The Path-Dependent Volatility Model : Efficient Particle Method

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1 Introduction

The following method proposed by [1] deals with an efficient particle method for computing the price of a forward starting call spreads asset considering a path-dependent volatility model (PDV) where the instantaneous volatility depends on the path followed by the asset price so far.

2 Theoretical framework

We consider the following path-dependent volatility model, :

$$dS_t = S_t \cdot \sigma(t, (S_u, u \leq t)) \cdot dW_t$$

We have chosen to take zero interest rates and dividends.

In this article, we choose a set of path dependent variables X_t and a function $\sigma(t, S_t, X_t)$ so that the path-dependent volatility is given by $\sigma(t, (S_u, u \leq t)) = \sigma(t, S_t, X_t)$.
In order to calibrate the model to the market smile of S ,we multiply $\sigma(t, S_t, X_t)$ by a leverage function $l(t, S_t)$.

The PDV model is now the following :

$$dS_t = S_t \cdot \sigma(t, S_t, X_t) \cdot l(t, S_t) \cdot dW_t$$

From Itô-Tanaka's formula, this model is exactly calibrated to the market smile if we have

$$\mathbb{E}[\sigma(t, S_t, X_t)^2 | S_t] \cdot l(t, S_t)^2 = \sigma_{Dup}(t, S_t)^2$$

As a result, the calibrated model satisfies the non-linear McKean differential equation :

$$dS_t = S_t \frac{\sigma(t, S_t, X_t)}{\sqrt{\mathbb{E}[\sigma(t, S_t, X_t)^2 | S_t]}} \sigma_{Dup}(t, S_t) \cdot dW_t$$

The particle method, explained in the next section, is a very efficient and elegant Monte Carlo method that computes the above conditional expectation, hence the leverage function :

$$l(t, S) = \frac{\sigma_{Dup}(t, S)}{\sqrt{\mathbb{E}[\sigma(t, S_t, X_t)^2 | S_t = S]}}$$

3 Numerical algorithm

3.1 Path-dependent volatility

The following path-dependent volatility that we will consider was proposed by [1]. It suggests that the volatility at t depends on the spot value at $t - \Delta$ through the following expression :

$$\sigma(t, S_t, X_t) = \bar{\sigma} 1_{\{\frac{S_t}{X_t} \leq 1\}} + \underline{\sigma} 1_{\{\frac{S_t}{X_t} > 1\}}$$

with $X_t = S_{t-\Delta}$ and $\bar{\sigma}$, $\underline{\sigma}$ parameters to be defined.
One question remains : How do we compute the conditional expectation ?

3.2 Particle method to compute $\mathbb{E}[a_t^2|S_t = s]$

We usually approximate $\mathbb{E}[a_t^2|S_t = s]$ by :

$$\mathbb{E}[a_t^2|S_t = s] = \frac{\sum_{i=1}^{M} (a_t^i)^2 \delta(S_t^i - s)}{\sum_{i=1}^{M} \delta(S_t^i - s)}$$

With δ being a regularizing kernel. Yet this method requires a computational time far too high if we do it for each Monte Carlo sample.

Thus instead of computing $\mathbb{E}[\sigma(t, S_t)^2|S_t = s]$ for all Monte Carlo samples. We decide to compute it only L times with $L << M$.
L is often called the number of bins : each bin being a same-sized interval containing $\frac{M}{L}$ spot values.
In order to choose which spot value in which bin, we classify the spot values S_t^i in the ascending order and each one of them will belong to a bin depending on its rank.
At t for the i^{th} sample, if S_t^i belongs to the L^{th} bin then we approximate the conditional expectation by :

$$\mathbb{E}[\sigma(t, S_t^i)^2|S_t] = \frac{\sum_{j:j \in L^{th} bin} \sigma(t, S_t^j)^2}{\frac{M}{L}}$$

This way we can compute the leverage function $L(t, S_t^i)$ for each sample with acceptable computation time.

