Some methods of Bidimensional Spread

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1.1 Model

The term "spread" is often used for the difference between two indices. Hence, the spread option refers to an option written as the difference between two indices and a fixed price, or, more generally, as a linear combination of a finite set of indices plus a certain constant ($\pm$ strike).

Spread options are traded in commodity markets, here is an example given in [3]. The soybean crush spread is traded on the Chicago Board of Trade (CBOT). The three underlying indices are the future contracts of soybean, soybean meal, and soybean oil. The spread option is based on the gross processing margin: the raw product is the soybean, and the manufactured products are soybean meal and soybean oil. As crushing a bushel of soybeans procuces 0.022 short tons of soybean meal and 11 lbs soybean oil in average, the GPM can be calculated as follows:

GPM = 0.022 * [meal($/short ton)] + 11 * [oil($/lb)] - [soybeans($/bushel)]

This crush spread is used by crushers to hedge cash positions: crushing is profitable only if the margin is greater than a constant which is the cost of the process, for instance.

In this section, we assume that the spread option has two underlying assets whose prices at time t are denoted by $S_1(t)$ and $S_2(t)$. $S_i(t)$, for $i = 1, 2$, follows geometric Brownian motion dynamic under the risk-neutral probability:

$$\begin{aligned} dS_1(t) &= S_1(t)[(r - q_1)dt + \sigma_1 dW_1(t)] \\ dS_2(t) &= S_2(t)[(r - q_2)dt + \sigma_2 dW_2(t)] \end{aligned} \tag{1}$$

where $W_1(t)$ and $W_2(t)$ are two Brownian motions correlated by $\rho \in [-1, 1]$. r is the risk-free interest rate and q_i the dividend or convenience yields. We denote the current forward price for delivery at maturity date $F_i := e^{(r-q_i)T} S_i(0)$.
And then, we have the following expression for $S_i(T)$:

$$S_i(T) = S_i(0)e^{(r-q_i-\sigma_i^2/2)T+\sigma_i W_i(T)} = F_i e^{-\sigma_i^2 T/2+\sigma_i W_i(T)} \tag{2}$$

The European type spread option grants the holder the right to pay the fixed price K instead of the difference $(S_1(t) - S_2(t))$ at the maturity date T. The spread option's pay-off at maturity is therefore:

$$(S_1(T) - S_2(T) - K)^+ \tag{3}$$

where $x^+ = \max(x, 0)$ denotes the positive part of x.
Then the price C of the Call-like spread option at the initial time 0 of this spread option is given under the risk-neutral probability:

$$C := e^{-rT}\mathbb{E}[(S_1(T) - S_2(T) - K)^+] \tag{4}$$

This quantity is what we want to compute using the different methods below.

Before we start, let us state the following identity which is the Call-Put parity. This identity is independent of the model chosen:

$$C + P = S_1(0)e^{-q_1T} - S_2(0)e^{-q_2T} - Ke^{-rT} \tag{5}$$

where C is the price of the spread option defined in (4), and P is equivalent of a Put option, it is the spread option with strike $-K$ and in which the roles of the assets 1 and 2 are reversed. For instance, the soy manufacturer would hedge his position using Put-like spread option in order to ensure he is paid more than K.

1.2 Exchange option - Margrabe Formula

In the case K is null, the pay-off of the spread option is $(S_1(T) - S_2(T))^+$. The holder is given the right to exchange the assets S_2 he possesses for the asset S_1 at T, that is why this particular case of spread option is named exchange option. Margrabe (1978) gave a method to compute explicitly the price of an exchange option in [9].

Proposition 1. *The price of an exchange option is given by:*

$$C = S_1(0)e^{-q_1T}N(d_1) - S_2(0)e^{-q_2T}N(d_2) \tag{6}$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_1(0)}{S_2(0)}\right) - (q_2 - q_1)}{\sigma\sqrt{T}} + \tfrac{1}{2}\sigma\sqrt{T} \\ d_2 &= \frac{\log\left(\frac{S_1(0)}{S_2(0)}\right) - (q_2 - q_1)}{\sigma\sqrt{T}} - \tfrac{1}{2}\sigma\sqrt{T} \\ \sigma &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \end{aligned} \tag{7}$$

and $N(\cdot)$ denotes the normal cumulative distribution function.

Proof. Margrabe used the change of numéraire technique, he expressed the price of the asset S_1 using the asset S_2 as numéraire. For the sake of simplicity, we suppose that the convenience yields q_1 and q_2 are equal to zero, the argument would be the same by considering the prices discounted by the corresponding factors $e^{q_i t}$.

Let $Z_1(t)$ and $Z_2(t)$ be two independent standard Brownian motions under the risk-neutral probability $\mathbb{P}$, S_1 and S_2 could be represented as

$$\begin{array}{l}(dS_1/S_1)(t) = rdt + \sigma_1(\rho dZ_1(t) + \rho' dZ_2(t)) \\ (dS_2/S_2)(t) = rdt + \sigma_2 dZ_2(t)\end{array} \tag{8}$$

or equivalently

$$\begin{array}{l}S_1(t) = S_1(0) * \exp[(r - \sigma_1^2/2)t + \sigma_1(\rho Z_1(t) + \rho' Z_2(t))] \\ S_2(t) = S_2(0) * \exp[(r - \sigma_2^2/2)t + \sigma_2 Z_1(t)]\end{array} \tag{9}$$

where $\rho \in]-1;+1[$ is the correlation factor and $\rho^2 + \rho'^2 = 1$ (if $|\rho| = 1$, the computation can be handled directly because we have a 1D problem).

