Quadratic interest rate model

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1 The quadratic interest rate model

1.1. Description of the model

In the quadratic interest rate model, the evolution of the spot interest rate $r(t)$ is described by the following SDE :

$$\begin{cases} dx(t) = (\alpha(t) - \beta\, x(t))\, dt + \sigma\, dW(t), \\ r(t) = \dfrac{1}{2} x(t)^2, \\ x(0) = \sqrt{2r(0)}, \end{cases}$$

where β and σ are constants. α is a time-dependent function determinated by the values of β, σ and the curve of the s-maturity zero-coupon prices at time $t = 0$. Notice that $(x(t),\, t \geq 0)$ is a gaussian process.

If $\mathbb{E}_t$ denote the conditional expectation at time t under the risk-neutral measure, for s-maturity zero-coupon bond at time t, we have :

$$\begin{aligned} P_s(t) &= \mathbb{E}_t \left[\exp\left(- \textstyle\int_t^s r(u) du\right)\right] \\ &= \exp\left(-\left(\frac{1}{2} B_s(t) x(t)^2 + b_s(t) x(t) + c_s(t)\right)\right), \end{aligned} \tag{1}$$

where $B_s(t)$, $b_s(t)$ and $c_s(t)$ are described in Section 2 and computed using equations given in Appendix (see **8.2.1**).

1.2. The T-forward risk adjusted measure

For options on bonds, caplet and call on futures, we will have to use the T-forward risk adjusted measure $\mathbb{E}_t^T$ defined by :

$$\mathbb{E}_t^T[Z(T)] = \mathbb{E}_t[\,\mathrm{e}^{-\int_t^T r(u)du}\, Z(T)]/P_T(t) \tag{2}$$

for all non-negative Itô process Z.

1.3. Notations

We write $Y \sim \Omega(B,\ b,\ c,\ \mu,\ V)$ if $Y = \frac{1}{2}BX^2 + bX + c$, where $X = \mu + VG$ and $G \sim \mathcal{N}(0,1)$ is a centered reduced gaussian. We also have $Y = \alpha + \beta(G + \sqrt{\lambda})^2$ where :

$$\alpha = c - \frac{1}{2}\frac{b^2}{B}, \quad \beta = \frac{1}{2}BV, \quad \lambda = \frac{(\mu + b/B)^2}{V}. \tag{3}$$

Notice that $(G+\sqrt{\lambda})^2$ is distributed as a non-central chi-square with 1 degree of freedom and non-centrality parameter λ. Let $\chi^2(y;\lambda,\beta)$ denote the cumulative distribution of $\beta(G+\sqrt{\lambda})^2$ and $\omega(y;B,b,\mu,V)$ the cumulative distribution of $\frac{1}{2}BX^2 + bX$. Hence, we have

$$\mathbb{P}\,(Y \le y) = \mathbb{P}\left(\beta(G+\sqrt{\lambda})^2 \le y - \alpha\right) = \chi^2(y-\alpha;\lambda,\beta),$$

as well as

$$\mathbb{P}\,(Y \le y) = \mathbb{P}\left(\frac{1}{2}BX^2 + bX \le y - c\right) = \omega(y-c;B,b,\mu,V).$$

In particular :

$$\omega(y;B,b,\mu,V) = \chi^2(y + \frac{1}{2}\frac{b^2}{B};\lambda,\beta).$$

For a function of two variables written as $f_s(t)$, we write $\dot{f}_s(t) \equiv \partial f_s(t)/\partial s$ and $\ddot{f}_s(t) \equiv \partial^2 f_s(t)/\partial s^2$.

We also use the following conventions :

- T denotes the maturity of an option,
- t denotes the maturity of a futures or a forward contract,
- s, or s' denote maturities of zero-coupon bonds,
- K denotes the strike of an option.

All prices are given at initial time $t = 0$. Hence, we have $0 \le T \le t \le s,\ s'$.

