Pricing american option under double Heston model: implementation in PREMIA

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Abstract

Using the modified asymptotic expansion method provided in [ZF18], we price the American option in a double Heston model. By introducing an explicit exercise rule, an asymptotic expansion of the solution to the partial differential equation is obtained for American put option, which is computed by the sum of the European option price and the early exercise premium. The early exercise premium is calculated by the difference between the approximated American and European option prices based on asymptotic expansion. The European option price is obtained by the efficient COS method. The asymptotic expansion method works well for small maturity and the approximation with 4-terms expansion is precise enough for maturity less than 1.

1 Model description and its PDE

The risk-neutral dynamic of the double Heston model is given as follows:

$$\begin{aligned} dS(t) &= (r-q)S(t)dt + \sqrt{V_1(t)}S(t)dW^{(1)}(t) + \sqrt{V_2(t)}S(t)dW^{(2)}(t), \\ dV_1(t) &= k_1(\theta_1 - V_1(t))dt + \sigma_1\sqrt{V_1(t)}dW^{(3)}(t), \\ dV_2(t) &= k_2(\theta_2 - V_2(t))dt + \sigma_2\sqrt{V_2(t)}dW^{(4)}(t), \end{aligned}$$

where r is the risk neutral interest rate, q is the dividend rate, $k_j, \theta_j, \sigma_j (j = 1, 2)$ are the mean-reverting rates, long-term volatilities and instantaneous volatilities of process $V_j(t)$, respectively. The variance processes $V_j(t)$ remain strictly positive if the Feller conditions, $2k_j\theta_j \geq \sigma_j^2$, are satisfied. Suppose $V_1(0) = V_1, V_2(0) = V_2, S(0) = S$. $dW^{(i)}(t)dW^{(j)}(t) = \rho_{ij}dt, i, j = 1, 2, 3, 4$.

Then the American put option price, denoted by $P(S, V_1, V_2, t)$, with strike price K and maturity date T satisfies the partial differential equation (PDE):

$$\begin{aligned} 0 &= P_t + (r-q)SP_S + \frac{1}{2}(V_1+V_2)S_2P_{SS} - rP + \rho_1\sigma_1V_1SP_{V_1S} + \rho_2\sigma_2V_2SP_{V_2S} \\ & \frac{1}{2}\sigma_1^2V_1P_{V_1V_1} + \frac{1}{2}\sigma_2^2V_2P_{V_2V_2} + k_1(\theta_1 - V_1)P_{V_1} + k_2(\theta_2 - V_2)P_{V_2}, \end{aligned} \tag{1.1}$$

with boudary conditions:

$$\begin{cases} P(\infty, V_1, V_2, t) = 0, \\ P(S(T)V_1, V_2, T) = \max(K - S(T, 0), \\ P(b(\tau), V_1, V_2, t) = K - b(\tau), \\ P_S(b(\tau), V_1, V_2, t) = -1. \end{cases}$$

A modified version of problem (1.1) with boundary condition replaced by an explicit exercise rule, the new problem is defined by the same PDE as in (1.1) with boundary conditions:

$$\begin{cases} P(\infty, V_1, V_2, t) = 0, \\ P(S(T), V_1, V_2, T) = \max(K - S(T), 0), \\ P(\tilde{S}(T-t), V_1, V_2, t) = \max(K - \tilde{S}(T-t), 0), \end{cases} \tag{1.2}$$

where $\tilde{S}(T-t) = K\mathrm{e}^{-y\sqrt{V_1+V_2}\sqrt{T-t}}$.

The unique solution to the above modified problem is the price of a barrier put option that is exercised as soon as the normalized moneyness $\theta := \frac{\ln(K/S)}{\sqrt{V_1+V_2}\sqrt{\tau}}$ reaches the barrier level y. If the barrier level y is chosen to approximate the exercise boundary of the American option, we expect the solution to the modified problem to be close to the true American option price.

