Premia software: Efficient one, and multiple, time-step Monte Carlo scheme for the SABR model

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March 3, 2020

1 Introduction

Monte Carlo methods are widely employed in the financial context because of their simplicity and applicability to general problems. In many situations (complex products, involved models, high-dimensionality), Monte Carlo-based techniques are the only possible choice, since analytic expressions or closed-form accurate approximations are not available. This is often the case when the *Stochastic Alpha Beta Rho (SABR)* model [4] is considered. Although an approximated closed-form formula (Hagan formula) was originally proposed, this formula is restricted to European options and, furthermore, performs wrongly under certain conditions.

The technique developed here is based on an efficient simulation of *SABR's integrated variance process*. The integrated variance process appears in the so-called *exact* SABR model simulation since it is part of the conditional cumulative distribution of the SABR forward asset dynamics. We base our approach on the derivation of the cumulative distribution function of the integrated variance and the use of a copulas to approximate the conditional distribution (integrated variance conditional on the SABR volatility process).

Premia 22

2 The SABR model

The SABR model [4] is an established SDE model which, in practice, is often used for interest rates and foreign-exchange (FX) modeling. It is based on a parametric local volatility component in terms of a model parameter, β, and reads

$$\begin{aligned} \mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_S(t), \quad S(0) = S_0 \exp\left(rT\right), \\ \mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_\sigma(t), \qquad \sigma(0) = \sigma_0. \end{aligned}$$

Here $S(t)$ denotes the forward price of the underlying asset $\bar{S}(t)$, with r an interest rate, S_0 the spot price and T the maturity. Further, $\sigma(t)$ represents a stochastic volatility process, with $\sigma(0) = \sigma_0$, $W_S(t)$ and $W_\sigma(t)$ are correlated Brownian motions with constant correlation coefficient ρ (i.e. $W_S W_\sigma = \rho t$). The parameters of the SABR model are $\alpha > 0$ (the volatility of volatility, *vol-vol*), $0 \leq \beta \leq 1$ (the variance elasticity) and ρ (the correlation coefficient).

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2.1 Almost exact SABR simulation

The forward dynamics of SABR model are governed by a Constant Elasticity of Variance (CEV) process. An analytic approximation for the cumulative distribution function (CDF) of the SABR conditional process (assuming independent Brownian motions $\mathrm{d}\hat{W}_S(t)$ and $\mathrm{d}\hat{W}_\sigma(t)$) is available [5]. For some generic time interval $[s,t]$, $0 \le s < t \le T$, with $S(s) > 0$, the conditional CDF for forward $S(t)$ with an absorbing boundary at $S(t) = 0$, given $\sigma(s)$, $\sigma(t)$ and $\int_s^t \sigma^2(z)\mathrm{d}z$, reads

$$Pr\left(S(t) \le K | S(s) > 0, \sigma(s), \sigma(t), \int_s^t \sigma^2(z)\mathrm{d}z\right) = 1 - \chi^2(a; b, c), \tag{1}$$

where

$$a = \frac{1}{\nu(t)}\left(\frac{S(s)^{1-\beta}}{(1-\beta)} + \frac{\rho}{\alpha}\left(\sigma(t) - \sigma(s)\right)\right)^2, \qquad c = \frac{K^{2(1-\beta)}}{(1-\beta)^2\nu(t)},$$
$$b = 2 - \frac{1 - 2\beta - \rho^2(1-\beta)}{(1-\beta)(1-\rho^2)}, \qquad \nu(t) = (1-\rho^2)\int_s^t \sigma^2(z)\mathrm{d}z,$$

and $\chi^2(x; \delta, \lambda)$ is the non-central chi-square cumulative distribution function.

This formula is exact in the case of $\rho = 0$ and constitutes an *approximation* otherwise. Based on Equation (1), an *approximately* exact simulation of SABR model is feasible by inverting the conditional SABR cumulative distribution when the conditional integrated variance is known.

