A Formula For Discrete Barrier Options

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Abstract

In the present paper we provide an analytical solution for pricing discrete barrier options in the Black-Scholes framework. We reduce the valuation problem to a Wiener-Hopf equation that can be solved analytically. We are able to give explicit expressions for the Greeks of the contract. The results from our formulae are compared with those from other numerical methods available in the literature. Very good agreement is obtained, although evaluation using the present method is substantially quicker than the alternative methods presented.

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1 Introduction

In this paper we study the valuation problem for discrete barrier options. Barrier options are common, extensively traded types of exotic derivatives. Barrier options are activated (knock-ins) or terminated (knock-outs) if a specific trigger is reached within the expiry date. There are now several papers dealing with the pricing of barrier options and a great number of valuation techniques have been proposed. In practice, barrier options differ from those studied in the academic literature in many respects. One of the most important is the monitoring frequency of the underlying assets, i.e. the frequency of observation of the triggering event. With discrete monitoring the trigger is checked at fixed times (e.g. weekly or monthly). As a consequence a knock-out (knock-in) option becomes less (more) expensive as the number of monitoring dates increases. In the case of continuous monitoring, several pricing formulae in the Black-Scholes framework are known, [G-Y], [K-I], [Ri]. Unfortunately, discrepancy between option prices under continuous and discrete monitoring can be huge. For this reason, several papers have proposed approximations based on either the continuous formula, [B-G-K], [B-G-K2], [Ho], or a variety of different numerical approaches, [A-L], [A-W-D-N], [B-B], [B-T], [D-D-G-S], [H-K], [Su]. A detailed discussion can be found in [F-R].

In the present paper, we consider the pricing problem of a down-out barrier option under a Geometric Brownian Motion (GBM) process with discrete monitoring rather than continuous monitoring. We then show how to reduce its evaluation to an integral equation of Wiener-Hopf type. The latter problem admits an *analytical solution* in the standard Black-Scholes framework, i.e. when the underlying asset evolves according to a GBM and the knock-out clause is activated by a constant barrier. The Wiener-Hopf technique has recently been used successfully to solve a number of different evaluation problems, especially within the framework of exotic derivatives [NN-Y], [B-L], [B-L2], [Ro] and in connection with Lévy Processes. However, the exotic contracts considered in these papers always assume continuous monitoring of the underlying asset and the Wiener-Hopf factorizations used there arise in a way not directly related to that used in our problem. Moreover, the analytical solutions obtained therein do not admit simple numerical implementation (typically the solution is given as a complex multiple integral with difficult integrands). The paper is organised as follows. In the following Section, the financial model is reduced to a scalar Wiener-Hopf integral equation of the second kind. The solution is obtained in terms of infinite sums of simple functions plus a single special function (11). These are summarised in Section 2.1. Section 2.2 offers the formal solution of the barrier option price as an inverse z-transform of the Wiener-Hopf solution, and also presents explicit formulae for the Greeks that makes our procedure really competitive with respect to Monte Carlo simulation. Numerical results are compared and contrasted with alternative numerical methods in Section 3, and final remarks are offered in Section 4.

2 The model

We work in a standard Black-Scholes framework where the underlying asset evolves, under the risk-neutral measure, according to a GBM process:

$$dx_t = rx_t dt + \sigma x_t dW,$$

with x_0 the initial stock price and where r and σ are respectively the constant risk-free rate and the constant instantaneous volatility. We want to price a down and out call option, i.e. a call option that expires worthless if a lower barrier has been hit at a monitoring date. The corresponding down and in call option can be priced subtracting from the price of a standard call option the price of the down-out call option. The barrier put option can be priced using the put-call transformation given in [Ha]. Let $0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$ be the monitoring dates, T the option maturity and l be the constant lower barrier active at all times t_n. The nth time interval is defined as $t_n < t < t_{n+1}$ and we denote the price of the barrier option in this interval as $C(x,t,n) \equiv C(x,t,n;l)$. Then, $C(x,t,n)$ satisfies the well known Black-Scholes partial differential equation (PDE):

$$-\frac{\partial C(x,t,n)}{\partial t} + rx\frac{\partial C(x,t,n)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C(x,t,n)}{\partial x^2} = rC(x,t,n). \qquad (1)$$