Denote $\tau := T-t$ and use the normalized moneyness $\theta := \frac{\ln(K/S)}{\sqrt{V_1+V_2}\sqrt{\tau}}$ instead of the underlying price S, we rewrite PDE (1.1) in terms of (θ, τ) instead of (S, t). Denote the price of the barrier put option with barrier level y as $P(\theta, V_1, V_2, \tau; y)$. The American put price $P(S, V_1, V_2, T)$ rewritten as $P(\theta, V_1, V_2, \tau)$ can be approximated from below by

$$P(\theta, V_1, V_2, \tau) \approx \max_{y \geq \theta} P(\theta, V_1, V_2, \tau; y) = P(\theta, V_1, V_2, \tau; \tilde{y}(\theta, \tau)), \tag{1.3}$$

where $\tilde{y}(\theta, \tau) = \arg\max_{y\geq\theta} P(\theta, V_1, V_2, \tau; y)$.

Using the definition of θ and setting $P(\theta, V_1, V_2, \tau) = P(S, V_1, V_2, T)$, the derivatives $P_t, P_S, P_{SS}, P_{V_j}, P_{V_jV_j}, P_{V_jS}$ can be transformed to $P_\theta, P_\tau, P_{\theta\theta}, P_{V_j\theta}$ by chain rule, the PDE (1.1) can be rewritten as

$$
\begin{aligned}
0 \;=\; & P_{\theta\theta} + \theta P_\theta - 2\tau P_\tau + \sqrt{\tau}\Bigg[\frac{(V_1+V_2+2(q-r))P_\theta}{\sqrt{V_1+V_2}} \\
& + \sum_{j=1}^{2} \rho_j\sigma_j V_j \left(-\frac{2P_{V_j\theta}}{\sqrt{V_1+V_2}} + \frac{P_\theta}{(V_1+V_2)^{3/2}} + \frac{\theta P_{\theta\theta}}{(V_1+V_2)^{3/2}}\right) \\
& + \tau\left\{\sum_{j=1}^{2}\Bigg[k_j(\theta_j - V_j)\left(2P_{V_j} - \frac{\theta P_\theta}{2(V_1+V_2)}\right)\right. \\
& \left. + \sigma_j^2 V_j\left(P_{V_jV_j} - \frac{\theta P_{V_j\theta}}{V_1+V_2} + \frac{\theta^2 P_{\theta\theta}}{4(V_1+V_2)^2} + \frac{3\theta P_\theta}{4(V_1+V_2)^2}\right)\Bigg] - 2rP\right\}.
\end{aligned} \tag{1.4}
$$

And the boundary condition (1.2) can be rewritten as

$$
\begin{cases}
P(\infty, V_1, V_2, \tau; y) = 0, \\
P(S(T), V_1, V_2, 0) = \max(K - S(T), 0), \\
P(y, V_1, V_2, \tau; y) = K\max(1 - \mathrm{e}^{-y\sqrt{V_1+V_2}\sqrt{\tau}}, 0) = K(1 - \mathrm{e}^{-y\sqrt{V_1+V_2}\sqrt{\tau}}).
\end{cases} \tag{1.5}
$$

The solution to (1.4) with boundary condition (1.2) has the following regular short-maturity asymptotic expansion:

$$
P(\theta, V_1, V_2, \tau; y) := \sum_{n=1}^{\infty} P_n(\theta, V_1, V_2; y)\tau^{n/2}, \tag{1.6}
$$

where $P_n(\theta, V_1, V_2; y), n = 1, 2, ...$ are the coefficients of the short-maturity asymptotic expansion in τ, its characterization will be given in next Section below. Putting (1.6) into (1.4) we have

