

Double exponential jump model (Kou)

DIA El Hadj Aly

Laboratoire Analyse et Mathématiques appliquées, Université de Marne-la-Vallée

March 3, 2020

Premia 22

The double exponential jump model, initiated by Steven KOU (see [1]), is an exponential Levy model, which is a compromise between reality and tractability. It gives an explanation of the two empirical phenomena which received much attention in financial markets : the asymmetric leptokurtic feature and the volatility smile. It permits to obtain analytical solutions to the prices of many derivatives : European call and put options; interest rate derivatives, such as swaptions, caps, floors, and bond options; as well as path-dependant options, such as perpetual American options, barrier, and lookback options .

1 The model

The behaviour of the asset price, S_t , under the risk neutral probability is modeled as followed :

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + d \left(\sum_{i=1}^{N_t} e^{Y_i} - 1 \right) \quad (1.1)$$

Where W is a standard brownian motion, N is a poisson process with rate λ , the constants μ and $\sigma > 0$ are drift and volatility of the diffusion part and the jump sizes $\{Y_1, Y_2, \dots\}$ are i.i.d random variables with a common asymmetric double exponential distribution, of density :

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} \mathbb{1}_{\{y < 0\}} \quad (1.2)$$

where $p, q \geq 0$ are constants, $p + q = 1$, $\eta_1 > 1$ and $\eta_2 > 0$.

The random processes $(W_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$, and random variables $\{Y_1, Y_2, \dots\}$ are independant. Furthermore we have $\mu = r - \lambda\xi$ with:

$$\xi = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1 \quad (1.3)$$

The condition on μ hold in order to obtain $(e^{-rt}S_t)_{t \geq 0}$ is a martingale. The characteristic exponent G of $\log(S_t)$ (i.e. $\mathbb{E}[e^{\theta \log(S_t)}] = e^{G(\theta)t}$) is defined as :

$$G(x) = x \left(r - \frac{1}{2} - \lambda \xi \right) + \frac{1}{2} x^2 \sigma^2 + \lambda \left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1 \right)$$

The equation $G(x) = \alpha$ has exactly four roots (see [2]) : $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$, where

$$0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \infty, \quad 0 < \beta_{3,\alpha} < \beta_{4,\alpha} < \infty. \quad (1.4)$$

2 European call and put

Let us define some special functions (see pp. 1094 and 1099 in[1]):

$$\begin{aligned} Hh_{-1}(x) &= e^{-\frac{x^2}{2}} \\ Hh_0(x) &= \sqrt{2\pi}\Phi(-x) \\ Hh_n(x) &= \int_x^{+\infty} Hh_{n-1}(y)dy = \frac{1}{n!} \int_x^{+\infty} (t-x)^n e^{-\frac{t^2}{2}} dt \quad \forall n \geq 0 \\ I_n(c; \alpha, \beta, \gamma) &= \int_c^{+\infty} e^{\alpha x} Hh_n(\beta c - \gamma) dx \quad \forall n \geq -1 \end{aligned}$$

where Φ is the standard normal cumulative distribution. Then we have :

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x) \quad \forall n \geq 1$$

And $\forall n \geq -1$:

$$\begin{aligned} I_n(c; \alpha, \beta, \gamma) &= -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left(\frac{\beta}{\alpha} \right)^{n-i} Hh_i(\beta c - \delta) + \left(\frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi \left(-\beta c + \delta + \frac{\alpha}{\beta} \right) \quad \beta > 0, \alpha \neq 0 \\ I_n(c; \alpha, \beta, \gamma) &= -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left(\frac{\beta}{\alpha} \right)^{n-i} Hh_i(\beta c - \delta) - \left(\frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi \left(\beta c - \delta - \frac{\alpha}{\beta} \right) \quad \beta < 0, \alpha < 0 \end{aligned}$$

