EFFICIENT WIENER-HOPF APPROACH FOR OPTION PRICING IN STOCHASTIC VOLATILITY MODELS

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Abstract. In the paper, we propose new efficient method for pricing barrier options in the Heston model, which is implemented into Premia 13. We use local consistency arguments to approximate the volatility process with a finite, but sufficiently dense Markov chain; then we apply this regime switching approximation to efficiently compute option prices using numerical Wiener-Hopf factorization method. The method can be extended for the case of the Bates model and other stochastic volatility Lévy models.

1. Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. Relaxing the assumption of a unique source of uncertainty leads to the stochastic volatility family of models, where the volatility parameter follows a separate diffusion. The important example is the process in Heston (1993) [15].

Relaxing the assumption of continuous sample paths, leads to the general Lévy models. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004) [11].

State-of-the-art pricing models combine the two approaches, producing models that incorporate both stochastic volatility and jumps (the most common being the one proposed in [2], see also [13]).

In the paper, we propose new efficient method for pricing barrier options in the Heston model. The method can be easily extended for the case of the Bates model and other stochastic volatility Lévy models. We use local consistency arguments to approximate the volatility process with a finite, but sufficiently dense Markov chain, following the approach of Chourdakis (2004) [8]. Then we use this regime switching approximation

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to efficiently compute option prices by using numerical Fast Wiener-Hopf factorization (FWHF). The FWHF method was introduced in Kudryavtsev and Levendorskiǐ (2009) [21], where a fast and accurate numerical method for pricing barrier option for a wide class of Lévy processes was constructed. The FWHF method is based on an efficient approximation of the Wiener-Hopf factors and the Fast Fourier Transform algorithm. The advantage of the Wiener-Hopf approach over finite difference schemes in terms of accuracy and convergence property was shown in [21]. In Kudryavtsev (2010) [22], the method was extended to the regime switching Lévy models.

Regime switching stochastic models have already enjoyed much success in interpreting the behavior of a number of economic and financial time-series in a concise, yet parsimonious way. A stochastic process is used as the instrument that models the financial market, where the parameters of this process are allowed to depend on the state of an unobserved Markov chain that lives in continuous time. The state space may represent general financial market trends and/or other economic factors (also called "states of the world" or "regimes").

2. Stochastic volatility models

2.1. **Definitions and approximation formulae.** The main example of the stochastic volatility model is the one introduced in Heston (1993). The Heston model [15] assumes that the stock price process S_t under risk neutrality is given as

$$dS_t = rS_t dt + \sqrt{V_t} dW_t^1, \tag{2.1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2, \tag{2.2}$$

where V_t denotes the variance, σ is volatility of the volatility, and Wiener processes W_t^1 and W_t^2 have correlation ρ. Note that the model for the variance (2.2) is the same as the one used in [6] for the short term interest rate. The parameter θ represents the long term variance, and κ is the rate of mean-reversion.

The general class of the stochastic volatility Lévy models including Heston and Bates models can be described as follows. Suppose that the data generating process, under risk neutrality, is summarized as follows.

$$d\log S_t = \mu(V_t)dt + \sigma(V_t)dW_t^1 + dX_t, \tag{2.3}$$

$$dV_t = \alpha(V_t)dt + \beta(V_t)dW_t^2, \tag{2.4}$$

where V_t is a notation for the state process (e.g. the variance process), which is assumed to be stationary, $\mu(V_t)$ and $\sigma(V_t)$ indicate the drift and volatility of the logprice process $\log S_t$, respectively, which are state-dependent; X_t is a pure jump Lévy process, Wiener processes W_t^1 and W_t^2 are allowed to be correlated with $d < W^1, W^2 >_t = \rho dt$.

Under a Lévy model with stochastic volatility, a standard option pricing problem is typically reduced to the numerical solving of the correspondent three dimensional partial integro-differential equation (PIDE). Hence, it can be a very computationally intensive task. We suggest to use an approximate regime switching model. Under the regime switching structure, a system of the two-dimensional PIDEs will have to be solved.