$$
\begin{aligned}
0 \;=\; & -nP_n + \theta P_{n,\theta} + P_{n,\theta\theta} + \frac{(V_1+V_2+2(q-r))P_{n-1,\theta}}{\sqrt{V_1+V_2}} - 2rP_{n-2} \\
& + \sum_{j=1}^{2}\Bigg[\rho_j\sigma_j V_j\left(-\frac{2P_{n-1,V_j\theta}}{\sqrt{V_1+V_2}} + \frac{P_{n-1,\theta}}{(V_1+V_2)^{3/2}} + \frac{\theta P_{n-1,\theta\theta}}{(V_1+V_2)^{3/2}}\right) \\
& + k_j(\theta_j - V_j)\left(2P_{n-2,V_j} - \frac{\theta P_{n-2,\theta}}{2(V_1+V_2)}\right) \\
& + \sigma_j^2 V_j\left(P_{n-2,V_jV_j} - \frac{\theta P_{n-2,V_j\theta}}{V_1+V_2} + \frac{\theta^2 P_{n-2,\theta\theta}}{4(V_1+V_2)^2} + \frac{3\theta P_{n-2,\theta}}{4(V_1+V_2)^2}\right)\Bigg],
\end{aligned} \tag{1.7}
$$

where the second subscript of P_n denotes the derivative of P_n with respect to the second subscript, i.e. for example $P_{n,\theta} = \frac{\partial P_n}{\partial \theta}$ and $P_{n-1,V_j\theta} = \frac{\partial^2 P_n}{\partial V_j \partial \theta}$.

2 Asymptotic expansion of American option price

Now we present the modified asymptotic expansion method given by [ZF18] to solve (1.7) and then obtain the American option price. Proposition 1 of [MS10] gives the characteristic of the coefficients in (1.6) as:

$$P_n(\theta, V_1, V_2; y) = C_n(V_1, V_2; y)[p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta)] + p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta), \tag{2.1}$$

where $p_n^0(\theta), q_n^0(\theta), p_n^1(\theta), q_n^1(\theta)$ are polynomials in θ with the coefficients depending on model parameters and $C_1(V_1, V_2; y), C_2(V_1, V_2; y), \cdots, C_{n-1}(V_1, V_2; y)$ are functions independent of θ but depending on model parameters V_1, V_2 and y, $\Phi(\theta)$ and $\phi(\theta)$ is the cumulative distribution function and probability distribution function of normal distribution respectively. $P_n(\theta, V_1, V_2; y)$ comprises a homogeneous part

$$P_n^0(\theta, V_1, V_2) := p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta), \tag{2.2}$$

which is the solution to the homogeneous part of the equation (1.7), i.e.

$$-nP_n^0(\theta, V_1, V_2) + \theta P_{n,\theta}^0(\theta, V_1, V_2) + P_{n,\theta\theta}^0(\theta, V_1, V_2) = 0, \tag{2.3}$$

By substitute (2.2) into the homogeneous equation (2.3), after some rearrangements, we have

$$\begin{aligned} &\left[-np_n^0(\theta) + \theta p_{n,\theta}^0(\theta) + p_{n,\theta\theta}^0(\theta)\right]\Phi(\theta) \\ &+ \left[-(n+1)q_n^0(\theta) - \theta q_{n,\theta}^0(\theta) + q_{n,\theta\theta}^0(\theta) + 2p_{n\theta}^0(\theta)\right]\phi(\theta) = 0. \end{aligned}$$

To ensure that for any θ satisfies the above equation, we have the coefficients of $\Phi(\theta)$ and $\phi(\theta)$ equals to 0, then we have the following equations to derive $p_n^0(\theta)$ and $q_n^0(\theta)$ as

$$\left\{ \begin{array}{l} -np_n^0(\theta) + \theta p_{n,\theta}^0(\theta) + p_{n,\theta\theta}^0(\theta) = 0, \\ -(n+1)q_n^0(\theta) - \theta q_{n,\theta}^0(\theta) + q_{n,\theta\theta}^0(\theta) + 2p_{n,\theta}^0(\theta) = 0. \end{array} \right. \tag{2.4}$$