Given that the trigger condition is checked only at fixed times, we need to update the initial condition at each of the monitoring dates t_n:

$$\begin{aligned} C(x, t_n, n) &= C(x, t_n, n-1)\,\mathbf{1}_{(x \geq l)}, \\ C(x, t_0, 0) &= (x - K)\,\mathbf{1}_{(x \geq \max(K,l))}, \end{aligned}$$

where K is the exercise price of the option and $\mathbf{1}_{(x \geq l)}$ is the indicator, or Heaviside, function:

$$\mathbf{1}_{(x \geq l)} = \begin{cases} 1 & \text{if} \quad x \geq l, \\ 0 & \text{if} \quad x < l. \end{cases}$$

We can use the standard change of variables (see Wilmott et al. [W-D-H] p. 98):

$$C(x, t, n) = w(z, t, n),$$

where:

$$z = \ln(x/l)\,; k = \ln(K/l)\,; m = r - \sigma^2/2,$$

to transform the partial differential equation (1) into

$$\begin{aligned} &-w_t + m w_z + \tfrac{\sigma^2}{2} w_{zz} = rw, \\ &w(z, t_n, n) = w(z, t_n, n-1)\,\mathbf{1}_{(z \geq 0)}, n = 1, 2, 3 \ldots, \\ &w(z, t_0 = 0, 0) = l\left(e^z - e^k\right)\mathbf{1}_{(z \geq \delta)}, \end{aligned}$$

with

$$\delta = \max(k, 0)\,. \tag{2}$$

With the second transformation:

$$w(z, t, n) = e^{\alpha z + \beta t} g(z, t, n),$$

where:

$$\alpha = -\frac{m}{\sigma^2}; c^2 = \frac{\sigma^2}{2}; \beta = \alpha m + \frac{\alpha^2 \sigma^2}{2} - r,$$

the function $g(z, t, n)$ satisfies the heat equation:

$$-g_t + c^2 g_{zz} = 0 \tag{3}$$

with initial conditions, for $n = 1, 2, 3, \ldots$,

$$g(z, t_n, n) = g(z, t_n, n-1)\,\mathbf{1}_{(z \geq 0)},$$

whilst when $n = 0$ the initial condition becomes:

$$g(z, 0, 0) = l e^{-\alpha z}\left(e^z - e^k\right)\mathbf{1}_{(z \geq \delta)}.$$

When $t_n < t < t_{n+1}$ and $z > 0$ the solution of the above partial differential equation (3) is given by, see e.g. Strauss [St] p. 47,

$$g(z, t, n) = \begin{cases} \int_0^{+\infty} S(z - \xi, t - t_n)\, g(\xi, t_n, n-1)\, d\xi, & n = 1, 2, \ldots, \\ l \int_0^{+\infty} S(z - \xi, t - t_n)\, e^{-\alpha \xi}\left(e^{\xi} - e^k\right)\mathbf{1}_{(\xi \geq \delta)} d\xi, & n = 0, \end{cases}$$

where the kernel $S(z, t)$ is the Gaussian:

$$S(z, t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-z^2/\left(4c^2 t\right)}.$$

