Monte Carlo option pricing for tempered stable (CGMY) processes

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Premia 22

Abstract

Lévy processes are popular models for stock price behavior since they allow to take into account jump risk and reproduce the implied volatility smile. In this paper, we focus on the tempered stable (also known as CGMY) processes, which form a flexible 6-parameter family of Lévy processes with infinite jump intensity. It is shown that under an appropriate equivalent probability measure a tempered stable process becomes a stable process whose increments can be simulated exactly. This provides a fast Monte Carlo algorithm for computing the expectation of any functional of tempered stable process. We use our method to price European options and compare the results to a recent approximate simulation method for tempered stable process by Madan and Yor [14].

Keywords: Monte Carlo, Option pricing, Lévy process, Tempered stable process, CGMY model

1 Introduction

Lévy processes are stochastic processes with stationary and independent increments that is to say if $(X_t)_{t\geq 0}$ is a Lévy process, then $X_t - X_s$ with $t > s$ is independent of the history of the process up to time s, and its law only depends on $t - s$.

Exponentials of Lévy processes constitute the simplest model of stock price behavior allowing to take into account price discontinuities and the implied volatility smile phenomenon [7]. Despite this simplicity and unlike the Black-Scholes model, closed option pricing formulas are not available in exponential Lévy models and one must use either deterministic numerical methods (partial integro-differential equations [8] or Fourier methods [4]) or Monte Carlo methods which form the object of the present paper.

For Monte Carlo methods, finite-intensity (compound Poisson) Lévy processes can be easily simulated on a computer, but for infinite intensity processes the situation is more difficult: explicit simulation methods are only available for stable processes [5] and a few other parametric classes (see [7] for a comprehensive survey). For other processes, approximate methods are available [1, 11], which are basically reduced to truncating small jumps and replacing them with something easy to simulate. These methods are not adapted for Monte Carlo pricing of European options and other options with discrete observation of stock price trajectory since they require a complete simulation of the process trajectory even if only the terminal value is used for the final computation.

For the tempered stable process (also known as CGMY model), an approximate simulation method based on Brownian subordination and a rejection method by Rosinski [11] was recently proposed by Madan and Yor [14] but, to the best of our knowledge, no exact simulation method for the increments of this process is known. In section 3 of this paper, we propose a method for Monte Carlo evaluation of any functional of the tempered stable process which avoids direct simulation of the increments of this process. Instead, we construct an equivalent probability measure under which the original tempered stable process becomes a stable process. Since the method for direct simulation of stable random variables is well-known [5] and the measure change is explicit, this provides the desired algorithm.

After describing the theoretical framework of our method in section 3, we apply it to the pricing of European options in section 4 and compare the results with the ones obtained using the approximate simulation method by Madan and Yor. A strong point of the latter method is that it is easily generalizable to multiple dimensions via Brownian subordination, whereas in our setting a multidimensional extension is not straightforward. On the other

hand, an important advantage of our method is that it provides unbiased estimators of option prices, while the approximate method introduces a bias, due to the truncation of small jumps, which is not easy to quantify. More importantly, for the parameter sets used in this paper, our method required 16 to 50 times less computer time to achieve the same precision than the approximate method.

2 Basic tools and definitions

In this section we review important properties of Lévy processes, stable processes and tempered stable processes that are used in the sequel. Unless otherwise mentioned, all proofs can be found in Sato [13]. We concentrate on the one-dimensional case.

Lévy-Khintchine representation Let $h : \mathbb{R} \to \mathbb{R}$ be a measurable function such that for every z, $\int |e^{izx} - 1 - izh(x)|\nu(dx) < \infty$. Such a function h is called a truncation function. The Levy-Khintchine representation with the truncation function h takes the form:

$$E[e^{izX_t}] = e^{t\psi(z)}, z \in \mathbb{R}, \tag{2.1}$$

$$\psi(z) = -\frac{1}{2}zAz + i\gamma_h z + \int_{\mathbb{R}^d}(e^{izx} - 1 - izh(x))\nu(dx). \tag{2.2}$$

A and ν do not depend on the choice of h but γ_h depends on this choice. If γ is the value of γ_h for the standard truncation function $h(x) = x1_{|x|\leq 1}$, γ_h for arbitrary h can be computed with

$$\gamma_h = \gamma + \int_{-\infty}^{\infty}(h(x) - x\mathbf{1}_{|x|\leq 1})\nu(dx).$$

In the sequel, the value of γ_h corresponding to the truncation function $h \equiv 0$ (drift) will be denoted by γ_0 and the value corresponding to $h \equiv x$ (center) will be denoted by γ_c.

In the sequel, we will use the following result on exponential moments of Lévy processes (see [13, theorem 25.17]):

Proposition 2.1. *Let X be a Lévy process with characteristic triplet (A, ν, γ_h) with respect to a truncation function h and let $\lambda \in \mathbb{R}$. Then $E[e^{\lambda X_t}] < \infty$ for some $t > 0$ or equivalently for all $t > 0$ if and only if*

$$\int_{|x|\geq 1} e^{\lambda x}\nu(dx) < \infty.$$

In this case, $E[e^{\lambda X_t}] = e^{t\psi(-i\lambda)}$ with ψ as in (2.2).