By [MS10] the polynomials $p_n^0(\theta), q_n^0(\theta), p_n^1(\theta), q_n^1(\theta)$ can be guessed to have the forms as

$$\begin{aligned} p_n^0(\theta) &= f_{n,0}\theta^n + f_{n,1}\theta^{n-2} + f_{n,2}\theta^{n-4} + \cdots + f_{n,\frac{n-m}{2}}\theta^m, \\ q_n^0(\theta) &= g_{n,0}\theta^{n-1} + g_{n,1}\theta^{n-3} + g_{n,2}\theta^{n-5} + \cdots + g_{n,\frac{n-m-1}{2}}\theta^m, \\ p_n^1(\theta) &= h_{n,0}\theta^{n-2} + h_{n,1}\theta^{n-4} + h_{n,2}\theta^{n-6} + \cdots + h_{n,\frac{n-m-2}{2}}\theta^m, \\ q_n^1(\theta) &= k_{n,0}\theta^{n-3} + k_{n,1}\theta^{n-5} + k_{n,2}\theta^{n-7} + \cdots + k_{n,\frac{n-m-3}{2}}\theta^m, \end{aligned} \tag{2.5}$$

where $m = \mathrm{mod}(n, 2)$, i.e. m is 1 when n is an odd number and 0 when n is an even number. Then the coefficients $f_{n,i}$ and $g_{n,i}$ can be determined by substituting (2.5) into (2.4) as for $\forall n \geq 1, i \geq 0$,

$$\begin{aligned} f_{n,i+1} &= \frac{(n-2i)(n-2i-1)}{2i+2}f_{n,i}, \quad f_{n,0} = 1, \\ g_{n,i+1} &= \frac{g_{n,i}(n-2i-1)(n-2i-2) + 2f_{n,i+1}(n-2i-2)}{2n-2i-2}, \quad g_{n,0} = 1. \end{aligned}$$

By substitute P_{n-1} and P_{n-2} of the form (2.1) and P_n of the form

$$P_n^1(\theta, V1, V2) := p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta) \tag{2.6}$$

into (1.7), we have the equations for $p_n^1(\theta)$ and $q_n^1(\theta)$ as in (18) and (19) of [ZF18].

Then the coefficients $h_{n,i}$ and $k_{n,i}$ can be solved by the putting (2.5) into equations (18) and (19) of [ZF18] as

$$\begin{aligned}
h_{n,i} \;=\; & \Bigg\{(n-2i)(n-2i-1)h_{n,i-1} + \sum_{j=1}^{2}\left[A_{2j}B_1(n-2i-1)h_{n-1,i-1,V_j} + A_{4j}h_{n-2,i-1,V_jV_j}\right] \\
& + \left[A_1C_{n-1} + \sum_{j=1}^{2}(A_{2j}B_1C_{n-1,V_j} + A_{2j}B_2C_{n-1}(n-2i-1))\right](n-2i-1)f_{n-1,i} \\
& + \left[A_1 + \sum_{j=1}^{2}A_{2j}B_2(n-2i-1)\right](n-2i-1)h_{n-1,i-1} \\
& + \left[\sum_{j=1}^{2}(n-2i-2)\left(A_{3j}B_4C_{n-2} + A_{4j}B_6C_{n-2}(n-2i) + 2A_{4j}B_4C_{n-2,V_j}\right)\right. \\
& \left. + \sum_{j=1}^{2}\left(2A_{3j}C_{n-2,V_j} + A_{4j}C_{n-2,V_jV_j}\right) - 2rC_{n-2}\right]f_{n-2,i} \\
& + \left[\sum_{j=1}^{2}\left(A_{3j}B_4 + A_{4j}B_6(n-2i)\right)(n-2i-2) - 2r\right]h_{n-2,i-1} \\
& + \sum_{j=1}^{2}\left[2A_{3j} + A_{4j}2B_4(n-2i-2)\right]h_{n-2,i-1,V_j}\Bigg\}/(2+2i), \quad \forall n \geq 2, i \geq 1,
\end{aligned}$$