We would like to express S_1 using S_2 as numéraire, so let us study their quotient's dynamic:

$$(S_1/S_2)(t) = (S_1/S_2)(0) \exp\left[\frac{\sigma_2^2 - \sigma_1^2}{2}t + (\sigma_1\rho - \sigma_2)Z_1(t) + \sigma_1\rho' Z_2(t)\right] \tag{10}$$

Itō's lemma provides:

$$\frac{d(S_1/S_2)}{(S_1/S_2)}(t) = (\rho\sigma_1 - \sigma_2)dZ_1(t) + \rho'\sigma_1 dZ_2(t) + \sigma_2(\sigma_2 - \rho\sigma_1)dt \tag{11}$$

If we denote by $\mathcal{F}_T$ the completion of the σ-algebra generated by the processes $\{Z_1(t), Z_2(t)\}_{0\leq t\leq T}$, we create a new probability measure $\mathbb{Q}$ whose Radon-Nikodým derivative with respect to $\mathbb{P}$ is given by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left[-\frac{1}{2}\sigma_2^2 T + \sigma_2 Z_1(T)\right] = \left(e^{-rT}\frac{S_2(T)}{S_2(0)}\right)^{-1} \tag{12}$$

Under this new probability $\mathbb{Q}$ and by Girsanov theorem, $Z_1'(t) := Z_1(t) - \sigma_2 t$ and Z_2 are Brownian motions. We have:

$$\frac{d(S_1/S_2)}{(S_1/S_2)}(t) = (\rho\sigma_1 - \sigma_2)dZ_1'(t) + \rho'\sigma_1 dZ_2(t) \tag{13}$$

In other words, the quotient S_1/S_2 is a geometric Brownian motion under $\mathbb{Q}$ with no drift, and its volatility is σ given in (7).
The price of the spread option is thus:

$$\begin{array}{rl}C & = e^{-rT}\mathbb{E}_{\mathbb{P}}(S_1(T) - S_2(T))^+] \\ & = e^{-rT}\mathbb{E}_{\mathbb{P}}\{[(S_1/S_2)(T) - 1]^+ \cdot S_2(T)\} \\ & = S_2(0)\mathbb{E}_{\mathbb{Q}}\{[(S_1/S_2)(T) - 1]^+\}\end{array} \tag{14}$$

The expectation is exactly the price of a European call option with no interest rate, strike 1 and volatility σ. The result is given by the standard Black-Scholes formula.

□

Computation of the Deltas

Recall that the price of the exchange option is $C = e^{-rT}\mathbb{E}[(S_1(T) - S_2(T))^+]$ which is a homogeneous function of degree 1 of variables $S_1(0)$ and $S_2(0)$, Euler theorem shows that C satisfies:

$$P = \frac{\partial C}{\partial S_1(0)} S_1(0) + \frac{\partial C}{\partial S_2(0)} S_2(0) \tag{15}$$

The comparison between (6) and (15) suggests that the factors $\pm e^{-q_1 T} N(d_i)$ are exactly the corresponding Deltas. The following proposition shows we were right by thinking that.

Proposition 2. *The Deltas of the exchange option is given by:*

$$\begin{aligned} \Delta_1 &= +e^{-q_1 T} N(d_1) \\ \Delta_2 &= -e^{-q_2 T} N(d_2) \end{aligned} \tag{16}$$

Proof. We compute the Deltas explicitly, but we do not give all the details.

To compute Δ_1, we note first that

$$\frac{\partial d_1}{\partial S_1(0)} = \frac{\partial d_2}{\partial S_1(0)} = (S_1(0)\sigma\sqrt{T})^{-1} \tag{17}$$

The chain rule gives

$$\frac{\partial C}{\partial S_1(0)} = e^{-q_1 T} N(d_1) + \frac{\left[S_1(0) e^{-q_1 T} N'(d_1) - S_2(0) e^{-q_2 T} N'(d_2)\right]}{S_1(0)\sigma\sqrt{T}} \tag{18}$$

where N' denotes the probability density of a standard normal distribution. Finally, we can replace d_2 by $(d_1 - \sigma\sqrt{T})$ in N' and expanding the identity $N'(a+b) = N'(a)\exp(-b^2/2 + ab)$, we show that all the terms between the square brackets vanish.

The computation of Δ_2 is similar.

□

The interest of the Margrabe formula is that it extends the Black-Scholes formula to 2D (the Black-Scholes formula is obtained by giving to σ_2 the value 0). Unfortunately, we cannot use this same artful technique once K is not equal to 0. So far, there is no closed formula of the spread option's price. In the next part, we present three approximation formulae.

1.3 Carmona-Durrleman

The Carmona-Durrleman procedure uses the following elementary but essential result.