Let us consider the function $g(z,t,n)$ only at the monitoring times t_n, and let τ be the fixed period between the monitoring dates so that $t_n + \tau = t_{n+1}$. We set $f(z,n) = g(z,t_n,n-1)$, i.e. the value of g at the upper end, t_n, of the $n-1$th time interval, so that we have:

$$f(z,n) = \int_0^{+\infty} \frac{e^{-(z-\xi)^2/\left(4c^2\tau\right)}}{\sqrt{4\pi c^2 \tau}} f(\xi, n-1)\, d\xi, \quad n = 2, 3, \ldots, \tag{4}$$

and for $n = 1$:

$$f(z,1) = l \int_0^{+\infty} S(z-\xi,\tau)\, e^{-\alpha\xi} \left(e^{\xi} - e^{k}\right) \mathbf{1}_{(\xi \geq \delta)} d\xi. \tag{5}$$

The last equation may be absorbed into the set in (4) by defining the additional function

$$f(z,0) = l e^{-\alpha z} \left(e^{z} - e^{k}\right) \mathbf{1}_{(z \geq \delta)}. \tag{6}$$

We now take the *z-transform* of the above difference equation (4) by multiplying both sides of (4) by q^n, $q \in \mathbb{C}$:

$$q^n f(z,n) = q \int_0^{+\infty} S(z-\xi,\tau)\, q^{n-1} f(\xi, n-1)\, d\xi,$$

and then summing over all n, $n \geq 1$. Assuming that we can interchange the order of integration and summation (which may be proved *a posteriori*), we obtain:

$$\begin{aligned} \sum_{n=1}^{\infty} q^n f(z,n) &= q \int_0^{+\infty} S(z-\xi,\tau) \sum_{n=1}^{\infty} q^{n-1} f(\xi, n-1)\, d\xi \\ &= q \int_0^{+\infty} S(z-\xi,\tau) \sum_{n=0}^{\infty} q^{n} f(\xi, n)\, d\xi. \end{aligned}$$

So, by defining:

$$F(z,q) = \sum_{n=0}^{+\infty} q^n f(z,n), \tag{7}$$

and adding $f(z,0)$ to both sides, we arrive at the following integral equation for $F(z,q)$:

$$F(z,q) = q \int_0^{+\infty} S(z-\xi,\tau)\, F(\xi,q)\, d\xi + f(z,0) \tag{8}$$

defined over the interval $0 < z < \infty$.

2.1 Solution of the Wiener-Hopf equation

We can recognize in (8) an integral equation of the second kind with a semi-infinite range, and a convolution structure, i.e. the kernel S depends on the difference $z - \xi$. This integral equation has to be solved with respect to the unknown function $F(z,q)$. Fortunately, given the form of the kernel, this can be recognized as a Wiener-Hopf equation; see [P-M, N]. We remark that if the integral were extended to the whole real line it would be sufficient to use a Fourier transform method. However, since the integral range, and the variable z range, is the positive real line, it is necessary here to employ the Wiener-Hopf method. The Wiener-Hopf technique has a long and illustrious history. It was originally invented in 1931 to solve a specific problem involving neutron diffusion and has been employed in many thousands of articles since then in the fields of statistics, physics, mathematics, engineering etc. Applications of the method are very diverse, as can be found from the bibliography lists in, say [A, A2], and include the topics of fracture mechanics, wave diffraction (electromagnetic, acoustic, water), geophysical problems, crystal growth and more recently mathematical finance. The method is one of very few approaches offering an *exact* solution to a phyisically relevant class of integral equations. Discussion on the Wiener-Hopf technique can be found in more advanced textbooks on complex variable methods or integral equations, but the interested reader is particularly referred to the classic book by Noble [N]. There are several key steps in obtaining an exact analytical solution to Wiener-Hopf equations. First, one must apply a Fourier transform to the integral equation, which converts it into a Riemman-Hilbert equation defined in a strip in the complex transform parameter plane. To solve this equation, the transformed kernel of the integral equation (or here more precisely $\delta(\xi) - qS(\xi, \tau)$, where $\delta(\xi)$ is the generalised delta function) must be decomposed into a product of two functions, one analytic in the upper half of the transform plane and the other analytic in an overlapping lower half plane. By this means, the equation can be rearranged so that the left (right) side has similar upper (lower) analyticity properties, and hence analytical continuation arguments, and Liouville's theorem, can be applied to obtain an explicit solution. All details of the solution procedure for equation (8) are omitted here and can be found in Fusai et al. [1]. The exact solution of the above integral equation is given by the following expression:

$$\begin{aligned}
&F(z,q)\\
&= -\tfrac{il\gamma}{2} e^{(1-\alpha)k} \textstyle\sum_{n=-\infty}^{\infty} \frac{e^{i\mu_n |z-k|/\gamma}}{\mu_n(\mu_n - i\alpha\gamma\ \mathrm{sgn}(z-k))(\mu_n - i(\alpha-1)\gamma\ \mathrm{sgn}(z-k))}\\
&+ l e^{-\alpha z} \left\{ \tfrac{e^z}{1-qe^{(\alpha-1)^2\gamma^2}} - \tfrac{e^k}{1-qe^{(\alpha\gamma)^2}} \right\} \mathbf{1}_{(z\geq k)}\\
&- \tfrac{il\gamma}{4} e^{(1-\alpha)k} \textstyle\sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n) e^{i\mu_n z/\gamma}}{\mu_n} \sum_{m=-\infty}^{\infty} \frac{L_+(\mu_m) e^{i\mu_m k/\gamma}}{\mu_m(\mu_m + i\alpha\gamma)(\mu_m + i(\alpha-1)\gamma)(\mu_m + \mu_n)},
\end{aligned} \tag{9}$$

valid for $k > 0$, where the sign of x is denoted by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases} \tag{10}$$

and from (**??**):

$$L_+(u) = \exp\left\{ \frac{u}{\pi i} \int_0^{+\infty} \frac{\ln\left(1 - qe^{-z^2}\right)}{z^2 - u^2} dz \right\}, \quad \Im(u) > 0. \tag{11}$$

Also, the complex coefficients are:

$$\mu_m = \sqrt{\ln q + 2m\pi i}, \quad -\infty < m < \infty, \tag{12}$$

which lie in the upper half plane as shown in Figure (2). Note that (9) is given by (**??**) and (**??**) with the constants there replaced by the original parameters $a = \alpha\gamma$, $b = \gamma$, and γ:

$$\gamma = c\sqrt{\tau} = \sigma\sqrt{\frac{\tau}{2}}. \tag{13}$$

Finally, for (9) to offer a unique solution to the Wiener-Hopf equation (8) it is shown in (**??**) that the z-transform parameter q satisfies

$$|q| < \exp\left\{ -(1-\alpha)^2\gamma^2 \mathbf{1}_{((1-\alpha)\gamma\geq 0)} \right\}. \tag{14}$$

In PREMIA we do not have considered the case $k \leq 0$, however for this case see Fusai et al. [1].

[INSERT FIGURE 2]

2.2 The analytical formula

The solution of the Wiener-Hopf equation gives the function $F(z, q)$, but we still have to invert the z-transform in order to recover the original function $f(z, n)$. It is easily shown that the inversion formula has the following integral representation:

$$f(z, n) = \frac{1}{2\pi\rho^n} \int_0^{2\pi} F\left(z, \rho e^{iu}\right) e^{-inu} du, \quad n = 0, 1, 2, \ldots, \tag{15}$$

where $\rho = |q|$ satisfies (14). Under the latter constraint, the function $F\left(z, \rho e^{iu}\right)$ is analytic for all $0 \leq u \leq 2\pi$ and so may be differentiated without difficulty. Formally, expression (15) joint with (9) and

(**??**), gives an analytical solution for the pricing problem of discrete barrier options. The barrier option price at a monitoring date t_n, i.e. $C(x, t_n, n-1)$, is given by:

$$C(x, t_n, n-1) = \left(\frac{x}{l}\right)^{\alpha} e^{\beta t_n} f\left(\ln\frac{x}{l}, n\right), \quad n = 1, 2, 3, \ldots. \tag{16}$$

