Computation of risk measures for variable annuities with additional earnings

Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371

Xiao Wei *
China Institute for Actuarial Science & School of Insurance
Central University of Finance and Economics
39 South College Road, Haidian District
Beijing 100081
P.R. China

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Abstract

An approximation scheme is proposed for the computation of the risk measures of Guaranteed Minimum Maturity Benefits (GMMBs) and Guaranteed Minimum Death Benefits (GMDBs) with additional earnings, based on the evaluation of single integrals under conditional moment matching. This procedure is computationally efficient and recovers the numerical results obtained by standard analytical methods in the absence of additional earnings.

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Key words: Variable annuity guaranteed benefits; risk measures; value at risk; conditional tail expectation; conditional moment matching; additional earnings.

*Corresponding author, weixiao@cufe.edu.cn

1 Introduction

Variable annuity benefits offered by insurance companies are usually protected via different mechanisms such as Guaranteed Minimum Maturity Benefits (GMMBs) or Guaranteed Minimum Death Benefits (GMDBs). The computation of the corresponding risk measures such as value at risk and conditional tail expectation is an important issue for the practitioners in risk management.

We work in the standard model in which the underlying equity value $(S_t)_{t\in\mathbb{R}_+}$ is modeled as a geometric Brownian motion

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B_t}, \qquad t \in \mathbb{R}_+, \tag{1.1}$$

with constant drift and volatility parameters μ and σ respectively, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Given an insurer continuously charging annualized mortality and expense fees at the rate m from the account of variable annuities, the fund value A_t of the variable annuity is defined as

$$A_t := A_0 \mathrm{e}^{-mt} \frac{S_t}{S_0} = A_0 \mathrm{e}^{(\mu - m)t + \sigma B_t}, \qquad t \in \mathbb{R}_+,$$

and the margin offset income M_t^x is given by

$$M_t^x := m_x A_t = m_x A_0 \mathrm{e}^{(\mu - m)t + \sigma B_t}, \qquad t \in \mathbb{R}_+, \tag{1.2}$$

where m_x is replaced by m_e in the GMMB model, and by m_d in the GMDB model.

The GMMB and GMDB riders provide minimum guarantees to protect the investment account of the policyholder. Namely, denoting by τ_x the future lifetime of a policyholder at the age x, the future payment made by the insurer is $(GA_0 - A_T)^+ \mathbb{1}_{\{\tau_x > T\}}$ at maturity T for GMMBs, and $(\mathrm{e}^{\delta\tau_x} GA_0 - A_{\tau_x})^+ \mathbb{1}_{\{\tau_x \leqslant T\}}$ at the time of death of the insured for GMDBs, where GA_0 is the guarantee level expressed as a fraction G of the initial fund value A_0, δ is a roll-up rate according to which the guarantee increases up to the payment time.

Variable Annuities with embedded guarantees can be priced by the Monte-Carlo method or PDE discretization, however those methods are generally computationally demanding and a precise estimation of risk measures is difficult with classical Monte Carlo simulation or grid approximation, cf. e.g. [BKR08] for a general framework.

In this framework, the evaluation of quantile risk measures and conditional tail expectations of the net liabilities

$$L_0 := \mathrm{e}^{-rT}(GA_0 - A_T)^+ \mathbb{1}_{\{\tau_x > T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^e \mathrm{d}s \tag{1.3}$$

of GMMBs relies on the knowledge of the probability density function of the time integral $\int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^e \mathrm{d}s$ of the geometric Brownian motion (1.2).