2 Calibration and computation of bond coefficients

2.1. Initial values of bond coefficients

To compute the time-dependent function α, we must first compute the forward interest rate at time $t = 0$ from the initial zero-coupon curve $P_s(0)$ as described in the Appendix. Now, for any s and t, using equations given in **8.1.1**, we can compute $B_s(0), \dot{b_s}(0)$, $\dot{c_s}(0)$ and $\alpha(t)$ to fit the initial yield curve. We get $b_s(0)$ and $c_s(0)$ for any s integrating $\dot{b_s}(0)$ and $\dot{c_s}(0)$ with means of trapezoidal rule.

2.2. Transport equations

Transport equations yield formulas for $B_s(t)$, $b_s(t)$ and $c_s(t)$ for any t and s using their initial values. These equations are given in **6.1.2**.

3 Closed formulae for european options on bonds

3.1. European call

$$\textit{Price} : \mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\left(P_s(T) - K\right)_+\right] = P_T(0)\,\mathbb{E}_0^T\left[\left(P_s(T) - K\right)_+\right].$$

Under the T-forward risk adjusted measure, we have $-\log(P_s(T)) \sim \Omega(B,\, b,\, c,\, \mu,\, V)$ where $B = B_s(T)$, $b = b_s(T)$, $c = c_s(T)$ and :

$$\mu = \sqrt{\dot{B}_T(0)}x(0) + \frac{\dot{b_T}(0)}{\sqrt{\dot{B}_T(0)}}, \quad V = \sigma^2 B_T(0). \tag{4}$$

Using (3), we compute the coefficients α, β and λ corresponding to B, b, c, μ and V. Then, the price of a T-maturity call option on the s-bond is given by:

$$\begin{aligned}\mathbb{E}_0&\left[\mathrm{e}^{-\int_0^T r(u)du}\left(P_s(T) - K\right)_+\right]\\ &= P_s(0)\chi^2(-\alpha - \log(K); \frac{\lambda}{1+2\beta}, \frac{\beta}{1+2\beta})\\ &\qquad -KP_T(0)\chi^2(-\alpha - \log(K); \lambda, \beta).\end{aligned}$$

3.2. Caplet

$$\textit{Price}: \mathbb{E}_0\left[e^{-\int_0^T r(u)du}\,(r(T)-K)_+\right] = P_T(0)\,\mathbb{E}_0^T\left[(r(T)-K)_+\right].$$

Under the T-forward risk adjusted measure, we have $r(T) \sim \Omega(B, b, c, \mu, V)$ with $B = 1$, $b = 0$, $c = 0$ and μ and V given by (4). Thanks to (3), we compute the coefficients α, β and λ corresponding to B, b, c, μ and V. Then, the price of a T-maturity caplet is given by:

$$\mathbb{E}_0\left[e^{-\int_0^T r(u)du}\,(r(T)-K)_+\right] = P_T(0)\left[\frac{1}{2}(r_T(0)-K) + C(K-\alpha;\lambda,\beta)\right],$$

where:

$$C(K-\alpha;\lambda,\beta) = \frac{1}{\pi}\int_0^{+\infty}\left[1-\Psi(\lambda, 2\xi^2\beta^2)\cos(\xi(K-\alpha)-\Phi(\lambda,\xi\beta))\right]\frac{d\xi}{\xi^2},$$

with

$$\Psi(\lambda,z) = (1+2z)^{-1/4}\exp(\frac{\lambda z}{1+2z}) \text{ and } \Phi(\lambda,z) = \frac{1}{2}\arctan(2z) + \frac{\lambda z}{1+4z^2}.$$

3.3. Exchange option

$$\textit{Price}: \mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\,(kP_s(T)-k'P_{s'}(T))_+\right] = P_T(0)\mathbb{E}_0^T\left[(kP_s(T)-k'P_{s'}(T))_+\right].$$