The approximation of diffusions using Markov chains is not a novel approach, see the papers [26, 24]. For the diffusion to be approximated, a continuous time Markov chain is constructed for a given state-space, with probabilities that preserve the instantaneous drift and volatility structure. For any given state, only the neighboring states need to be reached, resembling a trinomial tree in continuous time.

We approximate the process V_t by a Markov chain V_t^h which takes real values in a discrete set $\mathcal{V} = \{V_1^h, V_2^h, \ldots, V_N^h\}$, where $h > 0$ denotes the distance between adjacent values, N is a number of the states. Let $\Lambda = (\lambda_{kj})$ be the transition rate matrix of the Markov chain, which should satisfy the "local consistency" concept (it exhibits the same instantaneous drift and volatility as the given diffusion, see details in [24]). One such approximation scheme can be found in [26] (the points V_j need to be equidistant):

$$\begin{aligned} \lambda_{k,k-1} &= \frac{1}{2h^2}\beta(V_k^h)^2 + \frac{1}{h}\alpha_-(V_k^h), \\ \lambda_{k,k} &= -\frac{1}{h^2}\beta(V_k^h)^2 - \frac{1}{h}\Big|\alpha(V_k^h)\Big|, \\ \lambda_{k,k+1} &= \frac{1}{2h^2}\beta(V_k^h)^2 + \frac{1}{h}\alpha_+(V_k^h), \end{aligned} \tag{2.5}$$

where $0 < k < N$, $a_+ = \max\{0, a\}$, $a_- = \max\{0, -a\}$, h denotes the space between adjacent points of the grid $\mathcal{V}$. The first and the last states can be made reflective.

$$\begin{aligned} \lambda_{0,0} &= -\frac{1}{2h^2}\beta(V_k^h)^2 - \frac{1}{h}\alpha_+(V_k^h), \\ \lambda_{0,1} &= \frac{1}{2h^2}\beta(V_k^h)^2 + \frac{1}{h}\alpha_+(V_k^h), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \lambda_{N,N-1} &= \frac{1}{2h^2}\beta(V_k^h)^2 + \frac{1}{h}\alpha_-(V_k^h), \\ \lambda_{N,N} &= -\frac{1}{2h^2}\beta(V_k^h)^2 - \frac{1}{h}\alpha_-(V_k^h). \end{aligned} \tag{2.7}$$

Now, consider the general case, when the points V_j^h are not equidistant. For convenience we drop the index h at V_j. Assume that at time t, the state value is equal to V_j. Over a short time interval, there are three possibilities: we can remain at V_j, move down by h_d^j to $V_{j-1} = V_j - h_d^j$, or move up by h_u^j to $V_{j+1} = V_j + h_u^j$. The local consistency conditions lead to the following approximation scheme (see [10]).

$$\begin{aligned} \lambda_{k,k-1} &= \frac{1}{h_d^k(h_u^k + h_d^k)}\beta(V_k)^2 - h_u\alpha(V_k), \\ \lambda_{k,k+1} &= \frac{1}{h_u^k(h_u^k + h_d^k)}\beta(V_k)^2 - h_d\alpha(V_k), \\ \lambda_{k,k} &= -\lambda_{k,k-1} - \lambda_{k,k+1}. \end{aligned} \tag{2.8}$$

For Λ to be a valid transition rates matrix, the off-diagonal elements need to be nonnegative. For this reason, when formulae (2.8) produce invalid elements, one can replace them with a scheme that always remains valid, but it matches the instantaneous drift only.

$$
\begin{aligned}
\lambda_{k,k-1} &= \frac{1}{h_d^k(h_u^k+h_d^k)}\beta(V_k)^2+(h_u+h_d)\alpha_-(V_k),\\
\lambda_{k,k+1} &= \frac{1}{h_u^k(h_u^k+h_d^k)}\beta(V_k)^2+(h_u+h_d)\alpha_+(V_k),\\
\lambda_{k,k} &= -\lambda_{k,k-1}-\lambda_{k,k+1}.
\end{aligned}
\tag{2.9}
$$

Due to the paper [10], state dependent correlations can also be constructed, by rewriting the regime switching approximation for (2.3), given that $V_t^h = V_j^h$, as follows.