Introduce the following notation : For any given probability P , define :

$$\psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) = \mathbb{P}[Z_T \geq a] \quad (2.5)$$

where $Z_T = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and Y has a double exponential distribution with density as in (1.2), and N is a poisson process with rate λ . Theorem B.1. in [1] gives us :

$$\begin{aligned} \psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) &= \frac{e^{(\sigma\eta_1)^2 \frac{T}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^n P_{n,k} (\sigma\sqrt{T}\eta_1)^k I_{k-1} \left(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_1 \right) \\ &\quad + \frac{e^{(\sigma\eta_2)^2 \frac{T}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^n Q_{n,k} (\sigma\sqrt{T}\eta_2)^k I_{k-1} \left(a - \mu T; \eta_2, \frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_2 \right) \\ &\quad + \pi_0 \text{Phi} \left(-\frac{a - \mu T}{\sigma\sqrt{T}} \right) \end{aligned}$$

where

$$\begin{aligned}
P_{n,i} &:= \sum_{j=i}^{n-1} p^j q^{n-j} \binom{n-i-1}{j-i} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{j-i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-j}, \quad 1 \leq i \leq n-1 \\
Q_{n,i} &:= \sum_{j=i}^{n-1} q^j p^{n-j} \binom{n-i-1}{j-i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{j-i} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-j}, \quad 1 \leq i \leq n-1 \\
P_{n,n} &:= p^n; \quad Q_{n,n} := q^n, \quad \pi_n = \frac{e^{-\lambda T} \lambda^n}{n!}
\end{aligned}$$

Using theorem 2, in [1], we know that the price of european call at inception and with maturity T is :

$$S_0 \psi \left(r + \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \log \left(\frac{K}{S_0} \right), T \right) - K e^{-rT} \psi \left(r - \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, p, \eta_1, \eta_2; \log \left(\frac{K}{S_0} \right), T \right)$$

where

$$\tilde{p} = \frac{p}{1 + \xi} \frac{\eta_1}{\eta_1 - 1}, \quad \tilde{\lambda} = \lambda(1 + \xi), \quad \tilde{\eta}_1 = \eta_1 - 1, \quad \tilde{\eta}_2 = \eta_2 + 1$$

The put price can be obtain by using the call-put parity.

3 Finite time horizon american put option

Let $EuP(v, t)$ be the price of a european put option with initial stock price v and maturity t , $\mathbb{P}^v[S_t \leq K]$ the probability that the stock price at t is below K with initial stock price v , $z = 1 - e^{-rT}$, $\beta_3 \equiv \beta_{3,r/z}$, $\beta_4 \equiv \beta_{5,r/z}$, $C_\beta = \beta_3 \beta_4 (1 + \eta_2)$ (see (1.4)), $D_\beta = \eta_2 (1 + \beta_3) (1 + \beta_4)$, $v_0 \equiv v_0(t) \in (0, K)$ the unique solution to the equation

$$C_\beta K - D_\beta (v_0 + EuP(v_0, t)) = (C_\beta - D_\beta) K e^{-rT} \mathbb{P}^{v_0}[S_t \leq K] \quad (3.6)$$

and

$$\begin{aligned}
A &= \frac{v_0^{\beta_3}}{\beta_4 - \beta_3} \{ \beta_4 K - (1 + \beta_4) [v_0 + EuP(v_0, t)] + K e^{-rT} \mathbb{P}^{v_0}[S_t \leq K] \} > 0, \\
B &= \frac{v_0^{\beta_4}}{\beta_3 - \beta_4} \{ \beta_4 K - (1 + \beta_3) [v_0 + EuP(v_0, t)] + K e^{-rT} \mathbb{P}^{v_0}[S_t \leq K] \} > 0,
\end{aligned}$$

Then the price of a finite-horizon american put option with maturity t and strike K can be approximated by $\psi(S_0, t)$ which is given by (see §3 in [3])

$$\psi(v, t) = \begin{cases} EuP(v, t) + Av^{-\beta_3} + Bv^{-\beta_4}, & \text{if } v \geq v_0 \\ K - v, & \text{if } v \leq v_0 \end{cases}$$