The above formula can be used also for the computation of the Greeks, i.e. the derivatives with respect to x of the option price. Under the assumption that we can derive the corresponding series term by term, for the delta we have:

$$\Delta = \frac{\partial C}{\partial x}(x, t_n, n-1) = \frac{1}{l}(\frac{x}{l})^{\alpha-1} e^{\beta t_n}\left[\alpha f(\ln\frac{x}{l}, n) + \frac{\partial f}{\partial z}(z, n)\right]_{z=\ln\frac{x}{l}},$$

where:

$$\frac{\partial f}{\partial z}(z, n) = \frac{1}{2\pi\rho^n}\int_0^{2\pi}\frac{\partial F}{\partial z}(z, \rho e^{iu})e^{-inu}du$$

and $\frac{\partial F}{dz}(z, q)$ can be computed from (9) for $k > 0$:

$$\begin{aligned}
&\tfrac{\partial F}{dz}(z,q)\\
&= \tfrac{l}{2}e^{(1-\alpha)k}\ \mathrm{sgn}(z-k)\textstyle\sum_{n=-\infty}^{+\infty}\frac{e^{i\mu_n|z-k|/\gamma}}{(\mu_n-i\alpha\gamma\ \mathrm{sgn}(z-k))(\mu_n-i(\alpha-1)\gamma\ \mathrm{sgn}(z-k))}\\
&+le^{-\alpha z}\left[\tfrac{(1-\alpha)e^z}{1-qe^{(\alpha-1)^2\gamma^2}}+\tfrac{\alpha e^k}{1-qe^{\alpha^2\gamma^2}}\right]1_{(z\geq k)}\\
&+\tfrac{l}{4}e^{(1-\alpha)k}\textstyle\sum_{n=-\infty}^{+\infty}L_+(\mu_n)e^{i\mu_n z/\gamma}\sum_{m=-\infty}^{+\infty}\frac{L_+(\mu_m)e^{i\mu_m k/\gamma}}{\mu_m(\mu_m+i\alpha\gamma)(\mu_m+i(\alpha-1)\gamma)(\mu_m+\mu_n)}.
\end{aligned} \tag{17}$$

In a similar way we obtain:

$$\Gamma = \frac{\partial^2 C}{\partial x^2}(x, t_n, n-1) = \frac{1}{l^2}(\frac{x}{l})^{\alpha-2}e^{\beta t_n}\left[\alpha(\alpha-1)f(z,n) + (2\alpha-1)\frac{\partial f}{\partial z} + \frac{\partial^2 f}{\partial z^2}\right]_{z=\ln(\frac{x}{l})},$$

where:

$$\frac{\partial^2 f}{\partial z^2}(z, n) = \frac{1}{2\pi\rho^n}\int_0^{2\pi}\frac{\partial^2 F}{\partial z^2}(z, \rho e^{iu})e^{-inu}du$$

and when $k > 0$:

$$\begin{aligned}
&\tfrac{\partial^2 F}{\partial z^2}(z,q)\\
&= \tfrac{il\gamma}{2}e^{(1-\alpha)k}\textstyle\sum_{n=-\infty}^{+\infty}\frac{\mu_n e^{i\mu_n|z-k|}}{(\mu_n-i\alpha\gamma\ \mathrm{sgn}(z-k))(\mu_n-i(\alpha-1)\gamma\ \mathrm{sgn}(z-k)}\\
&+le^{-\alpha z}\left(\tfrac{(1-\alpha)^2e^z}{1-qe^{(\alpha-1)^2\gamma^2}}-\tfrac{\alpha^2 e^k}{1-qe^{\alpha^2\gamma^2}}\right)1_{(z\geq k)}\\
&+\tfrac{il}{4\gamma}e^{(1-\alpha)k}\textstyle\sum_{n=-\infty}^{+\infty}\mu_n L_+(\mu_n)e^{i\mu_n z/\gamma}\sum_{m=-\infty}^{+\infty}\frac{L_+(\mu_n)e^{i\mu_n k/\gamma}}{\mu_m(\mu_m+i\alpha\gamma)(\mu_m+i(\alpha-1)\gamma)(\mu_m+\mu_n)}
\end{aligned} \tag{18}$$

Note that convergence of the infinite sums in the above expressions is immediately apparent when $z \neq k$ due to the exponential factors. However, when $z = k$ or when z or k is small convergence is very slow. Also, in PREMIA we have not considered the corresponding expressions to (17), (18) when $k \leq 0$.