2.2 SABR Monte Carlo simulation

In order to apply an "almost exact" Monte Carlo method for the SABR model, several steps need to be performed, that are described in the following:

- *Simulation of the SABR volatility process, $\sigma(t)$ given $\sigma(s)$*:

$$\sigma(t) \sim \sigma(s)\exp(\alpha\hat{W}_\sigma(t-s) - \frac{1}{2}\alpha^2(t-s)).$$

- *Simulation of the SABR integrated variance process, $\int_s^t \sigma^2(z)\mathrm{d}z | \sigma(s), \sigma(t)$.*

- *Simulation of the SABR forward price process, $S(t) | S(s), \sigma(s), \sigma(t), \int_s^t \sigma^2(z)\mathrm{d}z$.*

The conditional integrated variance simulation is very challenging. In our work, we approximate the distribution of the integrated variance conditional on the volatility dynamics by joining the two involved marginal distributions, i.e. the SABR volatility and integrated variance distributions, by means of copula techniques. Once both marginal distributions are available, we use copulas to define a multivariate distribution which approximates the conditional distribution of the integrated variance given the stochastic volatility.

3 Copula-based simulation of $\int_s^t \sigma^2(z)\mathrm{d}z | \sigma(t), \sigma(s)$

In this section we summarize the algorithm for the simulation of the integrated variance given $\sigma(t)$ and $\sigma(s)$ by means of a copula. For simplicity, hereafter, we denote the SABR's integrated variance process by $Y(s,t) := \int_s^t \sigma^2(z)\mathrm{d}z$.

In order to obtain the copula joint distribution, we need the marginal distributions. In our case, the required CDFs are $F_{\log Y | \log \sigma(s)}$ and $F_{\log \sigma(t) | \log \sigma(s)}$ (in the log-space). While the second CDF is known ($\log \sigma(t)$ is normally distributed), an approximation of the CDF of the integrated variance, $F_{\log \hat{Y} | \log \sigma(s)}$, must be derived.

3.1 CDF of $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(s)$ using the COS method

We present a technique to approximate the CDF of $Y(s,t)|\sigma(s)$, i.e. $F_{Y|\sigma(s)}$. We will work in the log-space, so an approximated CDF of $\log Y(s,t)|\log\sigma(s)$, $F_{\log Y|\log\sigma(s)}$, will be estimated. We approximate $Y(s,t)$ by its discrete equivalent, i.e.

$$Y(s,t) := \int_s^t \sigma^2(z)\mathrm{d}z \approx \sum_{j=1}^{M} \Delta t \sigma^2(t_j) =: \hat{Y}(s,t) \tag{2}$$

where M is the number of intermediate or discrete time-points, $\Delta t = \frac{t-s}{M}$ and $t_j = s + j\Delta t$, $j = 1,\ldots,M$. $\hat{Y}(T)$ is subsequently transformed to the logarithmic domain, with

$$F_{\log\hat{Y}|\log\sigma(s)}(x) = \int_{-\infty}^{x} f_{\log\hat{Y}|\log\sigma(s)}(y)\mathrm{d}y, \tag{3}$$

and $f_{\log\hat{Y}|\log\sigma(s)}$ the *probability density function* (PDF) of $\log\hat{Y}(s,t)|\log\sigma(s)$.

Density $f_{\log\hat{Y}|\log\sigma(s)}$ is, in turn, found by approximating the associated characteristic function, $\phi_{\log\hat{Y}|\log\sigma(s)}$, and applying a Fourier inversion procedure. The characteristic function and the corresponding PDF form the so-called *Fourier pair*.

3.1.1 Recursive procedure to recover $\phi_{\log\hat{Y}|\log\sigma(s)}$

We have proposed a recursive procedure, as described in [8], originally employed to price arithmetic Asian options. This iterative method is based on the derivation of the characteristic function of the integrated variance process and Fourier techniques (COS method [2]) to recover the probability density function (PDF). We start by defining the sequence,

$$R_j = \log\left(\frac{\sigma^2(t_j)}{\sigma^2(t_{j-1})}\right), \quad j = 1,\ldots,M, \tag{4}$$

where R_j is the logarithmic increment of $\sigma^2(t)$ between t_j and t_{j-1}, $j = 1,\ldots,M$. As the volatility process follows log-normal dynamics, and increments of Brownian motion are independent and identically distributed, the R_j are also independent and identically distributed, i.e. $R_j \overset{\mathrm{d}}{=} R$. In addition, the characteristic function of R_j is well-known and reads, $\forall u, j$,