$$
\begin{aligned}
k_{n,i} \;=\;& \frac{1}{2n-2i-2}\{2(n-2-2i)h_{n,i}+(n-2i-1)(n-2i-2)k_{n,i-1}\\
&+D_1 f_{n-1,i+1}+D_0 f_{n-1,i+2}+D_1(n-2i-2)g_{n-1,i}+2D_0(n-2i-4)g_{n-1,i+1}\\
&+(-D_1+D_0)g_{n-1,i+1}-D_0 g_{n-1,i+2}+D_2 h_{n-1,i}+D_3 h_{n-1,i+1}+D_2(n-2i-2)k_{n-1,i-1}\\
&+2D_3(n-2i-4)k_{n-1,i}+(-D_2+D_3)k_{n-1,i}-D_3 k_{n-1,i+1}\\
&+\sum_{j=1}^{2}A_{2j}B_1\left[h_{n-1,i,V_j}+(n-2i-2)k_{n-1,i-1,V_j}-k_{n-1,i,V_j}\right]\\
&-D_3 C_{n-1}\left[2(n-2i-3)f_{n-1,i+1}+(n-2i-2)(n-2i-3)g_{n-1,i}\right]\\
&-D_3\left[2(n-2i-3)h_{n-1,i}+(n-2i-2)(n-2i-3)k_{n-1,i-1}\right]\\
&+D_4 g_{n-2,i}+D_5 g_{n-2,i+1}+D_6 g_{n-2,i+2}+D_7 f_{n-2,i+1}-D_6 f_{n-2,i+2}+D_7(n-2i-3)g_{n-2,i}\\
&-2D_6(n-2i-5)g_{n-2,i+1}+D_8 h_{n-2,i}-D_9 h_{n-2,i+1}+D_8(n-2i-3)k_{n-2,i-1}\\
&-2D_9(n-2i-5)k_{n-2,i}+(-D_8-D_9)k_{n-2,i}+D_9 k_{n-2,i+1}-2rk_{n-2,i-1}\\
&+\sum_{j=1}^{2}\left[2A_{4j}B_4\left(h_{n-2,i,V_j}+(n-2i-3)k_{n-2,i-1,V_j}\right)+2A_{3j}k_{n-2,i-1,V_j}-A_{4j}2B_4 k_{n-2,i,V_j}\right]\\
&+D_9\left[2C_{n-2}(n-2i-4)f_{n-2,i+1}+C_{n-2}(n-2i-3)(n-2i-4)g_{n-2,i}\right]\\
&+D_9\left[2(n-2i-4)h_{n-2,i}+(n-2i-3)(n-2i-4)k_{n-2,i-1}\right]\\
&\left.+\sum_{j=1}^{2}A_{4j}k_{n-2,i-1,V_jV_j}\right\}, \quad \text{for} \quad \forall n\geq 3, i\geq 1,
\end{aligned}
$$