$$
\begin{aligned}
d\log S_t &= \Big(\mu(V_j^h)-\rho\frac{\sigma(V_j^h)\alpha(V_j^h)}{\beta(V_j^h)}\Big)dt+\\
&+ \sqrt{1-\rho^2}\cdot\sigma(V_j^h)dW_t+\rho\frac{\sigma(V_j^h)}{\beta(V_j^h)}\Delta V_t^h+dX_t,
\end{aligned}
\tag{2.10}
$$

where W_t is a Wiener process, and the switch state process ΔV_t^h is defined by the formula:

$$
\Delta V_t^h = \begin{cases} +h_u, \text{with probability } \lambda_{k,k+1}dt+o(dt),\\ -h_d, \text{with probability } \lambda_{k,k-1}dt+o(dt),\\ 0, \text{with probability } 1+\lambda_{k,k}dt+o(dt).\end{cases}
\tag{2.11}
$$

Thus, correlations are accommodated into the formula (2.10), by allowing $d\log S_t$ to jump, whenever the variance switches. The process ΔV_t^h is therefore usually zero, except at the instances where the Markov chain changes the state.

The infinitesimal generator of the process (2.10), conditional on $\log S_0 = x$ and $V_0 = V_j$ is equal to:

$$
\begin{aligned}
L_jf(x,j) &= \lambda_{j,j}f(x,j)+\Big(\mu(V_j)-\rho\frac{\sigma(V_j)\alpha(V_j)}{\beta(V_j)}\Big)\partial_x f(x,j)+\\
&+ \frac{1}{2}(1-\rho^2)\cdot\sigma^2(V_j)\partial_x^2 f(x,j)+\lambda_{j,j+1}f\Big(x+\rho\frac{\sigma(V_j)}{\beta(V_j)}h_u, j+1\Big)+\\
&+ \lambda_{j,j-1}f\Big(x-\rho\frac{\sigma(V_j)}{\beta(V_j)}h_d, j-1\Big)+L_Xf(x,j),
\end{aligned}
\tag{2.12}
$$

where L_X is the infinitesimal generator of the Lévy process X_t.

2.2. **Lévy processes: general definitions.** A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999) [28]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1})\nu(dy), \tag{2.13}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2.14}$$

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\operatorname{Im}\xi \in (-1, 0)$ and continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$.

The infinitesimal generator of X, denote it L, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x) - y\mathbf{1}_{|y|\le 1}\frac{\partial u}{\partial x}(x))\nu(dy). \tag{2.15}$$

The infinitesimal generator L also can be represented as a pseudo-differential operator (PDO) with the symbol $-\psi(\xi)$, i.e. $L = -\psi(D)$, where $D = -i\partial_x$. Recall that a PDO $A = a(D)$ acts as follows:

$$Au(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi, \tag{2.16}$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx.$$

Note that the inverse Fourier transform in (2.16) is defined in the classical sense only if the symbol $a(\xi)$ and function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform by duality.

Further, if the riskless rate, r, is constant, and the stock does not pay dividends, then the discounted price process must be a martingale. Equivalently, the following condition (the EMM-requirement) must hold (see e.g. Boyarchenko and Levendorskiĭ (2002))

$$r + \psi(-i) = 0, \tag{2.17}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y\mathbf{1}_{|y|\le 1})\nu(dy). \tag{2.18}$$

Hence, the infinitesimal generator may be rewritten as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}(x) + \int_{\mathbf{R}} [u(x+y) - u(x) - (e^y - 1)\frac{\partial u}{\partial x}(x)]\nu(dy). \tag{2.19}$$

2.3. **Regular Lévy processes of exponential type.** Loosely speaking, a Lévy process X is called a *Regular Lévy Process of Exponential type* (RLPE) if its Lévy density has a polynomial singularity at the origin and decays exponentially at the infinity (see [4]). An almost equivalent definition is: the characteristic exponent is analytic in a strip $\operatorname{Im}\xi \in (\lambda_-, \lambda_+)$, $\lambda_- < -1 < 0 < \lambda_+$, continuous up to the boundary of the strip, and admits the representation

