - \kappa_m)(\kappa_n + \kappa_m)} + \frac{e^{-3\kappa_n \Delta_i - \kappa_m T_{i+1}}}{3\kappa_n(\kappa_n + \kappa_m)} \right)
\end{aligned}
$$

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