$$\phi_{R_j}(u) = \phi_R(u) = \exp(-iu\alpha^2\Delta t - 2u^2\alpha^2\Delta t), \tag{5}$$

with $i = \sqrt{-1}$ the imaginary unit. By the definition of R_j in Equation (4), we write $\sigma^2(t_j)$ as

$$\sigma^2(t_j) = \sigma^2(t_0)\exp(R_1 + R_2 + \cdots + R_j). \tag{6}$$

At this point, a backward recursion procedure in terms of R_j will be defined by which we can recover $\phi_{\log\hat{Y}|\log\sigma(s)}(u)$. We define

$$Y_1 = R_M, \quad Y_j = R_{M+1-j} + Z_{j-1}, \quad j = 2,\ldots,M. \tag{7}$$

with $Z_j = \log(1+\exp(Y_j))$.

By Equations (6) and (7), the discrete integrated variance can be expressed as

$$\hat{Y}(s,t) = \sum_{i=1}^{M} \sigma^2(t_i)\Delta t = \Delta t\sigma^2(s)\exp(Y_M). \tag{8}$$

From Equation (8) and by applying the definition of characteristic function, we determine $\phi_{\log\hat{Y}|\log\sigma(s)}$, as follows

$$\begin{aligned}\phi_{\log\hat{Y}|\log\sigma(s)}(u) &= \mathbb{E}[\exp(iu\log\hat{Y}(s,t))|\log\sigma(s)] \\ &= \exp\left(iu\log(\Delta t\sigma^2(s))\right)\mathbb{E}[\exp\left(iuY_M\right)|\log\sigma(s)] \\ &= \exp\left(iu\log(\Delta t\sigma^2(s))\right)\phi_{Y_M}(u).\end{aligned} \tag{9}$$

We have reduced the computation of $\phi_{\log\hat{Y}|\log\sigma(s)}$ to the computation of ϕ_{Y_M}. As Y_M is defined recursively, its characteristic function can be obtained by a recursion as well. In [6, 7], more details on the derivation of ϕ_{Y_M} are available.

3.1.2 Recovering $f_{\log \hat{Y}|\log \sigma(s)}$ by COS method

Once the approximation of ϕ_{Y_M}, $\hat{\phi}_{Y_M}$, has been efficiently derived, we can recover $f_{\log \hat{Y}|\log \sigma(s)}$ from $\phi_{\log \hat{Y}|\log \sigma(s)}$ by employing the COS method [2], as follows

$$f_{\log \hat{Y}|\log \sigma(s)}(x) \approx \frac{2}{\hat{b}-\hat{a}} \sum_{k=0}^{N-1'} C_k \cos\left((x-\hat{a})\frac{k\pi}{\hat{b}-\hat{a}}\right), \tag{10}$$

with

$$C_k = \Re\left(\phi_{\log \hat{Y}|\log \sigma(s)}\left(\frac{k\pi}{\hat{b}-\hat{a}}\right)\exp\left(-i\frac{\hat{a}k\pi}{\hat{b}-\hat{a}}\right)\right),$$

and

$$\begin{aligned}\phi_{\log \hat{Y}|\log \sigma(s)}\left(\frac{k\pi}{\hat{b}-\hat{a}}\right) &= \exp\left(i\frac{k\pi}{\hat{b}-\hat{a}}\log\left(\Delta t\sigma^2(s)\right)\right)\phi_{Y_M}\left(\frac{k\pi}{\hat{b}-\hat{a}}\right)\\ &\approx \exp\left(i\frac{k\pi}{b-a}\log\left(\Delta t\sigma^2(s)\right)\right)\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right),\end{aligned}$$

where N is the number of COS terms, $[\hat{a},\hat{b}]$ is the support[1] of $\log \hat{Y}(s,t)|\log\sigma(s)$ and the prime $'$ and $\Re$ symbols in Equation (10) mean division of the first term in the summation by two and taking the real part of the complex-valued expressions in the brackets, respectively. CDF $F_{\log \hat{Y}|\log \sigma(s)}$ can be obtained by integration, plugging the approximated $f_{\log \hat{Y}|\log \sigma(s)}$ from Equation (10) into Equation (3).