and

$$
\begin{aligned}
h_{n,0} \;=\;& \left\{A_1C_{n-1}+\sum_{j=1}^{2}\left[A_{2j}B_1C_{n-1,V_j}+A_{2j}B_2C_{n-1}(n-1)\right]\right\}(n-1)f_{n-1,0}/2\\
&+\left\{\sum_{j=1}^{2}\left[(n-2)\left(A_{3j}B_4C_{n-2}+A_{4j}B_6C_{n-2}n+A_{4j}2B_4C_{n-2,V_j}\right)\right.\right.\\
&\left.\left.+\;2A_{3j}C_{n-2,V_j}+A_{4j}C_{n-2,V_jV_j}\right]-2rC_{n-2}\right\}f_{n-2,0}/2, \quad \text{for} \quad \forall n\geq 2,\\
k_{n,0} \;=\;& \frac{1}{2n-2}\{2(n-2)h_{n,0}+D_1 f_{n-1,1}+D_0 f_{n-1,2}+D_1(n-2)g_{n-1,0}+2D_0(n-4)g_{n-1,1}\\
&+(-D_1+D_0)g_{n-1,1}-D_0 g_{n-1,2}+D_2 h_{n-1,0}+D_3 h_{n-1,1}+2D_3(n-4)k_{n-1,0}\\
&+(-D_2+D_3)k_{n-1,0}-D_3 k_{n-1,1}+\sum_{j=1}^{2}A_{2j}B_1\left(h_{n-1,0,V_j}+k_{n-1,0,V_j}\right)\\
&-D_3\left[C_{n-1}\left(2(n-3)f_{n-1,1}+(n-2)(n-3)g_{n-1,0}\right)+2(n-3)h_{n-1,0}\right]\\
&+D_4 g_{n-2,0}+D_5 g_{n-2,1}+D_6 g_{n-2,2}+D_7 f_{n-2,1}-D_6 f_{n-2,2}+2D_7(n-3)g_{n-2,0}\\
&-2D_6(n-5)g_{n-2,1}+D_8 h_{n-2,0}-D_9 h_{n-2,1}-2D_9(n-5)k_{n-2,0}\\
&+(-D_8-D_9)k_{n-2,0}+D_9 k_{n-2,1}+\sum_{j=1}^{2}2A_{4j}B_4\left[h_{n-2,0,V_j}-k_{n-2,0,V_j}\right]\\
&+D_9\left[2C_{n-2}(n-4)f_{n-2,1}+C_{n-2}(n-3)(n-4)g_{n-2,0}+2(n-4)h_{n-2,0}\right]\}, \quad \text{for} \quad \forall n\geq 3,
\end{aligned}
$$

where for $\forall n,\ f_{0,0}=g_{0,0}=h_{1,0}=k_{1,0}=k_{2,0}=0$, and the coefficients are

$$
\begin{aligned}
A_1 &= \frac{V_1+V_2+2(q-r)}{\sqrt{V_1+V_2}}, \quad A_{2j}=\rho_j\sigma_jV_j, \quad A_{3j}=k_j(\theta_j-V_j), \quad A_{4j}=\sigma_j^2V_j,\\
B_1 &= -\frac{2}{\sqrt{V_1+V_2}}, \quad B_2=\frac{1}{(V_1+V_2)^{3/2}}, \quad B_4=-\frac{1}{2(V_1+V_2)}, \quad B_6=\frac{1}{4(V_1+V_2)^2},
\end{aligned}
$$

$$
\begin{aligned}
D_1 &= A_1C_{n-1}+\sum_{j=1}^{2}\Big(A_{2j}B_1C_{n-1,V_j}+A_{2j}B_2C_{n-1}\Big), \quad D_0=-\sum_{j=1}^{2}A_{2j}B_2C_{n-1},\\
D_2 &= A_1+\sum_{j=1}^{2}A_{2j}B_2, \quad D_3=-\sum_{j=1}^{2}A_{2j}B_2, \quad D_4=\sum_{j=1}^{2}\Big(A_{3j}2C_{n-2,V_j}+A_{4j}C_{n-2,V_jV_j}\Big)-2rC_{n-2},\\
D_5 &= -\sum_{j=1}^{2}\Big(A_{3j}B_4C_{n-2}+A_{4j}B_5C_{n-2,V_j}+A_{4j}B_6C_{n-2}+3A_{4j}B_6C_{n-2}\Big), \quad D_6=\sum_{j=1}^{2}A_{4j}B_6C_{n-2},\\
D_7 &= \sum_{j=1}^{2}\Big(A_{3j}B_4C_{n-2}+A_{4j}2B_4C_{n-2,V_j}+A_{4j}3B_6C_{n-2}\Big), \quad D_8=\sum_{j=1}^{2}\left(A_{3j}B_4+A_{4j}3B_6\right),\\
D_9 &= \sum_{j=1}^{2}A_{4j}B_6.
\end{aligned}
$$