The marginal probability density of $\int_0^T S_t dt$, called the Hartman-Watson distribution, has been used in [FV12] for the evaluation of the risk measures of the net liabilities (1.3) by analytic methods. It allowed the authors to deal with the risk measures of the net liabilities

$$L_0' := \mathrm{e}^{-r\tau_x}(\mathrm{e}^{\delta\tau_x} GA_0 - A_{\tau_x})^+ \mathbb{1}_{\{\tau_x \leqslant T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^d \mathrm{d}s$$

of GMDBs, also written in discrete time as

$$L_0^{(n)} := \mathrm{e}^{-r\kappa_x^{(n)}} \left(\mathrm{e}^{\delta\kappa_x^{(n)}} GA_0 - A_{\kappa_x^{(n)}}\right)^+ \mathbb{1}_{\{\kappa_x^{(n)} \leqslant T\}} - \int_0^{T\wedge\kappa_x^{(n)}} \mathrm{e}^{-rs} M_s^d \mathrm{d}s,$$

when n is large enough, where $\kappa_x^{(n)} := \frac{1}{n}\lceil n\tau_x \rceil$ and $\lceil a \rceil$ is the integer ceiling of $a \geqslant 0$. More computationally efficient expressions for those risk measures have been obtained in [FV14] based on identities in law for the geometric Brownian motion with affine drift

$$S_t + a\int_0^t \frac{S_t}{S_s} ds, \qquad t \in \mathbb{R}_+,$$

where $a > 0$.

Here we propose to use moment matching for the computation of the risk measures of GMMBs and GMDBs. This allows us to derive single integral approximations which are significantly faster than the double integral expressions of [FV12], while approaching the performance of the single integral and series approximations of [FV14]. Moreover, we show that conditional moment matching can be applied to compute the risk measures of the GMDB and GMMB riders with Additional Earnings (AE), which cannot be treated via the approach of [FV14]. For this, we apply the stratified approximation method of [PY16] to GMDBs and GMMBs, which also allows us to take into account additional earning features as it is based on conditioning with respect to the terminal value of geometric Brownian motion.

2 GMMBs with additional earnings

In order to reduce incentives to lapse and reenter of the variable annuities, an Additional Earnings (AE) feature has been added to the basic riders, by increasing the benefit payout by a share ρ of the policyholder's variable annuities earnings, capped by the maximum additional payout C, cf. e.g. [MZ16] for details. Taking $\rho = 0$ recovers the plain GMMB and GMDB riders.

For a GMMB rider with AE feature, an extra payment

$$\min\left(C, \rho(A_T - GA_0)^+\right)$$

will be paid to the GMMB policyholder in addition to the guaranteed benefit, thus the net liability (1.3) of the GMMB rider with AE feature becomes

$$L_0 := \left(\mathrm{e}^{-rT}(GA_0 - A_T)^+ + \mathrm{e}^{-rT}\min\left(C, \rho(A_T - GA_0)^+\right)\right)\mathbb{1}_{\{\tau_x > T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^e \mathrm{d}s.$$

Risk measures on the net liability L_0 can still be expressed in terms of Hartmann-Watson distributions and double integral expressions as in [FV12], using the joint distribution of $\left(S_T, \int_0^T S_t ds\right)$, cf. [Yor92]. The closed form expressions of [FV14] do not apply to this setting as they rely on the particular distributional properties of geometric Brownian motion with affine drift. Here we propose to use conditional moment matching in order to deal with additional earnings while significantly improving computation speed in comparison with double integral expressions.

We do not consider negative liabilities, and restrict the risk tolerance level α to be greater than the probability ξ_m of non-positive liability, which is defined for GMMBs as

$$\xi_m := \mathbb{P}(L_0 \leqslant 0) = 1 - {}_Tp_x \mathbb{P}(L_0 > 0 \mid \tau_x > T) = 1 - {}_Tp_x P_\rho(T, G, 0),$$

where ${}_Tp_x$ is the probability that a policyholder at age x will survive T units of time, $x, T > 0$, and for $w \geqslant 0$, the key quantity $P_\rho(T, G, w)$ is defined as

$$P_\rho(T, G, w) := \mathbb{P}\left(\mathrm{e}^{-rT}(GA_0 - A_T)^+ + \mathrm{e}^{-rT} \min\left(C, \rho(A_T - GA_0)^+\right) - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s > w\right). \tag{2.1}$$