3.2 Pearson's correlation coefficient

For any copula some measure of the correlation between the marginal distributions is needed. In our case, we have employed the Pearson's correlation coefficient for $\log Y(s,t)$ and $\log\sigma(t)$. For this quantity, an approximated analytic formula can be derived (more details in [6, 7]). The approximated Pearson's correlation coefficient reads

$$\mathcal{P}_{\log Y,\log\sigma(t)} \approx \frac{t^2-s^2}{2\sqrt{\left(\frac{1}{3}t^4+\frac{2}{3}ts^3-t^2s^2\right)}}. \tag{11}$$

3.3 Copula simulation

We have approximations for the components required to apply our copula-based technique for the integrated variance simulation. The algorithm to sample $\int_s^t \sigma^2(z)\mathrm{d}z$ given $\sigma(t)$ and $\sigma(s)$ then consists of the following steps:

1. Determine $F_{\log\sigma(t)|\log\sigma(s)}$ (known analytically).
2. Determine $F_{\log \hat{Y}|\log\sigma(s)}$ by Equation (3).
3. Determine the correlation between $\log Y(s,t)$ and $\log\sigma(t)$ by Equation (11).
4. Generate correlated uniform samples, $U_{\log\sigma(t)|\log\sigma(s)}$ and $U_{\log \hat{Y}|\log\sigma(s)}$ from the chosen copula.
5. From $U_{\log\sigma(t)|\log\sigma(s)}$, invert $F_{\log\sigma(t)|\log\sigma(s)}$ to get the samples $\tilde{\sigma}_n$ of $\log\sigma(t)|\log\sigma(s)$.
6. From $U_{\log \hat{Y}|\log\sigma(s)}$, invert $F_{\log \hat{Y}|\log\sigma(s)}$ to get the samples $\tilde{y}_n$ of $\log \hat{Y}|\log\sigma(s)$.
7. The samples σ_n of $\sigma(t)|\sigma(s)$ and y_n of $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t),\sigma(s)$ are obtained by simply taking exponentials as
$$\sigma_n = \exp(\tilde{\sigma}_n), \quad y_n = \exp(\tilde{y}_n).$$

[1] It can be calculated given the support of Y_M, $[a,b]$.

	Gaussian	Student $\mathbf{t}$ ($\nu = 5$)	Gumbel
Set I	5.0323×10^{-3}	5.0242×10^{-3}	3.8063×10^{-3}
Set II	3.1049×10^{-3}	3.0659×10^{-3}	4.5703×10^{-3}
Set III	5.9439×10^{-3}	6.0041×10^{-3}	4.3210×10^{-3}

Table 1: Generic GOF. See [6], for more details on the parameter sets configuration.

4 Simulation of $S(t)$ given $S(s)$, $\sigma(s)$, $\sigma(t)$ and $\int_s^t \sigma^2(z)\mathrm{d}z$

We complete the mSABR method by the conditional sampling of $S(t)$. The most commonly used techniques can be classified in two categories: direct inversion of the SABR distribution function given in Equation (1) and discretization schemes. The direct inversion procedure has a higher computational cost because of the evaluation of the non-central χ^2 distribution. However some recent developments make this computation affordable. In [1], the authors proposed a forward asset simulation based on a combination of moment-matching (Quadratic Gaussian) and enhanced direct inversion procedures. A second approach is the use of advanced discretization schemes, such as the so-called *Log-Euler+*, which allows the incorporation of the conditional integrated variance term. It reads

$$S(t) = S(s)\exp\left(-\frac{1}{2}S^{2(\beta-1)}(s)\int_s^t \sigma^2(z)\mathrm{d}z + S^{\beta-1}(s)\frac{\rho}{\alpha}\left(\sigma(t)-\sigma(s)\right)\right.$$
$$\left. + S^{\beta-1}(s)\sqrt{1-\rho^2}\int_s^t \sigma^2(z)\mathrm{d}W_S(z)\right).$$

5 One time-step SABR simulation

The one-step SABR simulation is a particular case within our general framework. Our copula-based simulation for the conditional integrated variance becomes much simpler. Since at the initial time, $s = 0$ and the final time $t = T$, the process $\log(S)$ turns into a constant value and the marginal distributions involved in the copula are then $F_{\log\sigma(T)}$ and $F_{\log\hat{Y}(T)}$. The computation of the corresponding characteristic function, $\phi_{\log\hat{Y}(T)}$, is much easier than for the general multiple time-step approach, and very fast.