Lemma 1. *Let X be a real valued random variable on a probability space $(\Omega, \mathcal{H}, P)$, and let $\mathcal{A} \subset \mathcal{H}$. Then,*

$$\mathbb{E}[X^+] \geq \sup_{A \in \mathcal{A}} \mathbb{E}[X \mathbf{1}_A]$$

Moreover, the inequality becomes an equality iff $\{X \geq 0\} \in \mathcal{A}$.

Proof. For any $A \subset \mathcal{H}$,

$$\begin{aligned} \mathbb{E}[X\mathbf{1}_A] = \int_A X dP &= \int_{A \cap \{X \geq 0\}} X dP + \int_{A \cap \{X < 0\}} X dP \\ &\leq \int_{A \cap \{X \geq 0\}} X dP \\ &\leq \int_{\{X \geq 0\}} X dP = \mathbb{E}[X^+] \end{aligned}$$

The equality is obtained by taking $A = \{X \geq 0\}$. □

This lemma is used as follows:
The main idea of [2] is to find a class of strategies $\mathcal{A}$ for which the expectation $\mathbb{E}[X\mathbf{1}_A]$ has an explicit formula for each $A \in \mathcal{A}$, then maximize this lower bound with respect to A in order to approximate the real price of the spread option.

Recall that we are interested in computing the expectation of (3), which has the same distribution as X^+ with:

$$X := \alpha e^{\beta[\sin\phi Z_1 + \cos\phi Z_2] - \beta^2/2} - \gamma e^{\delta Z_2 - \delta^2/2} - \kappa \tag{19}$$

and

$$\begin{aligned} \alpha &= S_1(0)e^{-q_1 T} \\ \beta &= \sigma_1\sqrt{T} \\ \gamma &= S_2(0)e^{-q_2 T} \\ \delta &= \sigma_2\sqrt{T} \\ \kappa &= Ke^{-rT} \end{aligned} \tag{20}$$

where Z_1 and Z_2 are two independent $N(0,1)$ random variables. Naturally, in such representation, we use the phase angle ϕ that satisfy $\cos\phi = \rho$.

Given any other phase angle $\theta \in \mathbb{R}$, we introduce the random variable

$$Y_\theta = \sin\theta Z_1 - \cos\theta Z_2 \tag{21}$$

and we choose the set $\mathcal{A} = (A_{\theta,d})_{(\theta,d)\in\mathbb{R}^2}$ where $A_{\theta,d} = (\{Y_\theta \leq d\})_{(\theta,d)\in\mathbb{R}^2}$
The expectation $\mathbb{E}[X\mathbf{1}_{A_{\theta,d}}]$ could be computed explicitly:

$$\mathbb{E}[X\mathbf{1}_{A_{\theta,d}}] = \alpha N(d + \beta\cos(\theta+\phi)) - \gamma N(d + \delta\cos(\theta)) - \gamma N(d) \tag{22}$$

For this choice of $\mathcal{A}$, the event $\{X \geq 0\}$ is not necessarily an element of $\mathcal{A}$, we are only computing a lowerbound.

By denoting $\Pi(\theta, d)$ the expectation above, we will use the following pricing formula which could be computed by maximizing a function of two variables:

Pricing Formula.

$$\begin{aligned} C \approx \Pi^* : &= \sup_{(\theta,d)\in\mathbb{R}^2} \Pi(\theta, d) \\ &= \sup_{(\theta,d)\in\mathbb{R}^2} \alpha N(d + \beta\cos(\theta+\phi)) - \gamma N(d + \delta\cos(\theta)) - \gamma N(d) \quad (23) \\ &= \hat{\Pi}(S_1(0), S_2(0), \theta^*, d^*) \end{aligned}$$

where $\hat{\Pi}$ *is defined by*

$$\begin{aligned} \hat{\Pi}(x_1, x_2, \theta, d) = \quad & +x_1 e^{-q_1 T} N(d + \sigma_1\cos(\theta+\phi)\sqrt{T}) \\ & -x_2 e^{-q_2 T} N(d + \sigma_2\cos(\theta)\sqrt{T}) \\ & -K e^{-rT} N(d) \end{aligned} \quad (24)$$

and θ^* *and* d^* *denote the maximizing argument in the third line.*

The numerical results show that the lower bound we use is very accurate. The explanation is probably that the approximation formula turns out to be exact formula in many special cases, as shown in the following proposition, whose proof is not given here.

Proposition 3. $\Pi^* = \mathbb{E}[X^+]$ *whenever* $\alpha = 0$, *or* $\gamma = 0$, *or* $\kappa = 0$, *or* $\rho = -1$, *or*
$\rho = +1$ *and* $\kappa(\beta - \delta) > 0$
or
$\rho = +1$, *and* $\kappa(\beta - \delta) < 0$, *and* $(\frac{\gamma\delta}{\alpha\beta})^{\frac{\delta}{\beta-\delta}} \leq \frac{\kappa\beta}{\gamma(\delta-\beta)} e^{-\delta(\beta+\delta)/2}$

Computation of the Deltas

Proposition 4. *We compute the deltas with the following formulae which give a sub-hedge for the option:*

$$\begin{aligned} \Delta_1 &= +e^{-q_1 T} N(d^* + \sigma_1\cos(\theta^* + \phi)\sqrt{T}) \\ \Delta_2 &= -e^{-q_2 T} N(d^* + \sigma_2\cos(\theta^*)\sqrt{T}) \end{aligned} \quad (25)$$

Proof. We do not give the entire proof of the proposition above, but we will state a computation rule to compute all the Greeks.