Value at Risk for GMMBs

The Value at Risk (VaR)

$$V_\alpha(L_0) := \inf\left\{y \ : \ \mathbb{P}(L_0 \leqslant y) \geqslant \alpha\right\}$$

with risk tolerance level $\alpha > \xi_m$ for the net liability L_0 of GMMB is determined implicitly from the relation

$$1 - \alpha = {}_Tp_x P_\rho(T, G, V_\alpha(L_0)). \tag{2.2}$$

Conditional Tail Expectation for GMMBs

The Conditional Tail Expectation (CTE)

$$\mathrm{CTE}_\alpha(L_0) := \mathbb{E}[L_0 \mid L_0 > V_\alpha(L_0)]$$

at the level of risk tolerance level $\alpha > \xi_m$ for the net liability L_0 of the GMMB with AE feature is given by

$$\mathrm{CTE}_\alpha(L_0) = \frac{{}_Tp_x}{1 - \alpha} Z_\rho(T, G, V_\alpha(L_0)), \tag{2.3}$$

where

$$Z_\rho(T, G, w) := \mathbb{E}\left[\left(\mathrm{e}^{-rT}(GA_0 - A_T)^+ + \mathrm{e}^{-rT} \min(C, \rho(A_T - GA_0)^+) - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s\right) \mathbb{1}_{E_T(w,G)}\right], \tag{2.4}$$

$w, T \geqslant 0$, and $\mathbb{1}_{E_T(w,G)}$ is the indicator function of the event

$$E_T(w, G) := \left\{\mathrm{e}^{-rT}(GA_0 - A_T)^+ + \mathrm{e}^{-rT} \min(C, \rho(A_T - GA_0)^+) - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s > w\right\}.$$

3 GMDBs with additional earnings

In the case of GMDBs the extra payment is $\min\Big(C, \rho(A_{\tau_x} - GA_0 e^{\delta\tau_x})^+\Big)$ and the net liability of the GMDB rider with AE feature becomes

$$L_0^{'} := e^{-r\tau_x}\left(\left(e^{\delta\tau_x}GA_0 - A_{\tau_x}\right)^+ + \min\left(C, \rho(A_{\tau_x} - GA_0 e^{\delta\tau_x})^+\right)\right)\mathbb{1}_{\{\tau_x \leqslant T\}} - \int_0^{T\wedge\tau_x} e^{-rs} M_s^d \mathrm{d}s.$$

If the benefits of GMDBs with AE feature are payable on a discrete-time basis, their net liability is

$$\begin{aligned} L_0^{(n)} : &= e^{-r\kappa_x^{(n)}}\left(\left(e^{\delta\kappa_x^{(n)}}GA_0 - A_{\kappa_x^{(n)}}\right)^+ + \min\left(\rho\Big(A_{\kappa_x^{(n)}} - GA_0 e^{\delta\kappa_x^{(n)}}\Big)^+, C\right)\right)\mathbb{1}_{\{\kappa_x^{(n)} \leqslant T\}} \\ &\quad - \int_0^{T\wedge\kappa_x^{(n)}} e^{-rs} M_s^d \mathrm{d}s. \end{aligned}$$

The probability of non-positive liability for GMDB riders with AE feature is given by

$$\xi_d := \mathbb{P}(L_0^{(n)} \leqslant 0) = 1 - \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \; {}_{1/n}q_{x+(k-1)/n} P_\rho(k/n, e^{\delta k/n}G, 0),$$

where $P_\rho(k/n, e^{\delta k/n}G, w)$ is defined in (2.1), and ${}_{1/n}q_{x+(k-1)/n}$ is the probability that a policyholder at age of $x + (k-1)/n$ will die in $1/n$ periods.

Value at Risk for GMDBs

The value at risk $V_\alpha\Big(L_0^{(n)}\Big)$ with $\alpha > \xi_d$ for the net liability of the GMDB is similarly given implicitly from the relation

$$1 - \alpha = \sum_{k=1}^{\lceil nT \rceil} {}_{(k-1)/n}p_x \; {}_{1/n}q_{x+(k-1)/n} P_\rho\Big(k/n, e^{\delta k/n}G, V_\alpha\Big(L_0^{(n)}\Big)\Big), \tag{3.1}$$

cf. e.g. Proposition 3.9 of [FV12] when $\rho = 0$.