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{2.20}$$

where $\phi(\xi)$ stabilizes to a positively homogeneous function at the infinity:

$$\phi(\xi) \sim c_\pm |\xi|^\nu, \quad \text{as } \operatorname{Re}\xi \to \pm\infty, \quad \text{in the strip } \operatorname{Im}\xi \in (\lambda_-, \lambda_+), \tag{2.21}$$

where $c_\pm > 0$. "Almost" means that the majority of classes of Lévy processes used in empirical studies of financial markets satisfy conditions of both definitions. These classes are: Brownian motion, Kou's model [18], Hyperbolic processes [14], Normal Inverse Gaussian processes [1], and extended Koponen's family. Koponen (1995) [17] introduced a symmetric version; Boyarchenko and Levendorskiǐ (2000) [3], gave a non-symmetric generalization; later a subclass of this model appeared under the name CGMY – model in Carr *et al.* (2002), [7], and Boyarchenko and Levendorskiǐ (2002) [4] used the name KoBoL family.

The important exception is Variance Gamma Processes (VGP; see, e.g.[25]). VGP satisfy the conditions of the first definition but not the second one, since the characteristic exponent behaves like $const \cdot \ln|\xi|$, as $\xi \to \infty$.

Example 2.1. The characteristic exponent of a pure jump CGMY model is given by

$$\psi(\xi) = -i\mu\xi + C\Gamma(-Y)[G^Y - (G + i\xi)^Y + M^Y - (M - i\xi)^Y], \tag{2.22}$$

where $C > 0$, $\mu \in \mathbf{R}$, $Y \in (0, 2), Y \neq 1$, and $-M < -1 < 0 < G$.

Example 2.2. If Lévy measure of a jump diffusion process is given by normal distribution:

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left(-\frac{(x-\gamma)^2}{2\delta^2}\right)dx,$$

then we obtain Merton model. The parameter λ characterizes the intensity of jumps. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\left(1 - \exp\left(-\frac{\delta^2\xi^2}{2} + i\gamma\xi\right)\right), \tag{2.23}$$

where $\sigma, \delta, \lambda \geq 0$, $\mu, \gamma \in \mathbf{R}$.

There are two important degenerate cases:

- If the intensity of jumps $\lambda = 0$, then we obtain Black-Scholes model with $\mu = r - \frac{\sigma^2}{2}$ fixed by the EMM-requirement;
- If the intensity of jumps $\lambda > 0$ but $\delta = 0$, then we obtain a jump diffusion process with a constant jump size γ; the drift term $\mu = r - \frac{\sigma^2}{2} + \lambda\Big(1 - e^{\gamma}\Big)$ is fixed by the EMM-requirement.

3. Pricing of options under regime switching Lévy models

3.1. **The regime switching Lévy process.** Let $I = \{1, 2, \dots, d\}$ be the space of all financial market states. Consider a continuous-time Markov chain Z_t, taking values in I. Denote the generator of Z_t with the transition rate matrix $\Lambda = (\lambda_{kj})$, where k, j belong to I. Notice that the off-diagonal elements of Λ must be non-negative and the diagonal elements must satisfy $\lambda_{kk} = -\sum_{j \neq k} \lambda_{kj}$.

Recall, given that the process Z_t starts in a state k at time t_1, it has made the transition to some other state j at time t_2 with probability given by

$$P(Z_{t_2} = j | Z_{t_1} = k) = \{\exp((t_2 - t_1)\Lambda)\}_{kj}.$$

We will assume that the underlying asset price takes the form $S_t = S_0 e^{X_t}$, where the log-price process X_t will be constructed from a collection of Lévy processes, as follows.

