Documentation for : "Unbiased simulation of SDEs"

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Premia 22

1 The problem to solve

Pricing of an option in a local volatility model.

The underlying

It follows (under the pricing measure) the SDE

$$dS_t = S_t\,\mu\;dt + S_t\,\mathrm{lv}(t, S_t)\;dW_t = S_t\,\mu\;dt + \sigma(t, S_t)\;dW_t,$$

where the diffusion coefficient $\sigma(t, s) = s\,\mathrm{lv}(t, s)$. In the model,

- $\mu = r - q$, where r is the risk-free interest rate and q is the dividends rate,
- S starts at the spot price S_0,
- the local volatility function is given by either

$$\mathrm{lv}(t, s) = \frac{15}{s} \qquad \text{or} \qquad \mathrm{lv}(t, s) = 0.01 + 0.01\,t + 0.1\,e^{-s/100}.$$

The payoff

A vanilla call or put, of payoff $p(S_T) = (S_T - K)^+$ or $(K - S_T)^+$ respectively, where

- T is the maturity and
- K is the strike.

The price

The price sought is given by

$$V_0 = e^{-rT} E[p(S_T)]$$

2 The algorithm

The above problem is a particular case of having to compute $V_0 = E[g(X_T)]$ where X follows an SDE

$$\begin{aligned} X_0 &= x_0 \\ dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t. \end{aligned}$$

The standard technique is to discretize the SDE to obtain a computable (and simulatable) approximation $\widetilde{X}_T$ for X_T and then do a Monte Carlo integration of the random variable $g(\widetilde{X}_t)$. The algorithms presented in [1] instead provide a computable/simulatable random variable ψ such that $V_0 = E[\psi]$. Therefore, given a number $M \in \mathbb{N}^*$ of samples, and random variables $(\psi^m)_{m=1\ldots M}$ i.i.d. $\sim \psi$, the estimator

$$\widehat{V}_0^M = \frac{1}{M}\sum_{m=1}^{M} \psi^m$$

is an unbiased estimator for V_0.

General idea

Generally speaking, ψ is constructed in the following way. Let $\beta > 0$ and let $(\tau_k)_{k\in\mathbb{N}^*}$ be i.i.d. $\sim \mathcal{E}(\beta)$. We associate to these inter-arrival times the arrival times T_k defined by $T_0 = 0$ and then $T_{k+1} = (T_k + \tau_{k+1}) \wedge T$, as well as the Poisson counting process $(N_t)_{t\in[0,T]}$.

The idea is then to simulate the switching-diffusion $\widehat{X}$ starting at x_0, with drift and diffusion coefficients $\widehat{b}, \widehat{\sigma} : ([0,T]\times\mathbb{R})^2 \longrightarrow \mathbb{R}$, such that on each interval $[T_k, T_{k+1}]$,

$$d\widehat{X}_t = \widehat{b}(T_k, \widehat{X}_{T_k}, t, \widehat{X}_t)dt + \widehat{\sigma}(T_k, \widehat{X}_{T_k}, t, \widehat{X}_t)dW_t.$$

One then computes weights $\overline{\mathcal{W}}_k$ depending on $T_{k-1}, \widehat{X}_{T_{k-1}}, T_k, \widehat{X}_{T_k}, \Delta W_{T_{k+1}}, \Delta T_{k+1}$, where $\Delta W_{T_{k+1}} = W_{T_{k+1}} - W_{T_k}$ and $\Delta T_{k+1} = T_{k+1} - T_k$. Finally, ψ is defined by

$$\psi = e^{\beta T}\Big[g(\widehat{X}_T) - g(\widehat{X}_{N_T})1_{\{N_T>0\}}\Big]\prod_{k=1}^{N_T}\overline{\mathcal{W}}_k.$$

Clearly, in order to have an exactly simulatable ψ, one must ensure that the SDE chosen for $\widehat{X}$ is solvable explicitly, and that $\widehat{X}_{T_{k+1}}$ is a function of $T_k, \widehat{X}_{T_k}$ and $\Delta W_{T_{k+1}}$. A major issue is that, if constructed without care, the random variable ψ is likely to be integrable but of infinite variance. So $\widehat{V}_0^M$ is then not a good estimator for V_0.

The case of scalar driftless SDEs

For this type of SDE, where $b = 0$ and σ is arbitrary, an appropriate set of switching-diffusion, weights and thus ψ is given in [1].

The idea for $\widehat{X}$ is to take a higher-order Euler–Maruyama scheme, with $\widehat{b} = 0$ and $\widehat{\sigma}(s, y, t, x) = \sigma(s, y) + \sigma_x(s, y)(x - y)$, so the SDE for $\widehat{X}$ is, over $[T_k, T_{k+1}]$,

$$d\widehat{X}_t = \Big(\sigma(T_k, \widehat{X}_{T_k}) + \sigma_x(T_k, \widehat{X}_{T_k})(\widehat{X}_t - \widehat{X}_{T_k})\Big) dW_t = \Big(c_1^k + c_2^k \widehat{X}_t\Big) dW_t,$$

where

$$c_1^k = \sigma(T_k, \widehat{X}_{T_k}) - \sigma_x(T_k, \widehat{X}_{T_k})\widehat{X}_{T_k} \qquad \text{and} \qquad c_2^k = \sigma_x(T_k, \widehat{X}_{T_k}).$$

The solution is given by

$$\widehat{X}_{T_{k+1}} = \begin{cases} \text{if } c_2^k = 0 \text{ then } \widehat{X}_{T_{k+1}} = \widehat{X}_{T_k} + \sigma(T_k, \widehat{X}_{T_k})\Delta W_{T_{k+1}} \\ \text{if } c_2^k \neq 0 \text{ then } \widehat{X}_{T_{k+1}} = -\dfrac{c_1^k}{c_2^k} + \Big(\dfrac{c_1^k}{c_2^k} + \widehat{X}_{T_k}\Big) e^{c_2^k \Delta W_{T_{k+1}} - \frac{1}{2}(c_2^k)^2 \Delta T_{k+1}}. \end{cases}$$