Conditional Tail Expectation for GMDBs

The conditional tail expectation

$$\mathrm{CTE}_\alpha\Big(L_0^{(n)}\Big) := \mathbb{E}\left[L_0^{(n)} \;\Big|\; L_0^{(n)} > V_\alpha\Big(L_0^{(n)}\Big)\right]$$

with risk tolerance level $\alpha > \xi_d$ for the net liability $L_0^{(n)}$ of the GMDB with AE feature is given by

$$\mathrm{CTE}_\alpha\Big(L_0^{(n)}\Big) = \frac{1}{1-\alpha}\sum_{k=1}^{\lceil nT \rceil} Z_\rho\Big(k/n, Ge^{k\delta/n}, V_\alpha(L_0^{(n)})\Big)\mathbb{P}\Big(\kappa_x^{(n)} = k/n\Big), \tag{3.2}$$

where $Z_\rho\Big(k/n, e^{k\delta/n}G, V_\alpha(L_0^{(n)})\Big)$ is defined by (2.4) for any $k, n \geqslant 0$.

4 Conditional moment matching

In this section we propose a conditional moment matching approximation for the estimation of the key quantities $P_\rho(T,G,w)$ and $Z_\rho(T,G,w)$ by approaching the probability density function of the time integral

$$\Lambda_T := \int_0^T \tilde{S}_t \mathrm{d}t = \frac{1}{A_0 m_x}\int_0^T \mathrm{e}^{-rt} M_t^x \mathrm{d}t$$

where $\tilde{S}_t := \mathrm{e}^{(\mu-m-r)t+\sigma B_t}$, $t \in \mathbb{R}_+$, using a lognormal distribution, conditionally to the terminal value $\tilde{S}_T = z$, as in [PY16]. We approximate the conditional probability density of Λ_T given $\tilde{S}_T = z$ by the lognormal density function with parameters $(-\mu_T^z(\sigma_T^z)^2T/2, (\sigma_T^z)^2T)$ as

$$f_{\Lambda_T|\tilde{S}_T=z}(x;\mu_T^z,(\sigma_T^z)^2) \approx \frac{1}{x\sigma_T^z\sqrt{2\pi T}}\mathrm{e}^{-(\mu_T^z(\sigma_T^z)^2T/2+\log x)^2/(2(\sigma_T^z)^2T)}, \tag{4.1}$$

where μ_T^z and σ_T^z are also derived by conditional moment matching by taking

$$(\sigma_T^z)^2 := \frac{1}{T}\log\left(\frac{2}{\sigma^2 a_T^z}\left(\frac{b_T^z}{a_T^z}-1-z\right)\right) \quad \text{and} \quad \mu_T^z := 1-\frac{2}{(\sigma_T^z)^2T}\log a_T^z,$$

cf. Proposition 3.2 of [PY16]. In Proposition 4.1 we use the lognormal approximation (4.1) to evaluate the key quantity $P_\rho(T,G,w)$ used in the compuation (2.2) of VaR, by single numerical integrations.

Proposition 4.1. *Under the conditional lognormal approximation the key quantity $P_\rho(T,G,w)$ in the calculation (2.2) of VaR can be estimated by the single integrals*

$$P_\rho(T,G,w) \approx \int_0^{\frac{\mathrm{e}^{-rT}GA_0-w}{A_0}} \Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}GA_0-w-zA_0}{A_0m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz \tag{4.2}$$

$$+\int_{\frac{\rho\mathrm{e}^{-rT}GA_0+w}{\rho A_0}}^{\frac{\mathrm{e}^{-rT}}{\rho A_0}(\rho GA_0+C)} \Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zA_0-\mathrm{e}^{-rT}\rho GA_0-w}{A_0m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz \tag{4.3}$$

$$+\int_{\frac{\mathrm{e}^{-rT}}{\rho A_0}(\rho GA_0+C)}^{\infty} \Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}C-w}{A_0m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz. \tag{4.4}$$

Similarly, we get the following approximation result of the key quantity $Z_\rho(T,G,w)$ appearing in the CTE expression (2.3).