5.1 Copula analysis

In [6], a copula analysis based on the one-step SABR simulation was carried out. Several types of copulas were considered: Gaussian, Student $\mathbf{t}$ and Archimedean (Clayton, Frank and Gumbel). In order to select the most suitable one for our purposes, we have evaluated the so-called *goodness-of-fit (GOF)* of a copula, i.e. one test for the Archimedean copulas and another one for the general overall assessment (i.e. see Table 1). Resulting was that Gaussian and Archimedean Gumbel copulas perform best under our conditions. As a general strategy, the Gumbel copula is the most robust choice, but the Gaussian copula may be a satisfactory alternative for short maturities.

5.2 Numerical tests

The one-step SABR simulation is appropriate for pricing European options under SABR dynamics. In Table 2, the convergence of our method when the number of samples, n is increased is shown. Regarding the computational time, also in Table 2, the execution times of the one-step SABR method are shown. We can see that, up to $n = 100000$, the number of MC paths does not affect the performance.

According to these results and others presented in [6], we conclude that the one-step SABR simulation we have proposed is a fast alternative to the well-known Hagan formula. It overcomes the known issues of the formula, like in the case of low strikes and high volatility. When longer maturities or more complex products (not only European style option) are encountered, however, the multiple time-step version should be employed.

	$n = 1000$	$n = 10000$	$n = 100000$	$n = 1000000$
	Gaussian (Set I, X_1)			
Error	519.58(204.02)	132.39(68.03)	37.42(16.55)	16.23(7.66)
Time	0.3386	0.3440	0.3857	0.5733
	Gumbel (Set I, X_1)			
Error	151.44(199.36)	−123.76(86.33)	34.14(17.03)	11.59(6.58)
Time	0.3492	0.3561	0.3874	0.6663

Table 2: Convergence in n: mean and standard deviation of the error (basis points) and time (*sec.*). See [6], for more details on the parameter set configuration.

Samples	Without SCMC	With SCMC		
		$N_{\hat{Y}} = N_\sigma = 3$	$N_{\hat{Y}} = N_\sigma = 7$	$N_{\hat{Y}} = N_\sigma = 11$
100	1.0695	0.0449	0.0466	0.0660
10000	16.3483	0.0518	0.0588	0.0798
1000000	1624.3019	0.2648	0.5882	1.0940

Table 3: SCMC time in seconds.

6 Multiple time-step SABR simulation

By the Monte Carlo technique described in Section 3, we have obtained a *multiple time-step version* of our copula-based SABR simulation. In [7], we have presented the complete methodology, that we have called *mSABR simulation*. Unlike the one-step version, the direct application of the recursive procedure to derive a characteristic function for the integrated variance is more complicated and much more expensive in terms of computational cost. In any intermediate time step, the computation of characteristic function, $\phi_{\log \hat{Y}|\log\sigma(s)}$, (and, consequently, $F_{\log \hat{Y}|\log\sigma(s)}$) needs to be performed for each sample of $\log\sigma(s)$. This makes the conditional integrated variance sampling (that relies on the inversion of $F_{\log \hat{Y}|\log\sigma(s)}$) unafforable when the required number of samples is large. In order to overcome this issue, we have proposed the use of the *Stochastic Collocation Monte Carlo* (SCMC) sampler [3]. Briefly, the insight behind this method is that any number of samples can be obtained by inverting the distribution at hand in some predefined points (called *collocation points*) and then applying interpolation techniques (Lagrange interpolation in this case) together with the sampling of another *cheaper* distribution.