Equivalent measure changes for Lévy processes

Proposition 2.2 (see [13, Theorems 33.1 and 33.2]). *Let (X_t, P) and (X_t, P') be two Lévy processes on $\mathbb{R}$ with characteristics triplets (A, ν, γ) and (A', ν', γ'). Then $P|_{\mathcal{F}_t}$ and $P'|_{\mathcal{F}_t}$ are equivalent for all t (or equivalently for one $t > 0$) if and only if the three following conditions are satisfied:*

1. *$A = A'$*
2. *The Lévy measures are equivalent with*

$$\int_{-\infty}^{\infty} (e^{\Phi(x)/2} - 1)^2 \nu(dx) < \infty$$

   where $\Phi(x) = \ln(\frac{d\nu'}{d\nu})$

3. *If $A = 0$ then we must in addition have*

$$\gamma'_h = \gamma_h + \int_{-\infty}^{\infty} h(x)(\nu' - \nu)(dx)$$

When P and P' are equivalent , the Radon-Nikodym derivative is

$$\frac{dP'}{dP}|_{\mathcal{F}_t} = e^{U_t}$$

with

$$U_t = \eta X_t^c - \frac{\eta^2 A t}{2} - \eta\gamma t + \lim_{\epsilon \to 0} \left(\sum_{|\Delta X_s| > \epsilon} \Phi(\Delta X_s) - t \int_{|x|>\epsilon} (e^{\Phi(x)} - 1)\nu(dx) \right)$$

Here (X_t^c) is the continuous part of (X_t), i.e. the Lévy process with generating triplet $(A, 0, \gamma)$ and η is such that

$$\gamma'_h - \gamma_h - \int_{-\infty}^{\infty} h(x)(\nu' - \nu)(dx) = A\eta \qquad if \quad A > 0$$

and zero if $A = 0$.

U_t is a Lévy process with characteristic triplet (A_U, ν_U, γ_U) given by:

$$a_U = \eta^2 \sigma^2$$

$$\nu_U = \nu \circ \Phi^{-1}$$

$$\gamma_U = -\frac{1}{2} A\eta^2 - \int_{-\infty}^{\infty} (e^y - 1 - h(y))(\nu\Phi^{-1})(dy)$$

Stable processes Real valued α-stable processes with $0 < \alpha < 2$ are Lévy processes with no continuous martingale part and Lévy measure of the form

$$\nu(x) = \frac{A}{|x|^{\alpha+1}} 1_{x>0} + \frac{B}{|x|^{\alpha+1}} 1_{x<0}.$$

If $1 < \alpha < 2$, the process has finite mean and its characteristic function has the following form:

$$\Phi_{X_t}(z) = \exp t \left\{ i\mu z + \int_{\mathbb{R}} (e^{izx} - 1 - izx)\nu(dx) \right\}$$

for some center $\mu \in \mathbb{R}$. In the case $0 < \alpha < 1$, the process has finite variation and we can write

$$\Phi_{X_t}(z) = \exp t \left\{ i\mu z + \int_{\mathbb{R}} (e^{izx} - 1)\nu(dx) \right\},$$

i.e., μ is now the drift. The characteristic function of a stable process may also be expressed as (see [13])

$$\begin{aligned} \Phi_{X_t}(z) &= \exp \left\{ -\sigma^\alpha |z|^\alpha t(1 - i\beta \operatorname{sgn} z \tan \frac{\pi\alpha}{2}) + i\mu z t \right\}, \quad \text{if } \alpha \neq 1, \\ \Phi_{X_t}(z) &= \exp \left\{ -\sigma |z| t(1 + i\beta \frac{2}{\pi} \operatorname{sgn} z \log |z|) + i\mu z t \right\}, \quad \text{if } \alpha = 1, \end{aligned} \tag{2.3}$$

where $\alpha \in (0, 2]$, $\sigma \geq 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$. A stable law with parameters $\alpha, \sigma, \beta, \nu$ is denoted by $S_\alpha(\sigma, \beta, \nu)$.

In the case $\alpha < 2$ the two parametrizations are linked by the following relations:

$$\begin{aligned} \sigma &= \left[-(A + B)\Gamma(-\alpha) \cos(\frac{\pi\alpha}{2}) \right]^{\frac{1}{\alpha}} \quad \text{when } \alpha \neq 1, \\ \sigma &= \frac{\pi}{2}(A + B) \quad \text{when } \alpha = 1, \\ \beta &= \frac{A - B}{A + B}. \end{aligned}$$

Furthermore, when $\alpha \neq 1$ and γ is the third parameter of the characteristic triplet of X for the standard truncation function $h(x) = x1_{|x|\leq 1}$ then

$$\mu = \gamma + \frac{B - A}{1 - \alpha}.$$

Simulation of stable random variables

Proposition 2.3 (see [5]). *Let Θ and W be independent with Θ uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and W exponentially distributed with mean 1, and let $0 < \alpha \leq 2$.*

- *The symmetric random variable*

$$Z = \begin{cases} \dfrac{\sin(\alpha\Theta)}{(\cos(\Theta))^{1/\alpha}} \left[\dfrac{cos(\alpha-1)\Theta}{W}\right]^{(1-\alpha)/\alpha}, & \alpha \neq 1 \\ \tan\Theta, & \alpha = 1 \end{cases}$$

  has a $S_\alpha(1,0,0)$ distribution.