By the boundary conditions

$$P(y,V_1,V_2,\tau;y)=K(1-\mathrm{e}^{-\sqrt{V_1+V_2}y\sqrt{\tau}}), \tag{2.7}$$

using the regular short maturity asymptotic expansion (1.6) to the left hand side of (2.7) and Taylor series expansion to the right hand side of (2.7), we have

$$\sum_{n=1}^{\infty}P_n(y,V_1,V_2)\tau^{n/2}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}K(V_1+V_2)^{n/2}y^n\tau^{n/2},$$

the two expansions equal only when the coefficients of $\tau^{n/2}$ in both expansion equal, that is

$$P_n(y,V_1,V_2)=\frac{(-1)^{n+1}}{n!}K(V_1+V_2)^{n/2}y^n,$$

from the characteristic expression of $P_n(y,V_1,V_2)$ given in (2.1), we have

$$C_n(V_1,V_2)[p_n^0(y)\Phi(y)+q_n^0(y)\phi(y)]+p_n^1(y)\Phi(y)+q_n^1(y)\phi(y)=\frac{(-1)^{n+1}}{n!}K(V_1+V_2)^{n/2}y^n,$$

then $C_n(V_1,V_2)$ can be solved from the above equation as

$$C_n(V_1,V_2)=\frac{\frac{(-1)^{n+1}}{n!}K(V_1+V_2)^{n/2}y^n-p_n^1(y)\Phi(y)-q_n^1(y)\phi(y)}{p_n^0(y)\Phi(y)+q_n^0(y)\phi(y)}. \tag{2.8}$$

3 Modified Approximation Expansion Method

To improve the accuracy, we modify the asymptotic expansions method by approximating an American option price by European option price plus the early exercise premium, that is

$$P(\theta, V_1, V_2, \tau) = P_E(\theta, V_1, V_2, \tau) + \epsilon,$$

where $P_E(\theta, V_1, V_2, \tau)$ represents the European put price with initial normalized moneyness θ and maturity τ, and ϵ represents the early exercise premium. The European put price $P_E(\theta, V_1, V_2, \tau)$ is computed by the efficient COS method proposed by [FO08], for the pricing formula for the $P_E(\theta, V_1, V_2, \tau)$ please refer to [FO08]. The early exercise premium ϵ can be approximated by

$$\epsilon = P(\theta, V_1, V_2, \tau; \tilde{y}) - P(\theta, V_1, V_2, \tau; \infty),$$

where $P(\theta, V_1, V_2, \tau; \tilde{y})$ and $P(\theta, V_1, V_2, \tau; \infty)$ are the price of barrier put option with barrier level of $\tilde{y}$ and ∞ respectively. $P(\theta, V_1, V_2, \tau; \tilde{y})$ is the approximation of the American option and can be computed by (3.1), $P(\theta, V_1, V_2, \tau; \infty)$ is approximation of European option by the expansion method, it can be computed as

$$P(\theta, V_1, V_2, \tau; \infty) \approx \lim_{y \to \infty} P(\theta, V_1, V_2, \tau; y).$$

As $y \to \infty$, we have from (2.8)

$$C_1(V_1, V_2) = K(V_1 + V_2)^{1/2}, \qquad C_2(V_1, V_2) = -K(V_1 + V_2)/2,$$
$$C_3(V_1, V_2) = \frac{K(V_1 + V_2)^{3/2}}{3! f_{30}}, \qquad C_4(V_1, V_2) = -\frac{K(V_1 + V_2)^2}{4! f_{40}}$$

then by (2.1)