For the problem at hand, we require samples from the integrated variance conditional on the initial volatility, i.e. $\log \hat{Y}(s,t)|\log\sigma(s)$. Therefore, we need to make use of the *2D version of the SCMC technique*. Two levels of collocation points need to be chosen, one for each dimension. If we denote them by $N_{\hat{Y}}$ and N_σ, respectively, the resulting number of inversions equals $N_{\hat{Y}} \cdot N_\sigma$. The formal definition of the 2D SCMC technique applied to our context reads

$$y_n|v_n \approx g_{N_{\hat{Y}},N_\sigma}(x_n) = \sum_{i=1}^{N_{\hat{Y}}}\sum_{j=1}^{N_\sigma} F^{-1}_{\log \hat{Y}|\log\sigma(s)=\bar{v}_j}(F_X(\bar{x}_i))\ell_i(x_n)\ell_j(v_n),$$

where x_n are the samples from $X \sim \mathcal{N}(0,1)$, which is used as the cheap variable, and v_n the samples of $\log\sigma(s)$; $\bar{x}_i$ and $\bar{v}_j$ are the collocation points for approximating variables $\log Y$ and $\log\sigma(s)$, respectively. The ℓ_i and ℓ_j terms represent the Lagrange polynomials.

In order to illustrate how powerful the SCMC method can be in our case, in Table 3, the execution times for generating different numbers of samples are presented, with and without the use of SCMC and for several choices of collocation points.

6.1 Numerical tests

A wide range of experiments for the mSABR method were carried out (see [7]), including convergence in the number of samples, convergence in the number of time steps, stability in terms of the SABR parameter ρ and pricing of barrier options. In all of these experiments, our method

Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
Antonov	73.34%	71.73%	70.17%	N/A	67.23%	65.87%	64.59%
$m = T/4$	71.34%	69.98%	68.65%	67.36%	66.11%	64.92%	63.80%
Error(bp)	−199.96	−174.93	−151.81	N/A	−111.81	−94.64	−78.93
$m = T/2$	71.90%	70.41%	68.95%	67.54%	66.20%	64.91%	63.70%
Error(bp)	−143.63	−132.23	−121.75	N/A	−103.55	−95.89	−89.16
$m = T$	73.05%	71.46%	69.92%	68.45%	67.03%	65.68%	64.42%
Error(bp)	−28.72	−26.40	−24.28	N/A	−20.28	−18.61	−17.10
$m = 2T$	73.24%	71.62%	70.06%	68.55%	67.11%	65.74%	64.45%
Error(bp)	−10.21	−10.66	−11.05	N/A	−12.29	−13.03	−14.07
$m = 12T$	73.38%	71.76%	70.19%	68.69%	67.25%	65.88%	64.59%
Error(bp)	4.25	3.52	2.73	N/A	1.55	0.88	0.22

Table 4: Implied volatility, increasing m: Antonov vs. mSABR. Set I.

provides high accuracy using only a few time steps. As an example, in Table 4 the convergence in the number of time-steps (m) is presented. We can observe a very fast convergence with very few time-steps.

References

[1] Bin Chen, Cornelis W. Oosterlee, and Hans van der Weide. A low-bias simulation scheme for the SABR stochastic volatility model. *International Journal of Theoretical and Applied Finance*, 15(2):1250016–1 – 1250016–37, 2012. 5

[2] Fang Fang and Cornelis W. Oosterlee. A novel pricing method for European options based on Fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, 31:826–848, 2008. 3, 4

[3] Lech A. Grzelak, Jeroen A. S. Witteveen, M. Suárez-Taboada, and Cornelis W. Oosterlee. The stochastic collocation Monte Carlo sampler: highly efficient sampling from "expensive" distributions. *Quantitative Finance*, 19(2):339–356, 2019. 6

[4] Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, and Diana E. Woodward. Managing smile risk. *Wilmott Magazine*, pages 84–108, 2002. 1

[5] Othmane Islah. Solving SABR in exact form and unifying it with LIBOR market model, 2009. Available at SSRN: http://ssrn.com/abstract=1489428. 2

[6] Álvaro Leitao, Lech A. Grzelak, and Cornelis W. Oosterlee. On a one time-step Monte Carlo simulation approach of the SABR model: application to European options. *Applied Mathematics and Computation*, 293:461–479, 2017. 3, 4, 5, 6

[7] Álvaro Leitao, Lech A. Grzelak, and Cornelis W. Oosterlee. On an efficient multiple time step Monte Carlo simulation of the SABR model. *Quantitative Finance*, 17(10):1549–1565, 2017. 3, 4, 6

[8] Bowen Zhang and Cornelis W. Oosterlee. Efficient pricing of European-style Asian options under exponential Lévy processes based on Fourier cosine expansions. *SIAM Journal on Financial Mathematics*, 4(1):399–426, 2013. 3