Proposition 4.2. *Under the conditional lognormal approximation, the key quantity $Z_\rho(T, G, w)$ in the CTE formula (2.3) can be estimated by the single integrals*

$$
\begin{aligned}
&Z_\rho(T,G,w)\\
&\approx \int_0^{\frac{\mathrm{e}^{-rT}GA_0-w}{A_0}} \left(\mathrm{e}^{-rT}GA_0 - A_0 z\right) \Phi\left(\frac{\mu_T^z \frac{(\sigma_T^z)^2 T}{2} + \log \frac{\mathrm{e}^{-rT}GA_0 - w - zA_0}{A_0 m_x}}{\sigma_T^z \sqrt{T}}\right) f_{\tilde{S}_T}(z)dz\\
&\quad - A_0 m_x \int_0^{\frac{\mathrm{e}^{-rT}GA_0-w}{A_0}} \mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2 T/2} \Phi\left(\frac{(\mu_T^z - 2)\frac{(\sigma_T^z)^2 T}{2} + \log \frac{\mathrm{e}^{-rT}GA_0 - w - zA_0}{A_0 m_x}}{\sigma_T^z \sqrt{T}}\right) f_{\tilde{S}_T}(z)dz\\
&\quad + \rho \int_{\mathrm{e}^{-rT}G + \frac{w}{\rho A_0}}^{\frac{\mathrm{e}^{-rT}}{\rho A_0}(\rho G A_0 + C)} \left(A_0 z - \mathrm{e}^{-rT} G A_0\right) \Phi\left(\frac{\mu_T^z \frac{(\sigma_T^z)^2 T}{2} + \log \frac{\rho z A_0 - \mathrm{e}^{-rT}\rho G A_0 - w}{m_x A_0}}{\sigma_T^z \sqrt{T}}\right) f_{\tilde{S}_T}(z)dz\\
&\quad - A_0 m_x \int_{\frac{\rho \mathrm{e}^{-rT} G A_0 + w}{\rho A_0}}^{\frac{\mathrm{e}^{-rT}}{\rho A_0}(\rho G A_0 + C)} \mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2 T/2} \Phi \left(\frac{(\mu_T^z - 2)\frac{(\sigma_T^z)^2}{2}\cdots}{\cdots}\right.\\
&\quad + \mathrm{e}^{-rT} C \int_{\frac{\mathrm{e}^{-rT}}{\rho A_0}(\rho G A_0 + C)}^{\infty} \Phi\left(\frac{(\mu_T^z - 2)\frac{(\sigma_T^z)^2 T}{2} + \log \frac{\mathrm{e}^{-rT} C - w}{m_x A_0}}{\sigma_T^z \sqrt{T}}\right) f_{\tilde{S}_T}(z)dz\\
&\quad - A_0 m_x \int_{\frac{\mathrm{e}^{-rT}}{\rho A_0}(\rho G A_0 + C)}^{w} \mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2 T/2} \Phi\left(\frac{(\mu_T^z - 2)\frac{(\sigma_T^z)^2 T}{2} + \log \frac{\mathrm{e}^{-rT} C - w}{m_x A_0}}{\sigma_T^z \sqrt{T}}\right) f_{\tilde{S}_T}(z)dz.
\end{aligned}
$$

5 Numerical examples

For GMMBs, the underlying asset of the variable annuities is assumed to follow (1.1) with $r = 4\%$, $\mu = 9\%$, and $\sigma = 30\%$. The variable annuities with GMMB and GMDB riders are designed for policyholders of age 65 with the product parameters $T = 10$, $m = 1\%$, and $m_e = 0.35\%$. The future life time table is the one published by the US Social Security Administration (Bell and Miller, 2005) in 2005, cf. Table 1 in [FV12]. The initial account value is set to be $A_0 = 100$, the guarantee level G and the risk measures VaR and CTE are represented in percentages of initial account value.