Under the T-forward risk adjusted measure : $-\log(P_s(T)) = \frac{1}{2}BX^2 + bX + c$ and $-\log(P_{s'}(T)) = \frac{1}{2}B'X^2 + b'X + c'$ where $X \sim \mathcal{N}(\mu, V)$, $B = B_s(T)$, $b = b_s(T)$, $c = c_s(T)$, $B' = B_{s'}(T)$, $b' = b_{s'}(T)$, $c' = c_{s'}(T)$ and μ, V given by (4). In particular, we have $-\log(P_s(T)) \sim \Omega(B,\, b,\, c,\, \mu,\, V)$ and $-\log(P_{s'}(T)) \sim \Omega(B',\, b',\, c', \mu, V)$. Then, the price of the exchange option to put k' s'-bonds and call k s-bond is :

$$\begin{aligned}&\mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\,(kP_s(T)-k'P_{s'}(T))_+\right]\\ &\quad = kP_s(0)\,\omega(\,c'-c-\log(\frac{k'}{k});\ B-B',\ b-b',\ \frac{\mu-bV}{1+BV},\ \frac{V}{1+BV})\\ &\quad - \ k'P_{s'}(0)\,\omega(\,c'-c-\log(\frac{k'}{k});\ B-B',\ b-b',\ \frac{\mu-b'V}{1+B'V},\ \frac{V}{1+B'V}).\end{aligned}$$

3.4. European call on forward contract

$$\textit{Price}^{1}: \mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\left(P_s(T)-KP_t(T)\right)_+\right] = P_t(0)\,\mathbb{E}_0^t\left[\left(\frac{P_s(T)}{P_t(T)}-K\right)_+\right].$$

Under the t-forward risk adjusted measure, we have $-\log(P_s(T)/P_t(T)) \sim \Omega(B, b, c, \mu, V)$ with $B = B_s(T) - B_t(T)$, $b = b_s(T) - b_t(T)$, $c = c_s(T) - c_t(T)$ and :

$$\mu = \sqrt{\frac{\dot{B}_t(0)}{\dot{B}_t(T)}}x(0) + \frac{\dot{b}_t(0)}{\sqrt{\dot{B}_t(0)\dot{B}_t(T)}} - \frac{\dot{b}_t(T)}{\dot{B}_t(T)}, \text{ and } V = \frac{V_t(0)-V_t(T)}{\dot{B}_t(T)}.$$

Thanks to (3), we compute the coefficients α, β and λ corresponding to B, b, c, μ and V. Then, the price of a call option on the t-delivery forward contract on the s-bond is given by:

$$\begin{aligned}&\mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\left(P_s(T)-KP_t(T)\right)_+\right]\\ &= P_s(0)\chi^2(-\alpha-\log(K);\frac{\lambda}{1+2\beta},\frac{\beta}{1+2\beta}) - KP_t(0)\chi^2(-\alpha-\log(K);\lambda,\beta).\end{aligned}$$

4 Closed formulae for futures and european options on futures

For any $0 \le T \le t \le s$, we set $F_{t,s}(T) \equiv \mathbb{E}_T[P_s(t)]$.

4.1. Futures

$$\textit{Price}: F_{t,s}(0) = \mathbb{E}_0[P_s(t)]$$

Under the risk-neutral measure, $-\log(P_s(t)) \sim \Omega(B, b, c, \mu, V)$ with $B = B_t(s)$, $b = b_t(s)$, $c = c_t(s)$, and

$$\mu = p_t + q_t x(0), \ \ V = v_t \tag{5}$$

Recall that $x(0) = \sqrt{2r_0}$). Subsection **8.2.1** of the Appendix gives expressions to compute $B_{t,s}(0)$, $b_{t,s}(0)$, $c_{t,s}(0)$, p_t and q_t. Then, the price of the t-delivery futures contract on the s-bond is given by :

$$F_{t,s}(0) = \mathbb{E}_0[P_s(t)] = \exp\left(-B_{s,t}(0)x(0)^2 + b_{s,t}(0)x(0) + c_{s,t}(0)\right). \tag{6}$$

[1] Evidently, the option to exchange two bonds as in section **3.4.** is equivalent to an option on a bond forward contract, as in section **3.4**. We thus have two different formulae for this option which agree under the asumption of the model.