3.3 Simulation scheme

We discretize $[0, T]$ on a regular grid of size N, with step size $\Delta = \frac{T}{N}$. We use the following exponential scheme :

$$S_{i,j+\Delta} = S_{i,j} \cdot \exp\left(\sigma(j, S_{i,j}) \cdot \frac{\sigma_{Dup}}{\sqrt{N \cdot EC_{i,j}}} \cdot W_{i,j} - \frac{1}{2} \frac{\sigma_{Dup}^2 \cdot \sigma(j, S_{i,j})^2}{EC_{i,j} \cdot N}\right)$$

with $S_{i,0} = S_0$, $EC_{i,j} = \mathbb{E}[\sigma(j, S_j^i, X_j^i)^2|S_t]$ and $(W_{i,j})$ being an independent brownian motion.

4 Numerical experiments

We test the algorithm on forward starting call spreads with payoff :

$$(\frac{S_T}{S_{T-1}} - K_1)_+ - (\frac{S_T}{S_{T-1}} - K_2)_+$$

with the following parameters :

S_0	$\bar{\sigma}$	$\underline{\sigma}$	K_1	K_2
1	0.32	0.08	0.95	1.05

We use $N = 120$ times steps, $T = 12$ months, $M = 10000$ particles and the local volatility $\sigma_{Dup} = 20\%$.
The conditional expectation is computed with $L = 100$ bins.
We get the following price at T : 0.02168

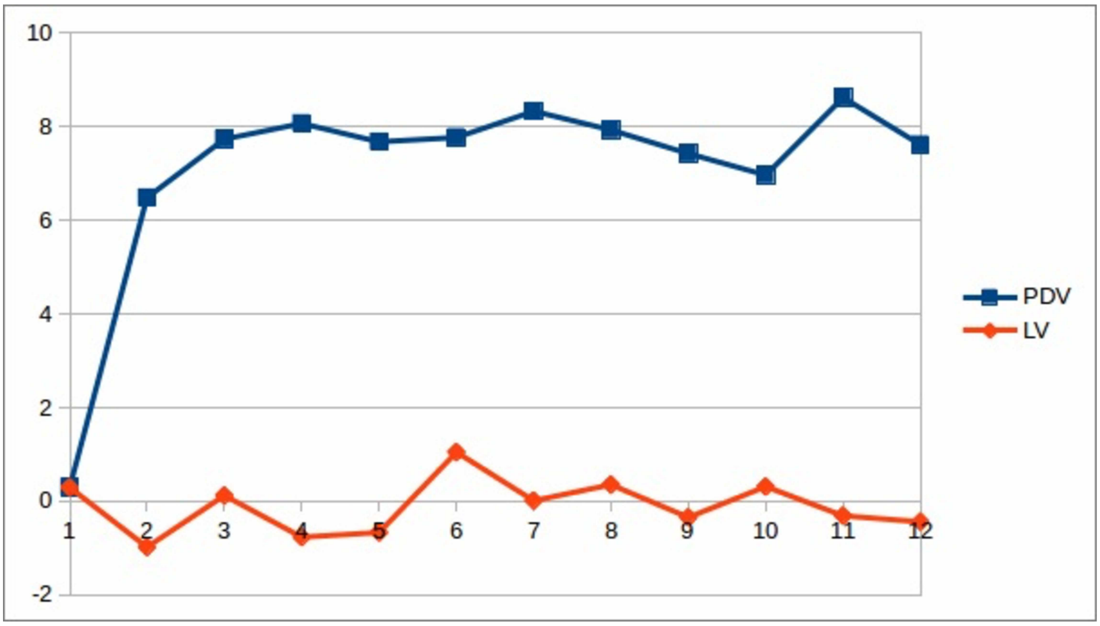
Furthermore we can compute the price in volatility points of the forward starting call spreads at each month. For the i^{th} month, we compute σ_i^{ATM} by dichotomy :

$$\mathbb{E}((\frac{S_{T_i}}{S_{T_{i-1}}} - 1)^+) = C_{BS}(1, \sigma_i^{ATM})$$

where $C_{BS}(K, \sigma)$ denotes the Black-Scholes price of a call with $S_0 = 1$, $r = q = 0$ and $T = 1$ month.
Then the price in volatility points at i is $100 \cdot \Delta\sigma$ such as $\Delta\sigma$ satisfies :

$$\mathbb{E}[(\frac{S_{T_i}}{S_{T_{i-1}}} - K_1)^+ - (\frac{S_{T_i}}{S_{T_{i-1}}} - K_2)^+] = C_{BS}(K_1, \sigma_i^{ATM} + \frac{\Delta\sigma}{2}) - C_{BS}(K_2, \sigma_i^{ATM} - \frac{\Delta\sigma}{2})$$

