mc_variancereduction

1 Computation of the price by Monte Carlo methods

We want to compute the price at time t $\in [0, T]$ of a "down and out" call option:

$$\Pi_{i=1}^{j} 1\left(S_{t_i} > L\right) \exp\left(-r(T-t)\right) E^* \left[\Pi_{i=j+1}^{m} 1\left(S_{t_i} > L\right) \left(S_T - K\right)_+ |F_t\right]$$

where

$$E^*[.]$$

denote the expectation under the risk neutral probability and

$$t_j \leq t \leq t_{j+1}$$

In order to simplify the notations, let us place at time 0. Our aim is then to approximate the quantity

$$P = E^* \left[\Pi_{i=1}^{m} 1\left(S_{t_i} > L\right) \left(S_T - K\right)_+ |F_t\right]$$

by using Monte Carlo methods, while attaching ourselve to reduce the variance of the estimator.

2 Variance Reduction

First, for an integrable random variable Y and an event A we can write

$$\begin{aligned} E\left[Y 1_A\right] &= P(A) E\left[Y \frac{1_A}{P(A)}\right] \\ &= P(A) E^Q\left[Y\right] \end{aligned}$$

where

$$\frac{dQ}{dP} = \frac{1_A}{P(A)}$$

So, to approximate

$$E\left[Y 1_A\right]$$

we can either simulate the natural estimator

$$Y 1_A$$

under the real probability or simulate

$$P\left(A\right) Y$$

under the probability Q, that is equivalent to simulate Y ($P\left(A\right)$ is a constant) conditionaly to the event A. Let Y_1 be a random variable such that

$$\pounds\left(Y_1\right) \equiv \pounds\left(Y|A\right)$$

It's quite straightforward that:

$$\sigma^2\left(P\left(A\right) Y_1\right) \leq \sigma^2\left(Y 1_A\right)$$

indeed we have

$$\sigma^2\left(P\left(A\right) Y_1\right) = \left(P\left(A\right)\right)^2 \left(E\left[Y^2|A\right] - \left(E\left[Y|A\right]\right)^2\right)$$

and

$$\sigma^2\left(Y 1_A\right) = P\left(A\right) E\left[Y^2|A\right] - \left(P\left(A\right)\right)^2 \left(E\left[Y|A\right]\right)^2$$

but

$$P(A) > \left(P(A)\right)^2$$

Let us now consider the price P of a "down and out" call option with barrier L and 2 monitoring instant t_1, t_2 :

$$\begin{aligned} P &= E^*\left[1\left(S_{t_1} > L\right) 1\left(S_{t_2} > L\right)\left(S_T - K\right)_+\right] \\ &= E^*\left[1\left(S_{t_1} > L\right) 1\left(S_{t_2} > L\right)\left(S_T - K\right)_+ | \left(S_{t_1} > L\right)\right] P\left(S_{t_1} > L\right) \\ &= E^*\left[1\left(S_{t_1} > L\right) 1\left(\frac{S_{t_2}}{S_{t_1}} S_{t_1} > L\right)\left(\frac{S_T}{S_{t_2}} \frac{S_{t_2}}{S_{t_1}} S_{t_1} - K\right)_+ | \left(S_{t_1} > L\right)\right] P\left(S_{t_1} > L\right) \end{aligned}$$

with

$$X_1 = S_{t_1},\ X_2 = \frac{S_{t_2}}{S_{t_1}},\ X_3 = \frac{S_T}{S_{t_2}}$$

$$P = E^* \left[1\left(X_2 X_1 > L\right) \left(X_3 X_2 X_1 - K\right)_+ | \left(X_1 > L\right) \right] P\left(X_1 > L\right)$$

Denote Y_1 a random variable independant of X_2 and X_3 such that

$$\pounds(Y_1) \equiv \pounds\left(X_1 | X_1 > L\right)$$

then we can write

$$\begin{aligned} P &= E^* \left[1\left(X_2 Y_1 > L\right) \left(X_3 X_2 Y_1 - K\right)_+ \right] P\left(X_1 > L\right) \\ &= E^* \left[E^* \left[\left(X_3 X_2 Y_1 - K\right)_+ | \left(X_2 > \frac{L}{Y_1} \right) \right] P\left(X_2 > \frac{L}{Y_1} \right) \right] P\left(X_1 > L\right) \end{aligned}$$