- *In the nonsymmetric case, for any $-1 \leq \beta \leq 1$, define*

$$\theta_0 = \arctan(\beta \tan(\pi\alpha/2))/\alpha$$

  when $\alpha \neq 1$. Then

$$Z = \begin{cases} \dfrac{\sin(\alpha(\Theta+\theta_0))}{(\cos(\alpha\Theta_0)\cos(\Theta))^{1/\alpha}} \left[\dfrac{cos(\alpha\Theta_0+(\alpha-1)\Theta}{W}\right]^{(1-\alpha)/\alpha}, & \alpha \neq 1 \\ \dfrac{\pi}{2}\left[(\dfrac{\pi}{2}+\beta\Theta)\tan\Theta - \beta\log\left(\dfrac{\frac{\pi}{2}W\cos(\Theta)}{\frac{\pi}{2}+\beta\Theta}\right)\right], & \alpha = 1, \end{cases}$$

  has a $S_\alpha(1,\beta,0)$ distribution.

Stable random variates with $\sigma \neq 1$ and $\mu \neq 0$ may be obtained from $S_\alpha(1,\beta,0)$ by scaling and translation.

Tempered stable processes A one-dimensional *tempered stable process* is obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis. Thus, a tempered stable process is a Lévy process on $\mathbb{R}$ with no Gaussian component and Lévy density of the form:

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0},$$

with parameters satisfy $c_- > 0$, $c_+ > 0$, $\lambda_- > 0$, $\lambda_+ > 0$ and $0 < \alpha < 2$. For greater generality one can allow different values of α for positive and

negative half-lines and speak of a *generalized tempered stable process* with Lévy measure of the form

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} 1_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} 1_{x<0}.$$

Tempered stable processes have been studied by many authors including [2, 3, 6, 10] under different names. In particular, the version when $c_+ = c_-$ and $\alpha_+ = \alpha_-$ was studied in [3] under the name CGMY process with Lévy measure

$$\nu_{CGMY}(x) = C \left[\frac{e^{-Mx}}{x^{1+Y}} 1_{x>0} + \frac{e^{-G|x|}}{|x|^{1+Y}} 1_{x<0} \right].$$

A multidimensional generalization of tempered stable processes is proposed in [12].

Because of the exponential tempering in the case of the tempered stable process big jumps need not be truncated and one can use the truncation function $h(x) = x$. In the general case ($\alpha_\pm \neq 1$ and $\alpha_\pm \neq 0$) the characteristic exponent $\psi(u) = t^{-1} \log E[e^{iuX_t}]$ then becomes

$$\begin{aligned}\psi(u) = iu\gamma_c &+ \Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ \left\{ \left(1 - \frac{iu}{\lambda_+} \right)^{\alpha_+} - 1 + \frac{iu\alpha_+}{\lambda_+} \right\} \\ &+ \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_- \left\{ \left(1 + \frac{iu}{\lambda_-} \right)^{\alpha_-} - 1 - \frac{iu\alpha_-}{\lambda_-} \right\}. \quad (2.4)\end{aligned}$$

In the sequel we will also need the characteristic exponent of the tempered stable process for $0 < \alpha < 1$ with $h \equiv 0$ as truncation function:

$$\psi(u) = iu\gamma_0 + \Gamma(-\alpha)[c_+((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) + c_-((\lambda_- + iu)^\alpha - (\lambda_-)^\alpha)]. \quad (2.5)$$

3 Monte Carlo evaluation of functionals of CGMY process using measure change

The following theorem is the main result of this paper and shows that under an appropriate change of measure the tempered stable process becomes a sum of two one-sided stable processes.

Theorem 3.1. *Let (X_t) be a (generalized) tempered stable Lévy process on the probability space $(\Omega, \mathcal{F}, P)$ with Lévy density*

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} \mathbf{1}_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} \mathbf{1}_{x<0},$$

and let (X_t^+) *and* (X_t^-) *be tempered stable Lévy processes such that* $X = X^+ + X^-$ *with characteristic triplets* $(0, \nu_+, \gamma^+)$ *and* $(0, \nu_-, \gamma^-)$ *where*

$$\nu_+(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} 1_{x>0} \quad \textit{and} \quad \nu_-(x) = \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} 1_{x<0}.$$

Then the following holds:

1. *There exists a unique constant* c *such that* e^{U_t} *is a* P*-martingale, where* $U_t = \lambda_+ X_t^+ - \lambda_- X_t^- + ct$.
   c *is given by*[1]