4.2. European call option on futures

$$\textit{Price}: \mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\,(F_{t,s}(T)-K)_+\right] = P_t(0)\,\mathbb{E}_0^t\left[(F_{t,s}(T)-K)_+\right]$$

Under the T-forward risk adjusted measure, $-\log(F_{t,s}(T)) \sim \Omega(B,b,c,\mu,V)$ with $B = B_{t,s}(T)$, $b = b_{t,s}(T)$, $c = c_{t,s}(T)$ and μ, V given by (4). The coefficients $B_{t,s}(T)$, $b_{t,s}(T)$, and $c_{t,s}(T)$ are computed from $B_{t,s}(0)$, $b_{t,s}(0)$, and $c_{t,s}(0)$, as described in subsection **6.2.2** of the appendix. Thanks to (3), we compute the coefficient α, β and λ corresponding to B, b, c, μ and V. Then, the price of a T-maturity european call option on the t-delivery futures on s-bond is :

$$\begin{aligned}
&\mathbb{E}_0\left[\mathrm{e}^{-\int_0^T r(u)du}\,(F_{t,s}(T)-K)_+\right] \\
&\qquad = P_T(0)\,\mathbb{E}_0^T\left[F_{t,s}(T)\right]\chi^2(-\alpha-\log(K);\frac{\lambda}{1+2\beta},\frac{\beta}{1+2\beta}) \\
&\qquad\qquad -KP_T(0)\chi^2(-\alpha-\log(K);\lambda,\beta),
\end{aligned}$$

with $\mathbb{E}_0^T\left[F_{t,s}(0)\right] = \mathrm{e}^{-Fx^2(0)-Gx(0)-H}$. Formula for F, G and H are given in section **8.2.3**.

4.3. Delivery option

$$\textit{Price}: \mathbb{E}_0\left[\min(kP_s(T), k'P_{s'}(T))\right]$$

Under the risk-neutral measure, we have $-\log(P_s(t)) \sim \Omega(B,\, b,\, c,\, \mu,\, V)$ and $-\log(P_{s'}(t)) \sim \Omega(B',\, b',\, c',\, \mu,\, V)$ with $B = B_s(t)$, $b = b_s(t)$, $c = c_s(t)$, $B' = B_{s'}(t)$, $b' = b_{s'}(t)$, $c' = c_{s'}(t)$, and μ, V given by (4). Then, the price of the T-maturity futures contract to deliver the cheapest between k s-bond or k' s'-bond is :

$$\begin{aligned}
&\mathbb{E}_0\left[\min(kP_s(T), k'P_{s'}(T))\right] \\
&\quad = \; kF_{t,s}(0)\,\omega(\,c-c'+\log(\frac{k'}{k});\; B'-B,\; b'-b,\; \frac{\mu-bV}{1+BV},\; \frac{V}{1+BV}) \\
&\quad + \; k'F_{t,s'}(0)\,\omega(\,c'-c-\log(\frac{k'}{k});\; B-B',\; b-b',\; \frac{\mu-b'V}{1+B'V},\; \frac{V}{1+B'V}).
\end{aligned}$$

Notice that $F_{s,t}(0)$ and $F_{s',t}(0)$ are given by (6).

5 Monte Carlo methods for european options on futures and bonds

For each option, we know that $(x(t), t \geq 0)$ is a gaussian process both under the risk-neutral measure and under the T-forward risk adjusted measure and we have expression for its mean and variance. Therefore, to compute Monte Carlo methods, we simulate the variable $x(t)$ and we simply use the relationships between $x(t)$ and the price of zero-coupon bond, the spot interest rate or the price of a futures.

For options on bonds, caplet, and call on futures, we use the distribution of x under the T-forward risk adjusted measure. Indeed, we have, $x \sim \mathcal{N}(\mu, V)$ with μ and V given by (4) and for any pay-off $X(T)$ at maturity T

$$\mathbb{E}_0\left[e^{-\int_0^T r(s)ds} X(T)\right] = P_T(0)\mathbb{E}_0^T[X(T)].$$

Hence, with a Monte Carlo method, we get the desired price.

For futures and for delivery options (section **4.1** and **4.3**), we directly get the desired price by a Monte Carlo method using the distribution of x in the risk-neutral measure : $x \sim \mathcal{N}(\mu, V)$ is again a gaussian process with μ, and V given by (5) for futures and by (4) for delivery options.