For instance, we want to compute $\Delta_1 = \dfrac{\partial \Pi^*}{\partial(S_1(0))}$, remembering that the price function Π^* depends on $S_1(0)$ through $\hat{\Pi}$, θ^*, and d^*. We have:

$$\frac{\partial \Pi^*}{\partial(S_1(0))} = \left.\frac{\partial \hat{\Pi}}{\partial x_1}\right|_{S(0),\theta^*,d^*} + \left.\frac{\partial \hat{\Pi}}{\partial \theta}\right|_{S(0),\theta^*,d^*} \frac{\partial \theta^*}{\partial(S_1(0))} + \left.\frac{\partial \hat{\Pi}}{\partial d}\right|_{S(0),\theta^*,d^*} \frac{\partial d^*}{\partial(S_1(0))} \quad (26)$$

As θ^* and d^* maximize Π^*, the corresponding differentials vanish both, so we finally obtain:

$$\frac{\partial \Pi^*}{\partial (S_1(0))} = \frac{\partial \hat{\Pi}}{\partial x_1}(S_1(0), S_2(0), \theta^*, d^*) \tag{27}$$

which is simple to compute, its expression is given in (25).

The previous remark is also used to compute Δ_2 and all the other Greeks. $\square$

1.4 Bjerksund-Stensland

Bjernstund and Stensland (2006) used the following expectation which constitutes a lower bound to the true value (see the lemma 1.):

$$e^{-rT}\mathbb{E}\left[(S_1(t) - S_2(t) - K)\cdot \mathbf{1}\left\{S_1(t) \geq \frac{a\cdot(S_2(t))^b}{\mathbb{E}[(S_2(t))^b]}\right\}\right] \tag{28}$$

where

$$\begin{array}{l} a = Y_0 = F_2 + K \\ b = F_2/a \end{array} \tag{29}$$

Which leads the following approximation formula.

Pricing Formula.

$$C \approx e^{-rT}\{F_1 N(d_1) - F_2 N(d_2) - K N(d_3)\} \tag{30}$$

where

$$\begin{array}{l} d_1 = \dfrac{1}{\sigma\sqrt{T}}\left(\log(F_1/a) + \dfrac{T}{2}(\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2)\right) \\ d_2 = \dfrac{1}{\sigma\sqrt{T}}\left(\log(F_1/a) + \dfrac{T}{2}(-\sigma_1^2 - 2\rho\sigma_1\sigma_2 + (b^2 - 2b)\sigma_2^2)\right) \\ d_3 = \dfrac{1}{\sigma\sqrt{T}}\left(\log(F_1/a) + \dfrac{T}{2}(-\sigma_1^2 + b^2\sigma_2^2)\right) \\ \sigma = \sqrt{\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2} \end{array} \tag{31}$$

In addition, this lower bound could be improved by maximizing the pricing formula with respect to a and b, which uses the same idea as in the Carmona-Durrleman method. Actually, Bjernskund and Stensland proved that such optimization is equivalent to the Carmona-Durrleman optmization. However, the numerical results show that the precision gained by such optimization procedure is not significant, which suggests the use of fixed values for a and b defined in (29).

1.5 Function usage

`static int Spread_CarmonaAn(...)`

Compute the Spread option price using Carmona-Durrleman formula

`static int Spread_BjerksundAn(...)`

Compute the Spread option price using Bjerksund-Stensland formula

The arguments are the following:

- `double s01`
  Initial price $S_1(0)$
- `double s02`
  Initial price $S_2(0)$
- `double K`
  Strike K
- `double t`
  Maturity T
- `double r`
  Risk-free interest r
- `double divid1`
  Convenience yield q_1 of the asset S_1
- `double divid2`
  Convenience yield q_2 of the asset S_2
- `double sigma1`
  Volatility σ_1 of the asset S_1
- `double sigma2`
  Volatility σ_2 of the asset S_2
- `double rho`
  Correlation factor ρ
- `double *ptprice`
  The pointer to the price computed using the function
- `double *ptprice`
  The pointer to the delta Δ_1 computed using the function
- `double *ptdelta1`
  The pointer to the delta Δ_1 computed using our functions
- `double *ptdelta2`
  The pointer to the delta Δ_2 computed using our functions

1.6 Numerical Performance

We are going to compare the method presented above.

Our benchmark will be the price computed using quasi Monte Carlo method (using the Halton sequence, see [10]) with 10^6 simulations. The theoretical upper bound of the errors is 10^{-2} (quasi Monte Carlo method in 2D converges at $(\log N)^2/N$, and the prices' variance is estimated at $\approx 10^2$), but in practice, we can consider that our benchmark has a 3, even 4 digit accuracy. The prices are computed for several values of K and ρ, for the parameters presented in Table 1.