$$\begin{aligned}
& P(\theta, V_1, V_2, \tau; \infty) \\
\approx \ & \Big\{C_1(V_1, V_2)\left[p_1^0(\theta)\Phi(\theta) + q_1^0(\theta)\phi(\theta)\right] + p_1^1(\theta)\Phi(\theta) + q_1^1(\theta)\phi(\theta)\Big\}\sqrt{\tau} \\
& + \Big\{C_2(V_1, V_2)\left[p_2^0(\theta)\Phi(\theta) + q_2^0(\theta)\phi(\theta)\right] + p_2^1(\theta)\Phi(\theta) + q_2^1(\theta)\phi(\theta)\Big\}\tau \\
& + \Big\{C_3(V_1, V_2)\left[p_3^0(\theta)\Phi(\theta) + q_3^0(\theta)\phi(\theta)\right] + p_4^1(\theta)\Phi(\theta) + q_2^1(\theta)\phi(\theta)\Big\}\tau^{2/3} \\
& + \Big\{C_4(V_1, V_2)\left[p_4^0(\theta)\Phi(\theta) + q_3^0(\theta)\phi(\theta)\right] + p_4^1(\theta)\Phi(\theta) + q_2^1(\theta)\phi(\theta)\Big\}\tau^2 + o(\tau^2) \\
= \ & \Big\{K\sqrt{V_1 + V_2}\left[p_1^0(\theta)\Phi(\theta) + q_1^0(\theta)\phi(\theta)\right] + p_1^1(\theta)\Phi(\theta) + q_1^1(\theta)\phi(\theta)\Big\}\sqrt{\tau} \\
& + \Big\{-K(V_1 + V_2)/2\left[p_2^0(\theta)\Phi(\theta) + q_2^0(\theta)\phi(\theta)\right] + p_2^1(\theta)\Phi(\theta) + q_2^1(\theta)\phi(\theta)\Big\}\tau \\
& + \left\{\frac{K(V_1 + V_2)^{3/2}}{3! f_{30}}\left[p_3^0(\theta)\Phi(\theta) + q_3^0(\theta)\phi(\theta)\right] + p_3^1(\theta)\Phi(\theta) + q_3^1(\theta)\phi(\theta)\right\}\tau^{3/2} \\
& + \left\{-\frac{K(V_1 + V_2)^2}{4! f_{40}}\left[p_4^0(\theta)\Phi(\theta) + q_4^0(\theta)\phi(\theta)\right] + p_4^1(\theta)\Phi(\theta) + q_4^1(\theta)\phi(\theta)\right\}\tau^2 + o(\tau^2)
\end{aligned}$$

substitute the polynomials of $p_n^0, q_n^0, p_n^1, q_n^1, n = 1, 2, 3, 4$ of (2.5) into the above equation, yields

$$
\begin{aligned}
& P(\theta, V_1, V_2, \tau; \infty) \\
\approx\ & K\sqrt{V_1 + V_2}\,[\theta\Phi(\theta) + \phi(\theta)]\,\sqrt{\tau} \\
& + \left\{-K(V_1 + V_2)/2\left[(\theta^2 + f_{2,1})\Phi(\theta) + \theta\phi(\theta)\right] + h_{2,0}\Phi(\theta)\right\}\tau, \\
& + \left\{\frac{K(V_1 + V_2)^{3/2}}{3!f_{30}}[(f_{30}\theta^3 + f_{31}\theta)\Phi(\theta) + (g_{30}\theta^2 + g_{31})\phi(\theta)] + h_{30}\theta\Phi(\theta) + k_{30}\phi(\theta)\right\}\tau^{3/2} \\
& + \left\{-\frac{K(V_1 + V_2)^2}{4!f_{40}}[(f_{40}\theta^4 + f_{41}\theta^2 + f_{42})\Phi(\theta) + (g_{40}\theta^3 + g_{41}\theta)\phi(\theta)]\right. \\
& \left. +(h_{40}\theta^2 + h_{41})\Phi(\theta) + k_{40}\theta\phi(\theta)\right\}\tau^2 + o(\tau^2)
\end{aligned}
$$

where $f_{2,1}, h_{2,0}, f_{30}, f_{31}, g_{30}, g_{31}, h_{30}, k_{30}, f_{40}, f_{41}, f_{42}, g_{40}, g_{41}, h_{40}, h_{41}, k_{40}$ are as given in the Section 2.

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