6 Algorithms

The functions below are common to all programs :

- `void bond_coeffs(ZCMarketData* ZCMarket, Data *data, double T, double beta, double sigma, double x0);`
  This function computes the coefficients $B_T(0)$, $b_T(0)$, $c_T(0)$, $\dot{B_T}(0)$, $\dot{b_T}(0)$ and stores them in structure `data` .

  Integrations are done with means of trapezoidal rule.

- `void transport(Omega *om, Data data1, Data data2, double alpha, double beta, double sigma, double x0)`
  This function computes the coefficients of $P_s(T)$ knowing those of $P_T(0)$ (contained in data1) and $P_s(0)$ (contained in data2).

  Results are stored in `om.B`, `om.b` and `om.c`.

- `void om2chn(Omega om, Chn *chn)`
  Transforms an `Omega` structure into a `Chn` strucure using equations (3).

7 Results and conclusions

To check the accuracy of the computed prices, we did the following tests.

- First, we checked the put prices computed with closed forms agree with the prices given by Pelsser in [2]. Computed prices are exactly the same.

- Then, to check the efficiency of the quadratic interpolation, we have taken a function for $P_s(0)$ ($P_s(0) = \exp\left(-s\left(0.08 - 0.05\mathrm{e}^{-.18s}\right)\right)$) and we have discretized it successively with a time-step of 0.05 and a time-step of 0.25. Prices for each kind of options are nearly the same : the error is always lower than 1 basis point.

- We also checked the prices using Monte Calo method. We launched the program 1000 times and around 95% of computed 95-per-cent-confidence intervals contain the closed form price.

- We also passed the following test : for α given and constant, we have computed prices of zero-coupon bond for several maturity. These prices were stored in the file `initialyield.dat`. Then, we launched the program for α time-dependent and we checked if computed prices with these values for bond were the same than for α constant. The prices are always the same with an error lower than one basis point.

8 Appendix

In all the following equations, we set $\gamma = \sqrt{\beta^2 + \sigma^2}$.

6.1. Bond coefficients

- **6.1.1. Equations to compute initial values of bond coefficients**

For all s and t :

$$B_s(0) = \frac{e^{2\gamma s} - 1}{(\gamma + \beta)e^{2\gamma s} + \gamma - \beta}$$

If α is constant, we have closed forms for $b_s(0)$ and $\dot{c}_s(0)$[2] :

$$h(s) = \left((\gamma+\beta)e^{2\gamma s} + \gamma - \beta\right)^{-1},$$

$$b_s(0) = \frac{\alpha}{\gamma} h(s)(\mathrm{e}^{\gamma s} - 1)^2,$$

$$\dot{c}_s(0) = \alpha b_s(0) + \frac{1}{2}\sigma^2 B_s(0) - \frac{1}{2}\sigma^2 b_s^2(0).$$

Else, for all s, the forward interest rate at time $t = 0$ is : $r_s(0) = -\dfrac{\partial \log(P_s(0))}{\partial s}$.
Then, we have :

$$\dot{b}_s(0) = \dot{B}_s(0)x(0) + \sqrt{\dot{B}_s(0)(2r_s(0) - \tfrac{1}{2}\sigma^2 B_s(0))}$$

$$\dot{c}_s(0) = \frac{1}{2}\left(\frac{(\dot{b}_s(0))^2}{\dot{B}_s(0)} + \sigma^2 B_s(0)\right)$$

$$\alpha(t) = (\dot{B}_t(0))^{-3/2}(\dot{B}_t(0)\ddot{b}_t(0) - \ddot{B}_t(0)\dot{b}_t(0)).$$