Consider a collection of independent Lévy processes X^k, $k \in I$. Given that $Z_t = k$, we assume that the joint stock price process S_t follows a one-dimensional exponential Lévy process $S_t = S_0 e^{X_t^k}$ with characteristic exponent ψ_k. The drift terms μ_k of each state are assumed prefixed by the EMM-requirement $\psi_k(-i) + r = 0$, where $r > 0$ is a riskless rate. The increments of the log-price process will switch between the d Lévy processes, depending on the state Z_t. Thus, this modeling assumption can be written as

$$dX_t = dX_t^{Z_t}. \tag{3.1}$$

3.2. **The system of the generalized Black-Scholes equations.** The price of any derivative contract, $V(t, X_t)$, will satisfy the Feynman-Kac formula, that is to say

$$(\partial_t + L - r)V(t, x) = 0, \tag{3.2}$$

where x denotes the (normalized) log-price, t denotes the time, and L is the infinitesimal generator (under risk-neutral measure).

For the sake of brevity, consider the down-and-out put option without rebate, with strike K, maturity T and barrier $H < K$, on a non-dividend paying stock S_t. Therefore, for the one-state Lévy process $X_t = \ln(S_t/H)$ with the generator (2.15), the derivative price, $V(t, X_t)$, will satisfy the following partial integro-differential equation (or more general pseudo-differential equation) with the appropriate initial and boundary conditions. See details in [4] and [11].

$$(\partial_t + L - r)V(t, x) = 0, \quad t < T, x > 0, \tag{3.3}$$

$$V(T, x) = (K - He^x)_+, x > 0 \tag{3.4}$$

$$V(t, x) = 0, \quad t \leq T, \; x \leq 0, \tag{3.5}$$

where $a_+ = \max\{a, 0\}$. In addition, V must be bounded.

If the characteristic exponent ψ is sufficiently regular (e.g. X_t belongs to the class of RLPE), then the general technique of the theory of PDO can be applied to show that a bounded solution, which is continuous on $\operatorname{supp} V \subset (-\infty, T) \times (0, +\infty)$, is unique – see, e.g., Kudryavtsev and Levendorskiǐ (2006) [20].

In a regime switching setting we will have to deal with the conditional (on the regime j) option values $V(t, x, j)$. Under the regime switching structure, a system of PIDEs will have to be solved.

$$(\partial_t + L_j - r)V(t, x, j) = 0,\ t < T, x > 0, \tag{3.6}$$

$$V(T, x, j) = (K - He^x)_+,\ x > 0, \tag{3.7}$$

$$V(t, x, j) = 0,\ t \le T, x \le 0. \tag{3.8}$$

Here, L_j represents the infinitesimal generator of the jth volatility state of the Lévy process (see (2.12)). It is possible to apply any of the usual finite-difference schemes to this system of PIDEs to solve the problem. However, as discussed earlier, it faces difficulties due to the non-local integral terms. Instead, we use the Fast Wiener-Hopf factorization algorithm [21] which is applicable to pricing barrier options under regime switching Lévy models (see details of the method in [22]).

In the case of American put the free boundary problem for the system of PIDEs has to be solved. A general approach to the pricing American options under a regime switching Lévy structure can be found in [?], see also the implementation of the FWHF-method in [23].

4. Implementation to the Premia 13

In the paper, we propose new efficient method for pricing barrier and American options in wide classes of stochastic volatility Lévy processes. We use local consistency arguments to approximate the volatility process with a finite, but sufficiently dense Markov chain as in [8, 10]. Then we apply this regime switching approximation to efficiently compute barrier option prices using FWHF-method from [22]. We price the derivatives as in [22] with a 2-times repeated Richardson extrapolation on discrete barrier options and vary the number of monitoring dates. In the case of American options we use the Geske-Johnson approximation.

We implemented into the program platform Premia the method for barrier and American options under the Heston model (see (2.1)-(2.2)). One can use the routine for the other types of stochastic volatility Lévy processes by replacing the corresponding part with the computation of the characteristic exponent.

Note that in the program implemented to Premia 13 one can manage by five parameters of the algorithm: the number of the volatility states N, the space step d, the scale of the logprice range k_x, the number of time steps M, the scale of the volatility range k_v. Parameter k_x, k_v control the size of the truncated region in x-space and in the volatility space, respectively. The typical values of the parameter are $k_x = 1$, $k_x = 2$ and $k_x = 4$, the same holds for k_v. To improve the results one should decrease d and/or increase M, N, when k_x, k_v are fixed.

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