   - *for* $0 < \alpha < 1$,

     $$c = -\lambda_+ \gamma_0^+ + \Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ + \lambda_- \gamma_0^- + \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_-.$$

   - *for* $1 < \alpha < 2$,

     $$c = -\lambda_+ \gamma_c^+ + \Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+(\alpha_+ - 1) + \lambda_- \gamma_c^- + \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_-(\alpha_- - 1).$$

2. *One can define a probability measure* Q *on* $(\Omega, \mathcal{F})$ *such that* $Q|_{\mathcal{F}_t} \sim P|_{\mathcal{F}_t}$ *for every* t *by* $\frac{dQ}{dP}|_{\mathcal{F}_t} = e^{U_t}$.

3. *Under* Q*, the processes* (X_t^+) *and* (X_t^-) *are stable processes with characteristic triplets* $(0, \tilde{\nu}_+, \tilde{\gamma}_+)$ *and* $(0, \tilde{\nu}_-, \tilde{\gamma}_-)$ *where*

   $$\tilde{\nu}_+(x) = \frac{c_+}{x^{1+\alpha_+}} 1_{x>0} \qquad \textit{and} \qquad \tilde{\nu}_-(x) = \frac{c_-}{|x|^{1+\alpha_-}} 1_{x<0}.$$

   and

   - *for* $0 < \alpha < 1$

     $$\tilde{\gamma}_0^+ = \gamma_0^+ \tag{3.1}$$
     $$\tilde{\gamma}_0^- = \gamma_0^- \tag{3.2}$$

   - *for* $1 < \alpha < 2$

     $$\tilde{\gamma}_c^+ = \gamma_c^+ + c_+ \int_0^\infty \frac{1 - e^{-\lambda_+ x}}{x^{\alpha_+}} dx = \gamma_c^+ - c_+ \lambda_+^{\alpha_+ - 1} \Gamma(1 - \alpha_+), \tag{3.3}$$
     $$\tilde{\gamma}_c^- = \gamma_c^- + c_- \int_{-\infty}^0 \frac{1 - e^{-\lambda_- |x|}}{|x|^{\alpha_-}} dx = \gamma_c^- - c_- \lambda_-^{\alpha_- - 1} \Gamma(1 - \alpha_-). \tag{3.4}$$

[1] Here and below $0 < \alpha < 1$ means that both $0 < \alpha_+ < 1$ and $0 < \alpha_- < 1$ and similarly for $1 < \alpha < 2$. The formulas for the other cases when, say $0 < \alpha_+ < 1$ and $1 < \alpha_- < 2$ are straightforward generalizations.

Remark 3.1. The processes X^+ and X^- are only unique up to a linear function: we can freely choose a constant k and put $X^{+'} = X^+ + kt$ and $X^{-'} = X^- - kt$. It is easy to see that the value of $-\lambda_+ X_t^+ + \lambda_- X_t^- - ct$ does not depend on k and therefore, the quantities that we want to compute, that is, expectations of functionals of X, do not depend on k.

Proof. 1. By proposition (2.1), $E[e^{\lambda_+ X_1^+}] < \infty$, $E[e^{-\lambda_- X_1^-}] < \infty$ and, since X^+ and X^- are independent,

$$E[e^{\lambda_+ X_1^+ - \lambda_- X_1^-}] = E[e^{\lambda_+ X_1^+}] E[e^{-\lambda_- X_1^-}] < \infty.$$

Let

$$c = -\ln E[e^{\lambda_+ X_1^+ - \lambda_- X_1^-}] \equiv -(\ln E[e^{\lambda_+ X_1^+}] + \ln E[e^{-\lambda_- X_1^-}]).$$

Since $\lambda_+ X_1^+ - \lambda_- X_1^-$ is a Lévy process,

$$e^{\lambda_+ X_t^+ - \lambda_- X_t^- + ct} = \frac{e^{\lambda_+ X_t^+ - \lambda_- X_t^-}}{E[\lambda_+ X_t^+ - \lambda_- X_t^-]}$$

is a martingale. The constant c can be computed as follows:

- If $0 < \alpha < 1$ we can use the truncation function $h \equiv 0$ and equation (2.5) yields

$$c = -\lambda_+ \gamma_0^+ + \Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ + \lambda_- \gamma_0^- + \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_-$$

- If $1 < \alpha < 2$ we can use the truncation function $h(x) = x$ and equation (2.4) yields:

$$c = -\lambda_+ \gamma_c^+ + \Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ (\alpha_+ - 1) + \lambda_- \gamma_c^- + \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_- (\alpha_- - 1)$$

2. Let

$$\Phi(x) = \lambda_+ x \mathbf{1}_{x>0} - \lambda_- x \mathbf{1}_{x<0}.$$

Since, as easily seen, $\int_{-\infty}^{\infty} (e^{\Phi(x)/2} - 1)^2) \nu(dx) < \infty$, by proposition 2.2, we can define a new probability $Q \sim P$ by:

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = e^{U_t}$$

with

$$
\begin{aligned}
U_t &= \lim_{\epsilon\to 0}\left(\sum_{|\Delta X_s|>\epsilon}\Phi(\Delta X_s) - t\int_{|x|>\epsilon}(e^{\Phi(x)}-1)\nu(dx)\right)\\
&= \lim_{\epsilon\to 0}\Bigg(\sum_{|\Delta X_s|>\epsilon}(\lambda_+\Delta X_s\mathbf{1}_{\Delta X_s>0} - \lambda_-\Delta X_s\mathbf{1}_{\Delta X_s<0})\\
&\qquad\qquad - t\int_{|x|>\epsilon}(e^{\lambda_+ x\mathbf{1}_{x>0}-\lambda_- x\mathbf{1}_{x<0}}-1)\nu(dx)\Bigg)\\
&= \lim_{\epsilon\to 0}\Big(\sum_{\Delta X_s>\epsilon}\lambda_+\Delta X_s - t\int_{x>\epsilon}(e^{\lambda_+ x}-1)\nu(dx)\Big)\\
&+ \lim_{\epsilon\to 0}\Big(\sum_{\Delta X_s<-\epsilon}(-\lambda_-)\Delta X_s - t\int_{x<-\epsilon}(e^{-\lambda_- x}-1)\nu(dx)\Big) \qquad (3.5)
\end{aligned}
$$

The first term in the right-hand side can be rewritten it as follows:

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\lambda_+\left(\sum_{\epsilon\le\Delta X_s}\Delta X_s - t\int_{\epsilon<x}h(x)\nu(dx)\right)\\
&\qquad\qquad - \lim_{\epsilon\to 0}\left(t\int_{x>\epsilon}(e^{\lambda_+ x}-1-\lambda_+ h(x))\nu(dx)\right)
\end{aligned}
$$

By the Lévy-Itô decomposition, the first term above equals

$$\lambda_+ X_t^+ - \lambda_+\gamma_h^+ t,$$

and by the Lévy-Khintchine representation (2.1), the second term gives

$$\ln E[e^{\lambda_+ X_1^+}] - \lambda_+\gamma_h^+.$$

Treating in the same way the second term in the right-hand side of (3.5) and assembling all terms, we finally get

$$U_t = \lambda_+ X_t^+ - \lambda_- X_t^- + ct.$$

3. By proposition (2.2), we can express the Lévy densities of X^+ and X^- under Q as follows:

$$
\begin{aligned}
\tilde{\nu}_+(dx) &= e^{\Phi(x)}\nu_+(dx) = e^{\lambda_+ x}\nu_+(dx) = \frac{c_+}{x^{1+\alpha_+}}1_{x>0},\\
\tilde{\nu}_-(dx) &= e^{\Phi(x)}\nu_-(dx) = e^{-\lambda_- x}\nu_-(dx) = \frac{c_-}{x^{1+\alpha_-}}1_{x<0}.
\end{aligned}
$$

Moreover, once again by proposition (2.2),

$$\tilde{\gamma}_h^+ - \gamma_h^+ = \int_{-\infty}^{\infty} h(x)(\tilde{\nu}_+ - \nu_+)(dx),$$

$$\tilde{\gamma}_h^- - \gamma_h^- = \int_{-\infty}^{\infty} h(x)(\tilde{\nu}_- - \nu_-)(dx).$$

Treating separately the cases $0 < \alpha < 1$ and $1 < \alpha < 2$, we obtain equations (3.1)–(3.4).

□

The expectation of any $\mathcal{F}_T$-measurable random variable H_T can be evaluated via

$$E^P[H_T] = E^Q[H_T e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT}].$$

In particular, if $H_T = f(X_T)$, then

$$E^P[H_T] = E^P[f(X_T)] = E^Q[f(X_T)e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT}].$$

The Monte Carlo estimator $\bar{H}$ of $E[H_T]$ is given by

$$\bar{H} = \frac{1}{N}\sum_{i=1}^{N} H_T^i \exp(-\lambda_+ X_T^{+,i} + \lambda_- X_T^{-,i} - cT),$$

where (X_T^i) for $i = 1, \ldots, N$ are independent realizations of X_T and H_T^i are corresponding realizations of H_T. We conclude this section with an upper bound for the variance of $\bar{H}$.