Both the Carmona-Durrleman and Bjerksund-Stensland methods turned out to be very accurate. Bjerksund-Stensland is prefered because it is faster, it uses a closed formula that can be computed in a single step whereas Carmona-Durrleman uses a two-dimensional optimization.

Note that in our implementation of the Carmona-Durrleman, in order to maximize the price function, we use the gradient descent method combined with the Brent's method (see [10]). More optimal methods exist, such as non-linear conjugate gradient method.

For the Carmona-Durrleman method, the errors should be zero when $\rho = -1$ and when $K = 0$ (see prop. 1). The non-null errors of our computation may be due to:

- the inaccuracy of our benchmark: we should keep in mind that the quasi-Monte Carlo method is still an approximation.
- the imperfection of our maximization method: the maximizing algorithm stops as soon as the limit of number of steps is reached OR the norm of the gradient of our price function is small enough. Both of these conditions can lead to an detectable error on the choice of θ^* and d^* and thus the price.

Besides, the first reason could explain why we obtained some positive signed errors using Carmona-Durrleman and Bjerksund-Stensland methods (recall that they provide lower bounds to the true value of the price, therefore their errors' signe should be negative).

Parameter	Value	Parameter	Value
$S_1(0)$	110	$S_2(0)$	100
q_1	0.03	q_2	0.02
σ_1	0.10	σ_2	0.15
r	0.05		

Table 1: Computation parameters

ρ / K	-1	-0.5	0	0.3	0.8	1
-20	29.6557	28.9945	28.3808	28.0699	27.7698	27.7536
	29.6561	28.9945	28.3802	28.0693	27.7689	27.7462
	29.6561	28.9946	28.3806	28.0693	27.769	27.7535
-10	21.8684	20.9048	19.8887	19.27	18.381	18.2437
	21.8686	20.9048	19.8884	19.2696	18.3805	18.241
	21.8686	20.9049	19.8888	19.27	18.3808	18.2435
0	15.1332	13.918	12.5238	11.5619	9.6325	8.8212
	15.1332	13.9179	12.5235	11.5618	9.6325	8.8196
	15.1332	13.918	12.5237	11.5618	9.6325	8.8212
5	12.2442	10.9564	9.4456	8.3677	5.9672	4.4542
	12.2441	10.9562	9.4453	8.3674	5.967	4.4533
	12.2441	10.9562	9.4453	8.3674	5.967	4.4542
15	7.5221	6.2425	4.7449	3.6802	1.3427	0.0488
	7.5218	6.2422	4.7444	3.6796	1.3419	0.0486
	7.5217	6.2421	4.7443	3.6796	1.3421	0.0479
25	4.2017	3.1301	1.9626	1.2204	0.1042	0
	4.2014	3.13	1.962	1.2197	0.0995	0
	4.2013	3.1298	1.9617	1.2194	0.1032	0

Table 2: Spread option price using different methods: quasi MC, Carmona-Durrleman and Bjerksund-Stensland

ρ / K	-1	-0.5	0	0.3	0.8	1
-20	0.0004 0.0004	0 0.0001	-0.0006 -0.0002	-0.0006 -0.0006	-0.0009 -0.0008	-0.0074 -0.0001
-10	0.0002 0.0002	0 0.0001	-0.0003 0.0001	-0.0004 0	-0.0005 -0.0002	-0.0027 -0.0002
0	0 0	-0.0001 0	-0.0003 -0.0001	-0.0001 -0.0001	0 0	-0.0016 0
5	-0.0001 -0.0001	-0.0002 -0.0002	-0.0003 -0.0003	-0.0003 -0.0003	-0.0002 -0.0002	-0.0009 0
15	-0.0003 -0.0004	-0.0003 -0.0004	-0.0005 -0.0006	-0.0006 -0.0006	-0.0008 -0.0006	-0.0002 -0.0009
25	-0.0003 -0.0004	-0.0001 -0.0003	-0.0006 -0.0009	-0.0007 -0.001	-0.0047 -0.001	0 0

Table 3: Pricing error of the different methods (approximation formula - quasi MC): Carmona-Durrleman and Bjerksund-Stensland

2 Fourier Transform Method

When the characteristic function of the log-spot $(\log S_1(T), \log S_2(T))$ is explicitly known, Fourier transform methods can be used to compute the price of the spread option.

2.1 Fourier transform formula

Without loss of generality, we assume that the strike price K is equal to 1 (if $K < 0$ we can use the Call-Put parity (5) to consider a spread option with a positive strike instead).
We denote $X_T = (\log S_1(T), \log S_2(T))$ and we make the simplifying assumption that the increment $X_t - X_0$ is independent of X_0 for any $t > 0$, which implies that the characteristic function of X_T factorizes as follows:

$$\mathbb{E}_{X_0}\left[e^{iu\cdot X_T}\right] = e^{iu\cdot X_0}\mathbb{E}\left[e^{iu\cdot(X_T - X_0)}\right] \tag{32}$$

where $\mathbb{E}_{X_0}[\cdot]$ is the conditional expectation given X_0.
We denote $\Phi := \mathbb{E}\left[e^{iu\cdot(X_T - X_0)}\right]$ and we assume this function is explicitly known. Hurd and Zhou (2010)[5] show that the price of the spread option can be written as a two-dimensional Fourier transform:

Theorem 1.

$$C = (2\pi)^{-2}e^{-rT}\int_{\mathbb{R}^2+i\varepsilon} e^{iu\cdot X_0}\Phi(u)\Psi(u)du \tag{33}$$

where Ψ is defined by :

$$\Psi(u_1, u_2) \mapsto \frac{\Gamma(i(u_1+u_2)-1)\Gamma(-iu_2)}{\Gamma(iu_1+1)} \tag{34}$$

And $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$ can be any ordered pair satisfying the constraints $\varepsilon_2 > 0$ and $\varepsilon_1 + \varepsilon_2 < -1$.