We get the following evolution of the price in volatility points, where PDV is the path-dependent volatility model and LV is the local volatility model ($dS_t = S_t \cdot \sigma_{Dup} \cdot dW_t$)



5 Correlated models

Let's apply this method for the correlated model where we have two rates S^1 and S^2 following local volatility dynamics and $S^{12} = \frac{S^1}{S^2}$ be the cross rate. We now have the following equations :

$$
\begin{aligned}
dS_t^1 &= \sigma_1(t, S_t^1) \cdot S_t^1 \cdot dW_t^1 \\
dS_t^2 &= \sigma_2(t, S_t^2) \cdot S_t^2 \cdot dW_t^2 \\
d < W^1, W^2 >_t &= \rho(t, S_t^1, S_t^2)
\end{aligned}
$$

The two driving processes W^1 and W^2 are two brownian motions ; they have a local instantaneous correlation $\rho(t, S^1, S^2) \in [0, 1]$.

We can demonstrate that this model is calibrated to the market smile of the cross rate S^{12} if and only if :

$$
\frac{\mathbb{E}[S_t^2(\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - 2\rho(t, S_t^1, S_t^2)\sigma_1(t, S_t^1)\sigma_2(t, S_t^2))|\frac{S_t^1}{S_t^2}]}{\mathbb{E}[S_t^2|\frac{S_t^1}{S_t^2}]} = \sigma_{12}^2(t, \frac{S_t^1}{S_t^2})
$$

The following ρ satisfies this condition :

$$
\rho(t, S_t^1, S_t^2) = \frac{\mathbb{E}[S_t^2(\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2))|\frac{S_t^1}{S_t^2}] - \sigma_{12}^2(t, \frac{S_t^1}{S_t^1})\mathbb{E}[S_t^2|\frac{S_t^1}{S_t^2}]}{2\mathbb{E}[S_t^2\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)|\frac{S_t^1}{S_t^2}]}
$$

For our problem we choose the following volatility dynamics :

$$
\sigma_1(t, S_t^1) = \sigma_1 + \delta\frac{S_t^1}{1 + S_t^1}
$$

$$
\sigma_2(t, S_t^2) = \sigma_2 + \delta\frac{S_t^2}{1 + S_t^2}
$$

$$
\sigma_{12}(t, \frac{S_t^1}{S_t^2}) = \sigma_1 + \delta\frac{\frac{S_t^1}{S_t^2}}{1 + \frac{S_t^1}{S_t^2}}
$$

Then we can simulate the rates of this correlated model with the following scheme model, similar to the previous one for the path-dependent volatility model.

5.1 Simulation scheme

$$
S_{i,j+\Delta}^1 = S_{i,j}^1 \exp\left(\frac{\sigma_1(j, S_{i,j}^1)}{\sqrt{N}}W_{i,j}^1 - \frac{1}{2}\frac{\sigma_1^2(j, S_{i,j}^1)}{N} - \frac{\sigma_1(j, S_{i,j}^1)\sigma_2(j, S_{i,j}^2)\rho(j, S_{i,j}^1, S_{i,j}^2)}{N}\right)
$$

$$
S_{i,j+\Delta}^2 = S_{i,j}^2 \exp\left(\frac{\sigma_2(j, S_{i,j}^2)}{\sqrt{N}}(\rho(j, S_{i,j}^1, S_{i,j}^2)W_{i,j}^1 + \sqrt{1 - \rho(j, S_{i,j}^1, S_{i,j}^2)^2}W_{i,j}^2) - \frac{1}{2}\frac{\sigma_2^2(j, S_{i,j}^2)}{N}\right)
$$

5.2 Numerical experiments

We test the algorithm for a put on worst with payoff :

$$g(S_T^1, S_T^2) = (K - \min(\frac{S_T^1}{S_0}, \frac{S_T^2}{S_0})_+$$

with the following strike $K = 0.95$, and $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, $\delta = 0.05$. All the other parameters M, N, L ... are the same as the previous section.
We finally get the following price at T for our model with the put on worst : 0.1304

Références

[1] J. Guyon Path-dependent volatility Quantitative Research Bloomberg L.P. , February 2014. 1, 2

[2] J. Guyon P. Henry-Labordere The smile calibration problem solved Risk, July 2011.