$G = 75\%$	[FV14]†	lognormal
$V_{95\%}/A_0$	12.177734	12.177230
$\mathrm{CTE}_{95\%}/A_0$	23.283517	23.283757

$G = 100\%$	[FV14]‡	lognormal
$V_{90\%}/A_0$	12.550367	12.550349
$\mathrm{CTE}_{90\%}/A_0$	30.296486	30.296445

$G = 120\%$	[FV14]‡	lognormal
$V_{80\%}/A_0$	0^*	0^*
$\mathrm{CTE}_{80\%}/A_0$	27.333610^*	27.333606^*

Table 1: Risk measure estimates in % for the GMMB rider with different levels of risk tolerance α.

The algorithms are implemented with the **https://github.com/pnlnum/pnlPNL scientific Library for special functions and numerical integration routines, while the original implementations of [FV12] and [FV14] for the inverse Laplace and Green function methods are using Maple. We applied the Newton-Raphson method with precision of 5 decimal places for the root search procedure to solve equations (2.2) and (3.1) for the computation of VaR for GMMBs and GMDBs. The conditional tail expectations of net liabilities** $\mathrm{CTE}_\alpha(L_0)$ **for GMMBs and** $\mathrm{CTE}_\alpha(L^{(n)})$ **for GMDBs are computed from**

$$\mathrm{CTE}_\alpha(L_0) := \frac{\mathbb{E}[L_0 \mathbb{1}_{\{L_0>0\}}]}{1-\alpha} = \frac{(1-\xi_m)\,\mathbb{E}[L_0 \mathbb{1}_{\{L_0>0\}}]}{1-\alpha} = \frac{(1-\xi_m)\mathrm{CTE}_{\xi_m}(L_0)}{1-\alpha}$$

as in [FV12].

The parameters of the products and the underlying asset (1.1) are the same as for GMMBs except that here $r = 7\%$**, and the roll-up rate per annum is** $\delta = 6\%$**. We take** $n = 1$**, but one can also take** $n \geqslant 2$ **and apply the fractional age assumption in order to consider payments more frequent than yearly payments.**

†Green function method.

*This value is computed using $L_0^* := \max(L_0, 0)$ when L_0 yields a negative risk measure.

$G = 75\%$	[FV14][†]	lognormal
$V_{80\%}/A_0$	0^*	0^*
$\mathrm{CTE}_{80\%}/A_0$	7.018559^*	7.018555^*

$G = 100\%$	[FV14][‡]	lognormal
$V_{90\%}/A_0$	2.135188	2.135182
$\mathrm{CTE}_{90\%}/A_0$	33.706297	33.706289

$G = 120\%$	[FV14][‡]	lognormal
$V_{95\%}/A_0$	50.732711	50.732661
$\mathrm{CTE}_{95\%}/A_0$	69.140653	69.140640

Table 2: Risk measure estimates in % for the GMDB rider with different levels of risk tolerance α.

The VaR $V_\alpha(L_0)$ is computed from (2.2) given $P_\rho(T, G, V_\alpha(L_0))$ approximated by (4.2) under the lognormal approximation. The CTE is similarly computed from (2.3) given $Z_\rho(T, G, w)$ evaluated as in Proposition 4.2. We take the risk tolerance level $\alpha = 90\%$, $G = 100\%$, and $C/A_0 = 100\%, 200\%, 250\%$ as in [MZ16], the other model and product parameters being the same as above. The computation time for VaR and CTE by stratified approximation is around 0.01 and 0.004 seconds respectively.

$C/A_0 = 100\%$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$
$V_{90\%}/A_0$	36.1990	53.5788	58.1323
$\mathrm{CTE}_{90\%}/A_0$	46.9541	57.5319	60.1738

Table 3: Risk measure estimates in % for the GMMB rider with AE feature and level of risk tolerance $\alpha = 90\%$ using the lognormal approximation.