4 Lookback option

The price of a lookback floating strike put option is given by :

$$\begin{aligned} LP(T) &= \mathbb{E} \left[e^{-rT} \left(\max \left\{ M, \max_{0 \leq t \leq T} S_t \right\} - S_T \right) \right] \\ &= \mathbb{E} \left[e^{-rT} \left(\max \left\{ M, \max_{0 \leq t \leq T} S_t \right\} \right) \right] - S_0 \end{aligned}$$

where $M \geq S_0$ is a fixed constant representing the prefixed maximum at time 0. The Laplace transform of the lookback put, using notations in [1.4](#), is given by (see theorem 1 in [\[3\]](#))

$$\int_0^{+\infty} e^{-\alpha T} LP(T) dT = \frac{S_0 A_\alpha}{C_\alpha} \left(\frac{S_0}{M} \right)^{\beta_{1,\alpha+r}-1} + \frac{S_0 B_\alpha}{C_\alpha} \left(\frac{S_0}{M} \right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S_0}{\alpha} \quad \forall \alpha > 0$$

where

$$\begin{aligned} A_\alpha &= \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1} \\ B_\alpha &= \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1} \\ C_\alpha &= (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r}) \end{aligned}$$

The put price is obtained by using an inversion of the Laplace transform. The call option price follows just by symmetry. For the lookback fixed strike, when we have $M \geq \max(S_0, K)$ for the put or $m \leq \min(S_0, K)$ for the call, we get similar results to those for floatings.

5 Barrier option

Since all eight types of barrier can be solved in similar way, we focus only on the price Up and In Call option defined as followed

$$UIC = \mathbb{E} \left[e^{-rT} (S_T - K)^+ \mathbf{1}_{\{\max_{0 \leq t \leq T} S_t \geq H\}} \right] \quad (5.7)$$

where $H > S_0$ is the barrier level. For any given probability P , define ;

$$\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) = \mathbb{P} \left[Z_T \geq a, \max_{0 \leq t \leq T} Z_t \geq b \right] \quad (5.8)$$

where $Z_T = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and Y has a double exponential distribution with density as in [\(1.2\)](#), and N is a poisson process with rate λ . Using formula [\(3.1\)](#) and the result before remark 3.1 in [\[2\]](#), we get

$$\int_0^{+\infty} e^{-\alpha T} \mathbb{P} \left[\max_{0 \leq t \leq T} Z_t \geq b \right] := \frac{1}{\alpha} \left(\frac{(\eta_1 - \beta_{1,\alpha})\beta_{2,\alpha}}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{1,\alpha}} + \frac{(\beta_{2,\alpha} - \eta_1)\beta_{1,\alpha}}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{2,\alpha}} \right)$$

By Inverting the Laplace transform we get $\mathbb{P}[\max_{0 \leq t \leq T} Z_t \geq b]$, which is useful for some types of barrier options. Let us now define some functions

$$H_i(a, b, c; n) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(\frac{1}{2}c^2 - b)t} t^{n+\frac{i}{2}} Hh_i\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) dt \quad i \geq -1, n \geq 0$$

$$\begin{aligned} A_\alpha &:= \mathbb{E}\left[e^{-\alpha\tau_b} \mathbf{1}_{X_{\tau_b}=b}\right] \\ &= \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}} \end{aligned}$$

$$\begin{aligned} B_\alpha &:= \mathbb{E}\left[e^{-\alpha\tau_b} \mathbf{1}_{X_{\tau_b}>b}\right] \\ &= \frac{(\eta_1 - \beta_{1,\alpha})(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} [e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}] \end{aligned}$$

where $\tau_b = \inf\{t \geq 0; X_t \geq b\}$. Hh functions are defined in § 2, and β variables in (1.4). For $i \geq 1$, under assumption that $b > 0$ and $c > -\sqrt{2b}$, we have

$$H_i(a, b, c; n) = \frac{1}{i} H_{i-2}(a, b, c; n+1) - \frac{c}{i} H_{i-1}(a, b, c; n+1) - \frac{a}{i} H_{i-1}(a, b, c; n)$$