By carefully combining $f(z, n)$ and its derivatives appearing in Δ and Γ, one can obtain expressions that are computationally of the same cost as that for the option price itself.

3 Numerical results

In this section we offer numerical results. To do this, a solution to the barrier option price (16) can be computed by combining (9) with a numerical approximation to the z-transform inverse (15). A simple and accurate algorithm, based on the Fourier-series method, can be found in Abate and Whitt [A-W]. They approximate the integral in (15) using the trapezoidal rule with a step size of π/n, and obtain the result:

$$\begin{aligned}
f(z,n) &\approx \tilde{f}(z,n) = \frac{1}{2n\rho^n}\sum_{j=1}^{2n}(-1)^j\,\Re\left(F\left(z, \rho e^{ji\pi/n}\right)\right) \\
&= \frac{1}{2n\rho^n}\left\{F(z,\rho) + (-1)^n F(z,-\rho) + 2\sum_{j=1}^{n-1}(-1)^j\,\Re\left(F\left(z,\rho e^{ji\pi/n}\right)\right)\right\}.
\end{aligned} \tag{19}$$

Note that the last expression is valid for all $n > 0$, where the sum term is taken as zero for $n = 1$. Abate and Whitt are able to provide an error bound when, in our case, ρ satisfies the constraint (14):

$$\left| f(z,n) - \tilde{f}(z,n) \right| \leq \frac{\rho^{2n}}{1-\rho^{2n}}.$$

For practical purposes, this error bound, when ρ^{2n} is small, is approximately equal to ρ^{2n}. Hence, to have accuracy to $10^{-\mu}$, say, we require $\rho = 10^{-\mu/2n}$, which lies within the uniqueness constraint. Note that in practice it is important to adjust ρ in this way for each value of n, so that ρ^n stays small but bounded away from zero as $n \to \infty$. Otherwise small computational errors in the numerical evaluation of the denominator in (19) is magnified and will lead to gross errors in the evaluation of $f(z,n)$. Note that the computational cost in calculating $f(z,n)$ from (19) increases linearly with monitoring frequency n.

In Table 1, we compare different numerical methods with the solution of the Wiener-Hopf equation. We consider a single barrier down-out call option for different barrier levels and different monitoring dates. The competing methods are the recursive integration method (RI) in [A-L] where a grid with 2000 nodes has been used, the continuous monitoring formula (CC) with a correction based on shifting the barrier level in [B-G-K2], the trinomial tree (TT) in [B-G-K], the Simpson recursive quadrature method (SQ) (grid spacing 2000 points) in [F-R], Monte Carlo simulation with 10^8 simulations with Mersenne twister pseudo random generator and antithetic variables in [B-B][4]. Note that, as we choose the spot price equal to the strike price in the Table, there is a small difficulty regarding the numerical convergence of the first sum in $F(z,q)$ in (9), which decays only as fast as $O\left(1/n^{3/2}\right)$ as $n \to \infty$. This is easily improved to any level of convergence by a standard rearrangement, as is given in Fusai et al .[1]. The WH solution in (9) has been computed using 20 terms in the first sum and 300 terms for each sum appearing in the third term (except when $n = 5$ and $l = 99$ when 700×700 terms are required for the given number of decimal places; for 300×300 the formula gives the value 4.48911). As we can see, the solution gives results comparable to other methods and our values can be considered exact to the figures quoted.