The proof of this theorem is computational and is provided in 2.7.

The computation of the complex gamma function can be handled conveniently, see [7] for more details.

If ε is well-chosen, the integrand function has an asymptotic exponential decay as $\|u\| \to \infty$ for the models treated in 2.5, **??** and 2.6. This is why this theorem enable us to implement methods that provide very accurate results with reasonable time of computation.

Computation of the Deltas

The Deltas are obtained by differentiating the integrand, for instance:

$$\begin{aligned}\Delta_1 &= (2\pi)^{-2}e^{-rT}\int_{\mathbb{R}^2+i\varepsilon}\frac{\partial(e^{iu\cdot X_0})}{\partial(S_1(0))}\Phi(u)\Psi(u)du \\ &= \frac{(2\pi)^{-2}e^{-rT}}{S_1(0)}\int_{\mathbb{R}^2+i\varepsilon} iu_1 e^{iu\cdot X_0}\Phi(u)\Psi(u)du\end{aligned} \tag{35}$$

2.2 Numerical integration

The integral over $\mathbb{R}^2$ is replace by a two-dimensional sum over the following lattice for each dimension of the u space:

$$u_0 = -\overline{u} < u_1 = -\overline{u} + \eta < \cdots < u_k = -\overline{u} + k\eta < \cdots < u_{N-1} = +\overline{u}$$

where N is a power of two for the convenience of FFT (see 2.4), $2\overline{u}$ is the width of the integration domain, and $\eta = \dfrac{2\overline{u}}{N}$ is the lattice spacing. N and $\overline{u}$ are used as fixed parameters.
On the other hand, we initially choose X_0 to lie on the reciprocal lattice, it's to say, for each dimension of the x space:

$$x_0 = -\overline{x} < x_1 = -\overline{x} + \eta^* < \cdots < x_l = -\overline{x} + l\eta^* < \cdots < x_{N-1} = +\overline{x}$$

where $\eta^* = \pi/\overline{u}$ is the lattice space, and $\overline{x} = \dfrac{N\eta^*}{2}$.

Pricing Formula.

$$C \approx (-1)^{l_1+l_2} e^{-rT} \left(\frac{\eta}{2\pi}\right)^2 \sum_{k\in\{0,\cdots,N-1\}^2} e^{2\pi k\cdot l/N} H(k) \tag{36}$$

where

$$H(k) = (-1)^{k_1+k_2} \Phi(u_k + i\varepsilon)\Psi(u_k + i\varepsilon) \tag{37}$$

and $l = (l_1, l_2)$ *is such that* $X_0 = x_l$

2.3 Choice of parameters

The approximation error is due to the following phenomena: the truncation and the discretization.
The truncation error is made by replacing the integration domain, $\mathbb{R}^2$, by $[-\overline{u}; +\overline{u}]$, the approximation is possible if the integrand decays rapidly at infinity on the u space. The discretization error is made by replacing the integral by a finite sum, the approximation if Φ is regular enough. In practice, we set the value of $\overline{u}$ and N.
We keep in mind that the truncation error is null when $\overline{u} \to \infty$ and the discretization error is null when $\eta \to 0$, or equivalently, with $\overline{u}$ fixed, when $N \to \infty$. For further details, see [5].

We choose $\varepsilon = (-5, 3)$, $\overline{u} = 40$ and $N = 256$ as default values, which gives moderate errors and good computation time in our numerical tests.

2.4 FFT algorithm

We notice that the sum over k in the pricing formula (36) is actually a discrete inverse Fourier transform of the function H. If we need to compute all the prices, for X_0 describing the entire reciprocal lattice ($x_l, l \in \{0, \cdots, N-1\}^2$), it is suitable to use the FFT algorithm. Numerical results show that the FFT algorithm is able to produce high accuracy approximation (with errors $\sim 10^{-8}$) for $N = 256$ in the models presented in 2.5, **??** and 2.6, see [5].
However, if we only need the price for a single value of X_0 (it's to say for a single value of l), then the direct computation of the sum can be done. That is why FFT algorithm is not used in our implementation of the Fourier transform method.

Now that we introduced the theoretical framework of the Fourier transform method, we are going to present some models of [5]. We implemented the Fourier transform method for these models. However, we will not focus on the numerical results as they are already discussed in [5], except for the two geometric Brownian model, which can be compared to the methods in the section 1.

We should keep in mind that Fourier transform method could be used for any other model for which we explicitly know (or at least we can approximate accurately and quickly) the characteristic function.