Proposition 3.2. *Suppose that in Theorem 3.1, (X_t) is such that $\alpha_+ \neq 1$ and $\alpha_- \neq 1$ and that the random variable H_T is bounded: $|H_T| \leq K$. Then:*

$$\begin{aligned}\sqrt{\mathrm{Var}^Q\,\bar{H}} &= \sqrt{\frac{1}{N}\,\mathrm{Var}^Q\left(H_T e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT}\right)} \\ &\leq \frac{K}{\sqrt{N}} \exp[\Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ T(2^{\alpha_+ - 1} - 1) + \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_- T(2^{\alpha_- - 1} - 1)].\end{aligned}$$

Proof. We have that

$$\begin{aligned}\mathrm{Var}^Q\, H_T e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT} &\leq E^Q[f(X_T)^2 \exp(-2\lambda_+ X_T^+ + 2\lambda_- X_T^- - 2cT)] \\ &\leq K^2 E^Q[\exp(-2\lambda_+ X_T^+ + 2\lambda_- X_T^- - 2cT)].\end{aligned}$$

But as X_T^+ and X_T^- are independent:

$$E^Q[\exp(-2\lambda_+ X_T^+ + 2\lambda_- X_T^- - 2cT)] = e^{-2cT} E^Q[\exp(-2\lambda_+ X_T^+)] E^Q[\exp(2\lambda_- X_T^-)].$$

Moreover:

$$\begin{aligned} E^Q[\exp(-2\lambda_+ X_T^+)] &= E^P[\exp(-\lambda_+ X_T^+ + \lambda_- X_T^- + cT)] \\ &= e^{cT} E^P[\exp(-\lambda_+ X_T^+)] E^P[\exp(-\lambda_- X_T^-)] \end{aligned}$$

and

$$E^Q[\exp(2\lambda_- X_T^-)] = e^{cT} E^P[\exp(\lambda_- X_T^-)] E^P[\exp(\lambda_+ X_T^+)].$$

Thus

$$\begin{aligned} &E^Q[\exp(-2\lambda_+ X_T^+ + 2\lambda_- X_T^- - 2cT)] \\ &\quad = E^P[\exp(-\lambda_+ X_T^+)] E^P[\exp(\lambda_- X_T^-)] E^P[\exp(-\lambda_- X_T^-)] E^P[\exp(\lambda_+ X_T^+)] \\ &\quad = \exp(T(\psi_+(i\lambda_+) + \psi_+(-i\lambda_+) + \psi_-(i\lambda_-) + \psi_-(-i\lambda_-))) \\ &\quad = \exp\left(\Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ T(2^{\alpha_+} - 2) + \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_- T(2^{\alpha_-} - 2)\right). \end{aligned}$$

□

4 Numerical results.

Overview of the direct simulation method Let us briefly review the approximate simulation method for the tempered stable process, introduced in [14]. This method applies to the 4-parameter CGMY family (tempered stable process with $c_- = c_+ = c$ and $\alpha_- = \alpha_+ = \alpha$). It is based on the fact that such a process can be represented as a time-changed Brownian motion with drift. Since some of the constants are not given explicitly in [14], we provide a short proof of this result here.

Proposition 4.1. *Let $c > 0$, $\alpha \in (0, 2)$, $\lambda_+ > 0$ and $\lambda_- > 0$ and let Z be a subordinator with zero drift and Lévy density*

$$\nu_Z(t) = \frac{c e^{\frac{t}{2}A^2 - \frac{t}{4}B^2} D_{-\alpha}(B\sqrt{t})}{t^{\frac{\alpha}{2}+1}},$$

where D is the parabolic cylinder function (see [9]), $A = \frac{\lambda_- - \lambda_+}{2}$ and $B = \frac{\lambda_+ + \lambda_-}{2}$. Then the process

$$X_t = AZ_t + W(Z_t),$$

where W is a standard Brownian motion, is a tempered stable process with Lévy density

$$\nu(x) = \frac{c e^{-\lambda_+ x} 1_{x>0} + c e^{-\lambda_- |x|} 1_{x<0}}{|x|^{1+\alpha}} \tag{4.1}$$

and center

$$\gamma_c = \int_{\mathbb{R}} x(1 - e^{-Ax})\nu(x)dx.$$

In the case $0 < \alpha < 1$*, the drift of* X *is equal to zero.*

Remark 4.1. In the case $\alpha \neq 1$ the center is given explicitly by

$$\gamma_c = c\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}).$$

Proof. By [13, Theorem 30.1], the Lévy density of X is given by

$$\nu_X(x) = \int_0^\infty p_t(x)\nu_Z(t)dt.$$

where

$$p_t(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-At)^2}{2t}}.$$

This integral corresponds to integral 7.728 in [9, page 837] and is equal to (4.1).

Once again, by [13, Theorem 30.1], the center of X (third component of the characteristic triplet with respect to the truncation function $h(x) \equiv x$) is given by

$$\int_0^\infty \nu_Z(t)dt \int_{\mathbb{R}} xp_t(x)dx = A\int_0^\infty t\nu_Z(t)dt.$$

On the other hand, by Fubini theorem,

$$\begin{aligned}\int_{\mathbb{R}} x(1-e^{-Ax})\nu(x)dx &= \int_{\mathbb{R}} x(1-e^{-Ax})\int_0^\infty p_t(x)\nu_Z(t)dtdx \\ &= \int_0^\infty \nu_Z(t)dt \int_{\mathbb{R}} x(1-e^{-Ax})p_t(x)dx = A\int_0^\infty t\nu_Z(t)dt.\end{aligned}$$