The weights $\overline{\mathcal{W}}_k$ are given as follows. Define the Malliavin weight

$$\mathcal{W}_k = -\sigma_x(T_k, \widehat{X}_{T_k}) \frac{\Delta W_{T_{k+1}}}{\Delta T_{k+1}} + \frac{\Delta W_{T_{k+1}}^2 - \Delta T_{k+1}}{(\Delta T_{k+1})^2},$$

as well as

$$\begin{aligned} \sigma_k &= \sigma(T_k, \widehat{X}_{T_k}), \qquad & \widetilde{\sigma}_k &= \sigma(T_{k-1}, \widehat{X}_{T_{k-1}}) + \sigma_x(T_{k-1}, \widehat{X}_{T_{k-1}})(\widehat{X}_{T_k} - \widehat{X}_{T_{k-1}}) \\ a_k &= \sigma_k^2, \qquad & \widetilde{a}_k &= \widetilde{\sigma}_k^2 \end{aligned}$$

and finally the weight

$$\overline{\mathcal{W}}_k = \frac{1}{\beta} \, \frac{a_k - \widetilde{a}_k}{2a_k} \, \mathcal{W}_k.$$

Define also

$$\mathrm{PF} = e^{\beta T} \, \Big(g(\widehat{X}_T) - g(\widehat{X}_{T_{N_T}}) 1_{\{N_T > 0\}}\Big).$$

The use of an antithetic variable is made over the last time-step, $[T_{N_T}, T]$. So, define

$$\widehat{X}_T^- = \begin{cases} \text{if } c_2^{N_T} = 0 \text{ then } \widehat{X}_T - = \widehat{X}_{T_{N_T}} + \sigma(T_{N_T}, \widehat{X}_{T_{N_T}}) \; (-1) \; \Delta W_{T_{N_T+1}} \\ \text{if } c_2^{N_T} \neq 0 \text{ then } \widehat{X}_T^- = -\dfrac{c_1^{N_T}}{c_2^{N_T}} + \Big(\dfrac{c_1^{N_T}}{c_2^{N_T}} + \widehat{X}_{T_{N_T}}\Big) e^{c_2^{N_T} \; (-1) \; \Delta W_{T_{N_T+1}} - \frac{1}{2}(c_2^{N_T})^2 \Delta T_{N_T+1}}, \end{cases}$$

as well as the Malliavin weight

$$\mathcal{W}_{N_T}^- = -\sigma_x(T_{N_T}, \widehat{X}_{T_{N_T}}) \frac{(-1)\Delta W_{T_{N_T+1}}}{\Delta T_{N_T+1}} + \frac{\big((-1)\Delta W_{T_{N_T+1}}\big)^2 - \Delta T_{N_T+1}}{(\Delta T_{N_T+1})^2},$$

the weight

$$\overline{\mathcal{W}}_{N_T}^- = \frac{1}{\beta} \, \frac{a_{N_T} - \widetilde{a}_{N_T}}{2a_{N_T}} \, \mathcal{W}_{N_T}^-,$$

and also

$$\mathrm{PF}^- = e^{\beta T} \, \Big(g(\widehat{X}_T^-) - g(\widehat{X}_{T_{N_T}}) 1_{\{N_T > 0\}}\Big).$$

With the above quantities defined (and computed), the random variable $\overline{\psi}$ used is given by

$$\overline{\psi} = \frac{\mathrm{PF}\ \overline{\mathcal{W}}_{N_T} + \mathrm{PF}^-\ \overline{\mathcal{W}}^-_{N_T}}{2} \prod_{k=1}^{N_T-1} \overline{\mathcal{W}}_k.$$

The case of a LV model with linear drift

The SDE of interest,

$$dS_t = S_t\,\mu\ dt + S_t \,\mathrm{lv}(t,S_t)\ dW_t = S_t\,\mu\ dt + \sigma(t,S_t)\ dW_t,$$

does not fit into the driftless SDE case. However, we can do a simple transform (a discounting) to obtain one. Define $X_t = e^{-\mu t} S_t$. It satisfies the SDE

$$\begin{aligned} dX_t &= 0\ dt + e^{-\mu t}\sigma(t,S_t)\ dW_t \\ &= e^{-\mu t}\sigma(t, e^{\mu t}X_t)\ dW_t \\ &= \widehat{\sigma}(t,X_t)\ dW_t, \end{aligned}$$

which is indeed driftless. So the algorithm described above can be applied to compute

$$V_0 = e^{-rT} E[p(S_T)] = e^{-rT} E[p(e^{\mu T} X_T)] = e^{-rT} E[g(X_T)].$$

3 The parameters in the PREMIA code

Model parameters

Variable name	Meaning
spot	spot price S_0
annual_dividend_rate	dividend rate Q, annual, in %, such that $q = \ln(1+Q/100)$
annual_interest_rate	interest rate R, annual, in %, such that $r = \ln(1+R/100)$
lv_type	integer giving the function lv (hence σ, $\widehat{\sigma}$) to use

Payoff parameters

Variable name	Meaning
strike	strike of the call/put
T	maturity
payoff_type	integer giving the payoff function p to use

Numerical parameters

Variable name	Meaning
M	number of MC samples
β	parameter β of the exponentially distributed times

References

[1] P. Henry-Labordère, X. Tan and N. Touzi, Unbiased simulation of stochastic differential equations. *arXiv:1504.06107v2*.