VaR and CTE of GMDBs with additional earnings

The VaR and CTE of GMDBs with additional earnings can be computed by the following C function, with alpha:=G/A_0.

```
int AP_GMDB_AE_Lognormal_VaR_CTE(double A0, double alpha, double maturity,
double r, double sigma, double risk_level, double rollup_rate, double me,
```

[†]Green function method.

[*]This value is computed using $L_0^{(n)*} := \max\left(L_0^{(n)}, 0\right)$ when $L_0^{(n)}$ yields a negative risk measure.

```
double mu, double m, int n_, double rho, double C, double *ptvar,
double *ptcte)
```

with the following parameters, in addition to A_0, r, σ, δ =rollup_rate, and maturity:

> **risk_level** $\in [0, 1]$
>
> **alpha : percentage of premium guaranteed**
>
> **me : margin offset**
>
> **mu : drift** μ
>
> **m : mortality & expense fees**
>
> **n_ : number of steps per year in the mortality table**
>
> **rho : share of the policyholder's variable annuities earnings**
>
> **C : maximum additional payout.**

The VaR $V_\alpha(L_0^{(n)})$ and CTE of the net liabilities can be similarly calculated implicitly from (3.1) for GMDBs.

$C/A_0 = 200\%$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$
$V_{90\%}/A_0$	14.735675	22.566120	28.094065
$\mathrm{CTE}_{90\%}/A_0$	38.180667	45.741347	53.113941

Table 4: Risk measure estimates in % for the GMDB rider with AE feature and level of risk tolerance $\alpha = 90\%$ using the lognormal approximation.

VaR and CTE of GMMBs with additional earnings

The VaR and CTE of GMMBs with additional earnings can be similarly computed by the following C function with alpha:=G/A_0.

```
int AP_GMMB_AE_Lognormal_VaR_CTE(double A0, double alpha, double maturity,
double r, double sigma, double risk_level, double me, double mu, double m,
double rho, double C, double *ptvar, double *ptcte)
```

with the following parameters, in addition to A_0, r, σ, and maturity:

> **risk_level** $\in [0, 1]$

alpha : **percentage of premium guaranteed**

me : **margin offset**

mu : **drift** μ

m : **mortality & expense fees**

rho : **share of the policyholder's variable annuities earnings**

C : **maximum additional payout.**

The risk measures of ordinary GMMBs and GMDBs without additional earnings can be computed by taking $\rho := 0$**.**

VaR and CTE of GMMBs and GMDBs by the spectral method

In the absence of additional earnings, the spectral method of [FV14] is implemented in C via the command

```
int AP_GMDB_Spectral_VaR_CTE(double A0, double alpha, double maturity, double r,
double sigma, double risk_level, double rollup_rate, double me, double mu, double m,
int n_, int N, double *ptvar, double *ptcte)
```

for GMDBs, with the following parameters, in addition to A_0, r, σ, δ =**rollup_rate, and maturity:**

risk_level $\in [0, 1]$

alpha : **percentage of premium guaranteed**

me : **margin offset**

mu : **drift** μ

m : **mortality & expense fees**

n_ : **number of steps per year in the mortality table**

N=7 : **parameter of the spectral method,**

and via

```
int AP_GMMB_Spectral_VaR_CTE(double A0, double alpha, double maturity, double r,
double sigma, double risk_level, double rollup_rate, double me, double mu,
double m, int N, double *ptvar, double *ptcte)
```

for GMMBs, with the following parameters, in addition to A_0, r, σ, δ =rollup_rate, and maturity:

$$\begin{aligned}&\text{risk\_level} \in [0,1]\\&\text{alpha : percentage of premium guaranteed}\\&\text{me : margin offset}\\&\text{mu : drift } \mu\\&\text{m : mortality \& expense fees}\\&\text{N=7 : parameter of the spectral method.}\end{aligned}$$

The PNL implementation of the inverse Laplace method was found to be computationally less stable than other methods, and highly dependent of the parameters chosen for the discretization of integrals.

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