The last statement of the theorem follows directly from [13, Theorem 30.1]. □

The paper [14] further shows that the density of the time change Z can be written as

$$\nu_Z(t) = f(t)\nu^0(t),$$

where $f(t) \leq 1$ and $\nu^0(t)$ is the Lévy density of a $\frac{\alpha}{2}$-stable subordinator:

$$\begin{aligned}f(t) &= \frac{2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2}+\frac{1}{2})e^{\frac{t}{2}A^2 - \frac{t}{4}B^2}}{\sqrt{\pi}} D_{-\alpha}(B\sqrt{t}), \\ \nu_0(t) &= 2^{-\frac{\alpha}{2}}\sqrt{\pi}\frac{\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}}{\Gamma(\frac{\alpha}{2}+\frac{1}{2})}\frac{c}{t^{1+\frac{\alpha}{2}}}1_{t>0} \equiv \frac{K}{t^{1+\frac{\alpha}{2}}}1_{t>0}.\end{aligned}$$

This allows to simulate paths of Z on the interval $[0, T]$ using Rosinski [11] rejection method as follows

Description of the algorithm

1. Approximate a trajectory of a stable subordinator with Lévy density ν_0 on $[0, T]$. The approximation consists in replacing this subordinator with a compound Poisson subordinator with Lévy density

$$\frac{K}{t^{1+\frac{\alpha}{2}}} 1_{t>\varepsilon}$$

and drift

$$d = \int_0^\varepsilon y \frac{K}{y^{\frac{\alpha}{2}+1}} dy = \frac{K\varepsilon^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}.$$

This amounts to replacing small jumps with their expectation. Note that an approximation bias is introduced at this stage. Denote by (T_i) the jump times and by (y_i) the jump sizes of the compound Poisson approximation (we do not give details since their simulation is straightforward).

2. Approximate the trajectory of the CGMY time change Z using the rejection method. In this setting this amounts to accept every jump y_i of the stable subordinator for which $f(y_i)$ is greater than an independent uniform random variable on $[0, 1]$. The approximated time change is given by

$$\hat{Z}_t = td + \sum_i y_i 1_{\Gamma_i \leq t} 1_{f(y_i)>U_i},$$

where (U_i) is an independent sequence of uniforms on $[0, 1]$.

3. Approximate the CGMY process by

$$\hat{X}_t = A\hat{Z}_t + W(\hat{Z}_t),$$

where W is a standard Brownian motion.

Financial model setup To compare the performance of the algorithm introduced in section 3 with that of the approximate simulation algorithm of [14], we use both methods to compute the prices of European put options in an exponential Lévy model driven by a tempered stable process. The European options were chosen, because their reference prices can be computed analytically which allows to estimate the bias in the approximate simulation method. In this model, the stock price is given by

$$S_t = S_0 e^{rt+X_t},$$

Strike	Price
80	1.7444
85	2.3926
90	3.2835
95	4.5366
100	6.3711
105	9.1430
110	12.7631
115	16.8429
120	21.1855

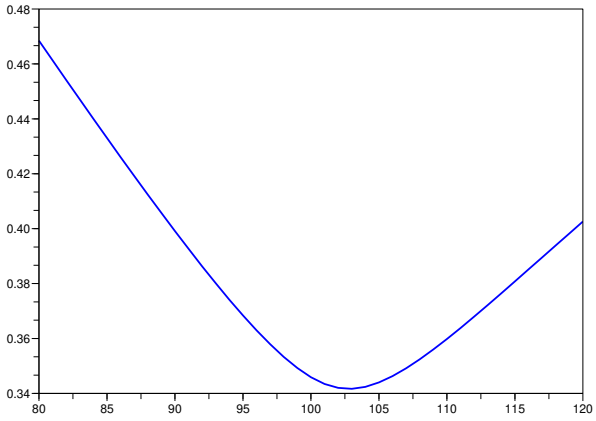

Table 1: Reference option prices computed using the Fourier transform method and the corresponding implied volatilities.

where X is a tempered stable process and r is the interest rate supposed constant. The European put option is a contract which pays to its holder $(K-S_T)^+$ at a future date T, where K is a fixed amount (strike). By the fundamental theorem of asset pricing, absence of arbitrage in a financial market is equivalent to the existence of a probability Q equivalent to the historical probability P, such that discounted prices of all assets are Q-martingales. In this paper we place ourselves directly under the probability Q. The process X must therefore satisfy the martingale condition $E[e^{X_t}] = 1$ and the price of a European put option at time $t = 0$ is

$$P(T, K) = e^{-rT} E^Q[(K - S_T)^+].$$

4.1 Results and comparisons

For numerical computations, we used the following artificial parameters of the tempered stable process: $\alpha = 0.5$, $c = 0.5$, $\lambda_+ = 3.5$ and $\lambda_- = 2$. The initial stock price was taken equal to 100, the option's maturity was $T = 0.25$, and the interest rate was 4%. The first step was to compute the reference prices using the Fourier transform method, described in [7]. The prices are given in table 1. The implied volatility smile corresponding to these prices is shown in the graph next to this table. This is a typical shape of implied volatility observed for options on stocks.