2.5 Two geometric Brownian case

The two geometric Brownian model treated in the section 1 can also be handled by the Fourier transform method.

With the notations of the section 1, the joint characteristic function of the log-spot is:

$$\begin{aligned}\Phi\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \exp\Bigg\{ iT(u_1,u_2)\left[\begin{pmatrix} r-q_1 \\ r-q_2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \end{pmatrix}\right] \\ - \frac{1}{2}(u_1,u_2)\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\Bigg\}\end{aligned} \tag{38}$$

Function usage

```
static int Spread_HurdZhou_bsAn(...);
```

The arguments are the same used in 1.5.

Numerical performance

The computed prices are presented in the Table 4, the parameters are the same as the ones used in section 1.

The Fourier transform method provides prices with very good accuracy for a 256^2-point integration, except for big values of K and $\rho = 1$. For the other values, the error is comparable to those of the Carmona-Durrleman and the

ρ / K	-1	-0.5	0	0.3	0.8	1
-20	29.6557 29.6561 0.0004	28.9945 28.9948 0.0003	28.3808 28.3811 0.0003	28.0699 28.0701 0.0002	27.7698 27.7701 0.0003	27.7536 27.7529 -0.0007
-10	21.8684 21.8686 0.0002	20.9048 20.905 0.0002	19.8887 19.8889 0.0002	19.27 19.2701 0.0001	18.381 18.3811 0.0001	18.2437 18.2466 0.0029
5	12.2442 12.2441 -0.0001	10.9564 10.9562 -0.0002	9.4456 9.4453 -0.0003	8.3677 8.3674 -0.0003	5.9672 5.9669 -0.0003	4.4542 4.4214 -0.0328
15	7.5221 7.5218 -0.0003	6.2425 6.2422 -0.0003	4.7449 4.7445 -0.0004	3.6802 3.6798 -0.0004	1.3427 1.3429 0.0002	0.0488 0.0118 -0.037
25	4.2017 4.2014 -0.0003	3.1301 3.13 -0.0001	1.9626 1.9621 -0.0005	1.2204 1.22 -0.0004	0.1042 0.1038 -0.0004	0 0.0597 0.0597
25	4.2017 4.2014 -0.0003	3.1301 3.13 -0.0001	1.9626 1.9621 -0.0005	1.2204 1.22 -0.0004	0.1042 0.1041 -0.0001	0 0.0017 0.0017

Table 4: Numerical performance of the Fourier transform method for the two geometric Brownian model, the parameters are ones used in the section 1. Note that the case $K = 0$ is not treated here. In each cell, the first value is obtained by the quasi-Monte Carlo method, the second one is computed by the Fourier transform method, the last one is their difference. All the values are computed using $N = 2^8 = 256$ and $\overline{u} = 40$ except for the ones in the last row, we used $\overline{u} = 80$ instead.

Bjerksund-Stensland methods.

For $K = 25$, we observe that the results are better when we choose $\overline{u} = 80$ instead of 40: we need to take the rate of decay of the characteristic functions into account by adapting the integration domain when parameters change. Naturally, all the results could be improved by increasing the value of N.

2.6 The Variance Gamma model

Madan and Seneta (1990) introduced the variance gamma(VG) process to model some prices.
A VG process Y is a pure jump Lévy process determined by its characteristic function:

$$\Phi_{Y(t)}(u) = \left[1 + i\left(\frac{1}{a_-} - \frac{1}{a_+}\right) + \frac{u^2}{a_+ a_-}\right]^{-\lambda t} \tag{39}$$

where a_+, a_- and λ are positive parameters.

We use here three independent VG processes Y_1, Y_2 and Y_3, with common parameters $a_\pm$, and $\lambda_1 = \lambda_2 = (1-\alpha)\lambda$, and $\lambda_3 = \alpha\lambda$.
The log-prices of the indices are assumed to be equal to:

$$\begin{aligned} X_1(t) &= X_1(0) + Y_1(t) + Y_3(t) \\ X_2(t) &= X_2(0) + Y_2(t) + Y_3(t) \end{aligned} \tag{40}$$

The parameter $\alpha \in [0;1]$ is interpreted as the correlation coefficient.
The joint characteristic function of $(X_1(T), X_2(T))$ is equal to:

$$\Phi(u) = \Phi_{Y_1}(u_1)\Phi_{y_2}(u_2)\Phi_{y_3}(u_1 + u_2) \tag{41}$$

Function usage

```
static int Spread_HurdZhou_vgAn(double s01, double s02, double K,
double t, double r, double a_m, double a_p, double alpha, double lambda,
double *ptprice, double *ptdelta1, double *ptdelta2);
```

- `double a_m`

  VG process parameter a_-

- `double a_p`

  VG process parameter a_+

- `double alpha`

  Correlation factor α

- `double lambda`

  VG process parameter λ

The other parameters are model-independent and are presented in 1.5.

Remark

The following condition is needed to avoid singularities in the u-space when we integrate: ε_1, ε_2 and $\varepsilon_1 + \varepsilon_2$ must lie in $]-a_+, a_-[$.