[INSERT TABLE 1 HERE]

In Table 2 we consider a numerically more difficult example. That is, increasing the number of monitoring dates and considering the barrier level approaching the spot price. The competing methods are the Markov Chain method (MCh) in [D-D-G-S], the Trinomial Tree method (TT) in [B-G-K2], the Simpson recursive quadrature method (grid spacing 2000 points) in [F-R] and the Monte Carlo simulation with 10^8 simulations with Mersenne twister pseudo random generator and antithetic variables in [B-B]. For the case of 25 monitoring dates and for a spot price far from the barrier (e.g. $l = 95$ in Table 2) we get an accurate solution. But as we move the barrier level closer to the spot price, the accuracy of our numerical solution in (9) deteriorates. This is due to the very weak convergence of the double summation when z and k are very small. The error also increases as we decrease the number of monitoring dates (from 125 to 25) because $1/\gamma = \sqrt{2n/T}/\sigma$ appears in the exponent of each sum (which is why we required 700×700 terms for $n = 5$ and $l = 99$ in Table 1). In general a very large number of terms are required, much greater than the 300 × 300 actually used in Table 2, to yield acceptable values when the barrier is close to the spot and strike prices. For this reason, we need an improved representation for the exact solution when k or z approach zero. A useful alternative exact form is derived in Fusai, Abrahams and Sgarra but has not been implemented in Premia.

[INSERT TABLE 2 HERE]

We examine in Table 4 the convergence of the discretely monitored barrier option price, Delta and Gamma to the corresponding continuously monitored quantities, i.e. the convergence for $N \to \infty$, given a fixed time to maturity. This Table shows the slow convergence of the option price and Greeks to their continuous time values. It therefore confirms the importance of an accurate procedure for pricing discretely monitored options and indicates the inaccuracy deriving from the use of a continuous monitoring

[4]We would like to thank M. Bertoldi and M. Bianchetti (Caboto SIM, Banca Intesa Group) for kindly providing the results of the Monte Carlo simulation.

formula. Note that the derivatives of the discrete option converge to the continuous values at the same rate as the option price itself, i.e. like $O\left(n^{-\frac{1}{2}}\right)$ as $n \to \infty$ (see Section **??**). This rate of convergence is confirmed from the numerical results in Table 4. For example the option Delta when $N = 1500$ is 1.05691 and for $N \to \infty$ is 1.073105. If we set $\Delta(N) = 1.073105 + (1.05691 - 1.073105)(1500/N)^{\frac{1}{2}}$, then we have $\Delta(1000) = 1.0533$, $\Delta(800)$ =1.0509 and $\Delta(500) = 1.0451$ that are very close to the results reported in Table 4. It is easily verified that the same slow convergence is true for the Gamma as well. Note that a small number of monitoring dates can even give a Gamma with an opposite sign to the continuous case. Clearly it would therefore be dangerous to use a continuous time formula to Delta-Gamma hedge a position with discrete barriers. Note that in Table 4, for the Monte Carlo estimates reported in [B-B], the Delta and Gamma have been computed using Malliavin calculus according to Fourniè et. al. [F-L-L-L].

[**INSERT TABLE 3**]