Next we study the bias of the approximate simulation method. To reduce

ε	**Price (std.dev)**	**Bias (std.dev)**	Time (10^7 traj.)
10^{-1}	8.4608(0.004)	2.090(0.004)	61 sec
10^{-2}	6.5834(0.004)	0.2123(0.004)	91 sec
10^{-3}	6.3784(0.004)	0.0073(0.004)	144 sec
10^{-4}	6.3658(0.004)	-0.0053(0.004)	233 sec

Table 2: Bias of the approximate simulation method.

Strike	**Price**	**Std. dev.**
80	1.745	0.007
85	2.378	0.008
90	3.278	0.009
95	4.544	0.011
100	6.349	0.012
105	9.131	0.013
110	12.765	0.014
115	16.855	0.015
120	21.192	0.016

Table 3: Option prices computed using the approximate simulation method. The computation time is about 23 seconds for each price.

the number of simulations, we only treat the strike $K = 100$. The results are shown in table 2. We see that for the parameter values used, the bias decreases rapidly to zero as $\varepsilon \to 0$. Moreover, since the computational complexity is proportional to $\frac{1}{\varepsilon^{\alpha/2}}$, decreasing ε by a factor of 10 reduces the bias by roughly the same factor but will only increase the complexity by a factor of $10^{1/4} \approx 1.77$. Therefore, in this case it is sufficient to take $\varepsilon = 10^{-4}$ and the Monte Carlo error will play the dominant role.

Armed with this knowledge, we set $\varepsilon = 10^{-4}$, the level at which the bias is insignificant for 10^7 trajectories, and compute the option prices by Monte Carlo using the approximate simulation method over 10^6 trajectories. This number of trajectories allows to estimate option prices with a precision of about one cent. The prices, standard deviations, and computation times are shown in table 3.

Next, we use the Monte Carlo method described in section 3 to compute the same prices. With 10^6 trajectories we obtain roughly the same variance for strike $K = 100$ as with the approximate method, whereas the computational time is reduced by a factor of 16. The results are shown in table 4.

Strike	Price	Std. dev.	Theor.bound
80	1.751	0.005	0.12
85	2.403	0.006	0.13
90	3.299	0.008	0.14
95	4.560	0.010	0.15
100	6.397	0.012	0.15
105	9.168	0.015	0.16
110	12.789	0.019	0.17
115	16.867	0.023	0.18
120	21.207	0.028	0.18

Table 4: Option prices computed using the method of section 3. The theoretical bound is that given in proposition 3.2. The computation time for one price is about 1.43 seconds. The errors are all of the same sign compared to table 1 because they have been computed over the same set of trajectories.

We also compute the theoretical bound on variance, given by proposition 3.2. The theoretical bound is quite far from the true variance since the option pay-off is estimated from above by the strike value (a very rough estimate).

As our last example, we compare the two algorithms in an infinite variation model with parameters $\alpha = 1.5$, $c = 0.1$, $\lambda_+ = 3.5$, $\lambda_- = 2$. This corresponds to roughly the same unit variance as in the previous case. We fix the strike equal to 100 and leave all the other parameters unchanged. The true price of the option (computed by Fourier transform) is equal to 8.3014 in this case.

First we analyze the bias of the approximate simulation algorithm. The results are shown in table 5. In this case, the computational time is much more sensitive to ε and the bias is, roughly, inversely proportional to the computational effort.

With the method of section 3, the standard deviation of about 0.011 is obtained for 1.8×10^6 trajectories, which requires about 2.62 seconds of machine time. Therefore, in this case our algorithm is about 50 times faster than approximate simulation, for the same precision.

To conclude, let us sum up the advantages and drawbacks of the two algorithms. First of all, we want to emphasize the fundamental difference between the two approaches: whereas the method of [14] is a simulation algorithm, our method is an algorithm for evaluating expectations, it cannot be used to, say, plot a trajectory of the tempered stable process for given parameter values.

The approximate simulation method by [14]

ε	**Price (std.dev)**	**Bias (std.dev)**	**Time (10^6 traj.)**
10^{-1}	10.081(0.013)	1.780(0.013)	1.5 sec
10^{-2}	8.683(0.012)	0.381(0.012)	5.17 sec
10^{-3}	8.368(0.011)	0.067(0.011)	24 sec
10^{-4}	8.304(0.011)	0.002(0.011)	132 sec

Table 5: Approximation bias, infinite variation case.

- Can be easily generalized to multiple dimensions by taking a multidimensional Brownian motion.
- Provides access to the entire trajectory of the process.
- Involves an approximation bias which may be important and is difficult to quantify.

On the other hand, our algorithm presented in section 3

- Is more general (in one dimension) in the sense that we allow different values of α_+ and α_- and different values of c_+ and c_-, which is not possible with Brownian subordination.
- Provides unbiased Monte Carlo estimates.
- Runs 16 to 50 times faster (for pricing European options).

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