- **6.1.2. Transport equations for bond coefficients**

For all T, and s, we have :

$$B_s(T) = \frac{B_s(0) - B_T(0)}{\dot{B}_T(0) - \sigma^2 B_T(0)(B_s(0) - B_T(0))},$$

$$\dot{B}_s(T) = \frac{\dot{B}_s(0)\dot{B}_T(0)B_s^2(T)}{(B_s(0) - B_T(0))^2},$$

$$b_s(T) = B_s(T)\sqrt{\dot{B}_s(T)}\left(\frac{b_s(0) - b_T(0)}{B_s(0) - B_T(0)} - \frac{\dot{b_T}(0)}{\dot{B_T}(0)}\right),$$

$$\dot{b}_s(T) = \frac{\dot{b}_s}{\sqrt{\dot{B}_T(0)}}\left(1 + \sigma^2 B_T(0)B_s(T)\right) + \dot{B}_s(T)\left(\sigma^2 B_T(0)(b_s(0) - b_T(0)\right) - \dot{b_T}(0)\Big),$$

$$c_s(T) = c_s(0) - c_T(0) - \tilde{c}(B_s(T),\, b_s(T),\, \dot{b_T}(0)/\sqrt{\dot{B}_T(0)},\, \sigma^2 B_T(0)).$$

with $\tilde{c}(B,\, b,\, a,\, V) = \dfrac{1}{2}\left(\log(1 + BV) + \dfrac{Ba^2 + 2ab - Vb^2}{1 + BV}\right)$.

6.2. Futures coefficients

[2] There is a misprint in the formula given by Jamshidian in [1] for $\dot{c}_s(0)$ which is corrected here.

- **6.2.1. Equations to compute initial values of futures coefficients**

For all s and t, we set:

$$p_t = \int_0^t \alpha(u)e^{-\beta(t-u)}du, \quad q_t = e^{-\beta t}, \quad v_t = \frac{\sigma^2(1-e^{-2\beta t})}{2\beta}.$$

Then, for all s and t, we have :

$$B_{t,s}(0) = \frac{q_t^2 B_s(t)}{1+B_s(t)v_t},$$
$$b_{t,s}(0) = \frac{q_t(b_s(t)+B_s(t)p_t)}{1+B_s(t)v_t},$$
$$c_{t,s}(0) = c_s(t) + \frac{1}{2}\log(1+v_tB_s(t)) + \frac{B_s(t)p_t^2 + 2b_s(t)p_t - v_tp_t^2}{2(1+v_tB_s(t))}.$$

- **6.2.2. Transport equations for futures coefficients**

For all s, t and T, we set :

$$B_{t,s}(T) = \frac{B_{t,s}(0)}{q_T^2 - v_TB_{t,s}(0)},$$
$$b_{t,s}(T) = B_{t,s}(T)\left(\frac{b_{t,s}(0)}{B_{t,s}(0)}q_T - p_T\right),$$
$$c_{t,s}(T) = c_{t,s}(0) - \frac{1}{2}\log(1+v_TB_{t,s}(T)) + \frac{B_{t,s}(T)p_T^2 + 2b_{t,s}(T)p_T - v_Tq_T^2}{2(1+v_TB_{t,s}(T))}.$$

For all t, s and T, we set :

- **6.2.3. Other formulas for futures**

For all s, t and T , we set :

$$p_t = \sqrt{\dot{B_T}(0)} \text{ and } q_t = \frac{\dot{b_T}(0)}{\sqrt{\dot{B_T}(0)}}.$$

Then, we have[3] : $\mathbb{E}_0^T\left[F_{t,s}(0)\right] = \mathrm{e}^{-Fx^2(0)-Gx(0)-H}$ with :

$$F = \frac{q_T^2B_{t,s}(T)}{1+B_{t,s}(T)v_T},$$
$$G = \frac{q_T(b_{t,s}(T)+B_{t,s}(T)p_T)}{1+B_{t,s}(T)v_T},$$
$$H = c_{t,s}(T) + \frac{1}{2}\log(1+v_tB_{t,s}(T)) + \frac{B_{t,s}(T)p_T^2 + 2b_{t,s}(T)p_T - v_Tp_T^2}{2(1+v_tB_{t,s}(T))}.$$

[3] We recall that $x(0) = \sqrt{2r(0)}$.

References

[1] Farshid Jamshidian, *Bond, futures and option evaluation in the quadratic interest rate model. Applied Mathematical Finance, 1996.* 9

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[3] William H. Press, Brian P. Flannery, Saul A. Teukolsky, William T. Vetterling. , *Numerical Recipes in C. The Art of Scientific Computing, Cambridge University Press, 1988.*

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