Denote Y_2 a random variable independant of X_3 such that

$$\pounds(Y_2) \equiv \pounds\left(X_2 | X_2 > \frac{L}{Y_1} \right)$$

then we can write

$$P = E^* \left[\left(X_3 Y_2 Y_1 - K\right)_+ P\left(X_2 > \frac{L}{Y_1} \right) \right] P\left(X_1 > L\right)$$

this leads to the new estimator

$$\left(X_3 Y_2 Y_1 - K\right)_+ P\left(X_2 > \frac{L}{Y_1} \right) P\left(X_1 > L\right)$$

We generalize this result to the case with n monitoring instant $t_1, ..., t_n$ and obtain the general form of the new estimator that we denote N

$$N = \left(X_{n+1} Y_n Y_{n-1} ... Y_1\right)_+ P\left(X_{n+1} > \frac{L}{Y_n Y_{n-1} ... Y_1} \right) ... P\left(X_2 > \frac{L}{Y_1} \right) P\left(X_1 > L\right)$$

where

$$\begin{aligned} X_1 &= S_{t_1} \\ X_i &= \frac{S_{t_i}}{S_{t_{i-1}}} \; 2 \leq i \leq n \\ X_{n+1} &= \frac{S_T}{S_{t_n}} \\ \pounds\left(Y_1\right) &= \pounds\left(X_1 | X_1 > L\right) \\ \pounds(Y_i) &= \pounds\left(X_i | X_i > \frac{L}{Y_{i-1} ... Y_1} \right) \; 2 \leq i \leq n \end{aligned}$$

3 Simulation

3.1 Simulation of N

To simulate N we proceed as follows : (1) We simulate Y_1 (2) Knowing the value of $\frac{L}{\mathrm{Y}_1}$ we can simulate Y_2 ... (i) Knowing the value of $\frac{L}{Y_{i-1}...Y_1}$ we can simulate Y_i for $2 \leq i \leq n$ ... (n+1) X_{n+1} being independant of $Y_1, ..., Y_n$ we can simulate it (n+2) Knowing the values of $Y_1, ..., Y_n$ we can compute $P\left(X_{n+1} > \frac{L}{Y_n Y_{n-1}...Y_1}\right) ... P\left(X_2 > \frac{L}{\mathrm{Y}_1}\right) P\left(X_1 > L\right)$

3.2 Simulation of $Y_n, Y_{n-1}, ..., Y_1$

In our example, the simulation of the random variable $Y_n, Y_{n-1}, ..., Y_1$ amounts to the simulation of random variables $(\mathrm{g}_i)_{i=1...n}$ normaly distributed with mean 0 and variance 1 conditionaly to the events $(g_i \in [a_i, +\infty])_{i=1...n}$ To do this, we use the following method: Let Z be a random variable such that

$$\pounds\left(Z\right) \equiv \pounds\left(g_i | g_i \in [a_i, +\infty]\right)$$

and

$$u \geq a_i$$

we can write

$$\begin{aligned} P(Z \leq u) &= P(X \leq u | X \in [a_i, +\infty]) \\ \Leftrightarrow F_Z(u) &= \frac{P(a_i \leq X \leq u)}{P(a_i \leq X)} \\ \Leftrightarrow F_Z(u) &= \frac{F_{g_i}(u) - F_{g_i}(a_i)}{1 - F_{g_i}(a_i)} \end{aligned}$$

it follows then

$$F_Z^{-1}(y) = F_{g_i}^{-1}\left(F_{g_i}(a_i) + y\left(1 - F_{g_i}(a_i)\right)\right)$$

But we know that if the random variable U is uniformly distributed on $[0, 1]$ then $\pounds\left(F_Z^{-1}(U)\right) \equiv \pounds\left(Z\right)$ has the same law than Z. Since we can easily simulate U and compute $F_{g_i}^{-1}\left(U\right)$ the simulation of Z doesn't raise any technical problem.