By knowing $H_{-1}(a, b, c; n)$ and $H_0(a, b, c; n)$, this recursive formula allows us to determine all values of H_i . Lemmas A.1 and A.2 in [2] give us

$$H_{-1}(a, b, c; n) = e^{-ac - \sqrt{2a^2b}} \sqrt{\frac{1}{2b}} \left(\sqrt{\frac{a^2}{2b}}\right)^n \sum_{j=0}^n \frac{(-n)_j (n+1)_j}{j! (-2\sqrt{2a^2b})^j}, \quad a \neq 0, n \geq 0$$

$$H_{-1}(a, b, c; n) = e^{-ac - \sqrt{2a^2b}} \sqrt{\frac{1}{2b}} \left(\sqrt{\frac{a^2}{2b}}\right)^n \sum_{j=0}^{n-1} \frac{(-n)_j (n+1)_j}{j! (-2\sqrt{2a^2b})^j}, \quad a \neq 0, n \leq -1$$

$$H_{-1}(0, b, c; n) = \frac{(2n)!}{n!(4b)^n} \frac{1}{2b}, \quad n \geq 0$$

$$H_0(a, b, c; n) = \frac{c}{2(n+1)} H_{-1}(a, b, c; n+1) - \frac{a}{2(n+1)} H_{-1}(a, b, c; n), \quad b = \frac{1}{2}c^2, n \geq 0$$

And $\forall n \geq 0$ et $b \neq \frac{1}{2}c^2$

$$H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} \sum_{i=0}^n \frac{(b - \frac{1}{2}c^2)^i}{i!} \left(\frac{a}{2} H_{-1}(a, b, c; i-1) - \frac{c}{2} H_{-1}(a, b, c; i)\right), \quad a > 0$$

$$H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} \left(1 + \sum_{i=0}^n \frac{(b - \frac{1}{2}c^2)^i}{i!} \left(\frac{a}{2} H_{-1}(a, b, c; i-1) - \frac{c}{2} H_{-1}(a, b, c; i)\right)\right), \quad a < 0$$

$$H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2}c^2)^{n+1}} \left(\frac{1}{2} + \sum_{i=0}^n \frac{(b - \frac{1}{2}c^2)^i}{i!} \frac{c}{2} H_{-1}(a, b, c; i)\right), \quad a = 0$$

where $(n)_j = n(n+1)\dots(n+j-1)$, with convention $(n)_0 = 1$.

We can now determine the exact expression of the Laplace transform of Ψ when $b > 0$ and $a \leq b$ (see theorem 4.1 in [2])

$$\begin{aligned}
\int_0^{+\infty} e^{-\alpha T} \mathbb{P} \left[Z_T \geq a, \max_{0 \leq t \leq T} Z_t \geq b \right] dT &= A_\alpha \int_0^{+\infty} e^{-\alpha T} \mathbb{P} [Z_T \geq a-b] dT \\
&\quad + B_\alpha \int_0^{+\infty} e^{-\alpha T} \mathbb{P} [Z_T + \xi^+ \geq a-b] dT \\
&= (A_\alpha + B_\alpha) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_0 \left(-h, \gamma_\alpha, -\frac{\mu}{\sigma}; n \right) \\
&\quad + e^{h\sigma\eta_1} \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{\lambda^n}{n!} (A_\alpha P_{n,j} + B_\alpha \bar{P}_{n,j}) \sum_{i=0}^{j-1} (\sigma\eta_1)^i H_i(h, \gamma_\alpha, c_+; n) \\
&\quad - e^{-h\sigma\eta_2} \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{\lambda^n}{n!} (A_\alpha Q_{n,j} + B_\alpha \bar{Q}_{n,j}) \sum_{i=0}^{j-1} (\sigma\eta_1)^i H_i(-h, \gamma_\alpha, c_-; n) \\
&\quad + e^{h\sigma\eta_1} B_\alpha \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{(\lambda)^n}{n!} (\sigma\eta_1)^i H_i(h, \gamma_\alpha, c_+; n) \\
&\quad + e^{h\sigma\eta_1} B_\alpha H_0(h, \gamma_\alpha, c_+; 0)
\end{aligned}$$