2.7 Proof of proposition 1

Proof. Let f be the function $x = (x_1, x_1) \mapsto (e^{x_1} - e^{x_2} - 1)^+$.
For all $\varepsilon = (\varepsilon_1, \varepsilon_2)$ satisfying $\varepsilon_1 + \varepsilon_2 < -1$ and $\varepsilon_2 > 0$, the function $x \mapsto e^{\varepsilon \cdot x} f(x)$ lies in $\mathbb{L}^2(\mathbb{R}^2)$ (under these conditions, all the integrals in the following computation will be well-defined). Therefore, by applying the Fourier inversion theorem to this function, we have:

$$e^{\varepsilon \cdot x} f(x) = \int_{\mathbb{R}^2} e^{iu \cdot x} \left(\int_{\mathbb{R}^2} e^{-iu \cdot x'} e^{\varepsilon \cdot x'} f(x') dx' \right) du \tag{42}$$

By arranging the terms:

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^2} e^{iu \cdot x} e^{-\varepsilon \cdot x} \left(\int_{\mathbb{R}^2} e^{-iu \cdot x'} e^{\varepsilon \cdot x'} f(x') dx' \right) du \\ &= \int_{\mathbb{R}^2} e^{i(u+i\varepsilon) \cdot x} \left(\int_{\mathbb{R}^2} e^{-i(u+i\varepsilon) \cdot x'} f(x') dx' \right) du \\ &= \int_{\mathbb{R}^2 + i\varepsilon} e^{iu \cdot x} \left(\int_{\mathbb{R}^2} e^{-iu \cdot x'} f(x') dx' \right) du \end{aligned} \tag{43}$$

We are going to show that the inner integral we are going to denote by $\Psi(u)$ is exactly the same function Ψ given in (34).
Let us compute this integral. We integrate f on the domain $\{(x_1, x_2),\ x_1 > 0, x_2 < \log(e^{x_1} - 1)\}$:

$$\begin{aligned} \Psi(u) &= \int_{\mathbb{R}^2} e^{-iu \cdot x} f(x) dx \\ &= \int_0^\infty e^{-iu_1 x_1} \left(\int_{-\infty}^{\log(e^{x_1}-1)} e^{-iu_2 x_2} [(e^{x_1} - 1) - e^{x_2}] dx_2 \right) dx_1 \\ &= \int_0^\infty e^{-iu_1 x_1} \left[\frac{e^{-iu_2 x_2}(e^{x_1} - 1)}{-iu_2} - \frac{e^{(-iu_2+1)x_2}}{-iu_2 + 1} \right]_{-\infty}^{\log(e^{x_1}-1)} dx_1 \end{aligned} \tag{44}$$

Recall that $\Re(-iu_2) = \Im(u_2) = \varepsilon_2 > 0$, which implies that the terms in the brackets vanish when $x_2 \to -\infty$.
We have the following expression after rearranging:

$$\begin{aligned} \Psi(u) &= \left[\frac{1}{-iu_2} - \frac{1}{-iu_2 + 1} \right] \int_0^\infty e^{-iu_1 x_1} (e^{x_1} - 1)^{-iu_2+1} dx_1 \\ &= \frac{1}{(-iu_2)(-iu_2 + 1)} \int_0^\infty e^{-iu_1 x_1} (e^{x_1} - 1)^{-iu_2+1} dx_1 \end{aligned} \tag{45}$$

The change of variable $z = e^{-x_1}$ leads to the following expression of the remaining integral:

$$\begin{aligned} \int_0^\infty e^{-iu_1 x_1} (e^{x_1} - 1)^{-iu_2+1} dx_1 &= \int_0^1 z^{i(u_1+u_2)-2} (1 - z)^{-iu_2+1} dz \\ &= B(i(u_1 + u_2) - 1, -iu_2 + 2) \end{aligned} \tag{46}$$

where the beta function B is defined by $B : (x, y) \mapsto \int_0^1 z^{x-1}(1-z)^{y-1} dz$ for all $(x, y) \in \mathbb{C}, \Re(x) > 0, \Re(y) > 0$ (it is the case here thanks to our conditions upon ε). This function is related to the gamma function by $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+$

y).
We have thus:

$$\Psi(u) = \frac{\Gamma(i(u_1+u_2)-1)}{\Gamma(iu_1+1)} \frac{\Gamma(-iu_2+2)}{(-iu_2)(-iu_2+1)} \tag{47}$$

Recall that the gamma function satisfies $\Gamma(x+1) = x\Gamma(x)$, we finally find the expected expression in (34).

Now that the essential part of the proof is done, let us compute the spread option's price:

$$\begin{aligned} C &= e^{-rT}\mathbb{E}[f(X_1(T), X_2(T))] \\ &= e^{-rT}\mathbb{E}\left[(2\pi)^{-2}\int_{\mathbb{R}^2+i\varepsilon} e^{iu\cdot X_T}\Psi(u)du\right] \\ &= e^{-rT}(2\pi)^{-2}\int_{\mathbb{R}^2+i\varepsilon} \mathbb{E}[e^{iu\cdot X_T}]\Psi(u)du \\ &= e^{-rT}(2\pi)^{-2}\int_{\mathbb{R}^2+i\varepsilon} e^{iu\cdot X_0}\Phi(u)\Psi(u)du \end{aligned} \tag{48}$$

□

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