where ξ^+ has an exponential law with rate η_1 , matrix P and Q are as defined in § 2, and

$$\begin{aligned}
\bar{P}_{n,1} &:= \sum_{j=i}^{n-1} Q_{n,i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^i, \quad \bar{P}_{n,i} := P_{n,i-1}, \quad 2 \leq i \leq n+1 \\
\bar{Q}_{n,i} &:= \sum_{j=i}^n \binom{n}{j} q^j p^{n-j} \binom{n-i}{j-i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{j-i} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-j+1}, \quad 1 \leq i \leq n \\
c_+ &:= \sigma\eta_1 + \frac{\mu}{\sigma}, \quad c_- := \sigma\eta_2 - \frac{\mu}{\sigma}, \quad \gamma_\alpha := \alpha + \lambda + \frac{\mu^2}{2\sigma^2}, \quad h := \frac{b-a}{\sigma}
\end{aligned}$$

For to get numerically $\mathbb{P} [Z_T \geq a, \max_{0 \leq t \leq T} Z_t \geq b]$ for a given T , i find that is better to inverse the right term in the first equality above, using some properties of the Laplace inversion. Note that $\mathbb{P} [Z_T \geq a-b]$ is given in § 2 and $\mathbb{P} [Z_T + \xi^+ \geq a-b]$ is given in [2] (pp. 528, formula B.5) :

$$\begin{aligned}
\mathbb{P} [Z_T + \xi^+ \geq a] &= \frac{e^{(\sigma\eta_1)^2 \frac{T}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^{n+1} \bar{P}_{n,k} \left(\sigma\sqrt{T}\eta_1 \right)^k I_{k-1} \left(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_1 \right) \\
&\quad + \frac{e^{(\sigma\eta_2)^2 \frac{T}{2}}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^n \bar{Q}_{n,k} \left(\sigma\sqrt{T}\eta_2 \right)^k I_{k-1} \left(a - \mu T; \eta_2, \frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_2 \right) \\
&\quad + \pi_0 \eta_1 \frac{e^{(\sigma\eta_1)^2 \frac{T}{2}}}{\sqrt{2\pi}} I_0 \left(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\eta_1 \sigma\sqrt{T} \right)
\end{aligned}$$

The price of the UIC option is obtained by, thanks to Kou and Wang (see theorem 2 in [3])

$$\begin{aligned} UIC = & S_0 \Psi \left(r + \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, p, \eta_1, \eta_2; \log \left(\frac{K}{S_0} \right), \log \left(\frac{H}{S_0} \right), T \right) \\ & - K e^{-rT} \Psi \left(r - \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, p, \eta_1, \eta_2; \log \left(\frac{K}{S_0} \right), \log \left(\frac{H}{S_0} \right), T \right) \end{aligned}$$

where

$$\tilde{p} = \frac{p}{1 + \xi} \frac{\eta_1}{\eta_1 - 1}, \quad \tilde{\lambda} = \lambda(1 + \xi), \quad \tilde{\eta}_1 = \eta_1 - 1, \quad \tilde{\eta}_2 = \eta_2 + 1$$

References

- [1] KOU, S. G. (2002). A Jump-Diffusion Model for Option Pricing. Management Science Vol. 48, No. 8, August 2002, pp. 1086-1101. [1](#), [2](#), [3](#)
- [2] KOU, S. G. AND WANG, H. (2003). First Passage Times Of A Jump Diffusion Process. Adv. Appl. Prob. 35, 504-531 (2003). [2](#), [4](#), [5](#), [6](#)
- [3] KOU, S. G. AND WANG, H. (2003). Option Pricing Under a Double Exponential Jump Diffusion Model. Management Science Vol 50, No. 9, September 2004, pp. 1178-1192.

[3](#), [4](#), [7](#)