4 Conclusion

In this paper we have introduced a Wiener-Hopf and z-transform approach to obtain an analytical solution to the single barrier problem under the hypothesis of a Geometric Brownian Motion evolution for an underlying asset. As shown, the solution thus derived is in a form suitable for numerical evaluation, and the results were compared and contrasted with other numerical/approximate techniques. The present method could also be used to find the solution under different assumptions for the evolution. For example, it is easy to check that this approach applies in all cases in which the process for the underlying asset has a Markov feature with a stationarity assumption. A wide class of evolution processes could then be solved with the present technique. For example, when the dynamics of the underlying asset is described by a Lévy process, the suggested procedure seems to admit an analytical solution although this represents work in progress. Two advantages of the exact solution for discrete barrier options are (i) the derivation of explicit expressions for the Greeks, and (ii) that the convergence to the result for continuous monitoring could be proved in the limit as the number of monitoring dates tends to infinity for a fixed time to maturity. This was performed in Section **??**, and further we proved there that the rate of convergence is very slow, behaving as $O(\sqrt{\tau})$ where τ is the fixed time interval between monitoring dates. This confirms the behaviour found numerically for the double barrier option results displayed in Figure 1. Finally, in the case of double barrier options, the pricing problem can be reduced to the solution of a Fredholm integral equation of the second kind with a difference kernel, which it is possible to solve in an L^2-frame. It is straightforward to pose the corresponding eigenvalue problem for the integral operator, and the solution admits a representation in terms of their eigenvalues and eigenfunctions. However, the exact calculations of the latter is sometimes involved and requires suitable numerical approximation. An alternative approach for double barrier options is to employ the modified Wiener-Hopf technique (or Jones' method) [N] and this is currently being investigated by the authors.

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Figure 1

The complex u-plane showing the locus of the zeros of $L(u)$, $\pm\mu_n$ $(-\infty < n < \infty)$, as $\arg(q)$ increases from 0 to 2π and with $|q| = 1/2$. The indicated positions on the path are when $\arg(q) = 0$. Also shown is the overlapping strip of analyticity, $\mathcal{D} \equiv \{u| \, (1-\alpha)\,\gamma < \Im(u) < \Im(\mu_0)\}$ (where in this example $(1-\alpha)\,\gamma > -\Im(\mu_0)$), inside which the Wiener-Hopf equation (**??**) is defined.

Table 1:

In the Table we price a single barrier down-out call option for different levels of l and different monitoring dates N. Parameters used are spot price=100, strike =100, r=0.10, σ=0.3, T=0.2. The competing methods are the recursive integration method (RI) in [A-L] where a grid with 2000 nodes has been used, the continuous monitoring formula (CC) with a correction based on shifting the barrier level in [B-G-K2], the trinomial tree (TT) in [B-G-K], the Simpson recursive quadrature method (SQ) (grid spacing 2000 points) in [F-R], Monte Carlo (MC) with 10^8 simulations with Mersenne twister Generator and antithetic variables according to the results reported in [B-B]. The Wiener-Hopf (WH) solution in (9) has been computed using formula (9) with 20 terms in the first sum and the accelerating procedure described in Appendix **??**, and 300 terms for each sum appearing in the third term. The inversion of the z-transform has been computed setting $\rho = 10^{-\mu/2N}$, $\mu = 8$ and N the number of monitoring dates.

l	N	WH+ZTI	RI	CC	TT	SQ	MC (s.e.)
89	5	6.28076	6.2763	6.284	6.281	6.2809	6.28092 (0.00078)
95	5	5.67111	5.6667	5.646	5.671	5.6712	5.67124 (0.00076)
97	5	5.16725	5.1628	5.028	5.167	5.1675	5.16739 (0.00073)
99	5	4.48917	4.4848	4.050	4.489	4.4894	4.48931 (0.0007)
89	25	6.20995	6.2003	6.210	6.21	6.2101	6.21059 (0.00078)
95	25	5.08142	5.0719	5.084	5.081	5.0815	5.08203 (0.00073)
97	25	4.11582	4.1064	4.113	4.115	4.11598	4.11621 (0.00067)
99	25	2.81244	2.8036	2.673	2.812	2.8128	2.81261 (0.00057)

Table 2

In the Table we price a single barrier down-out call option for different levels of l and different monitoring dates N. Parameters used are spot price=100, strike =100, r=0.10, σ=0.2, T=0.5. The competing numerical approximations are the Markov Chain (MCh) with a grid with 1001 points in [D-D-G-S], Monte Carlo (MC) with 10^8 simulations with Mersenne twister Generator and antithetic variables according to the results reported in [B-B] (in brackets we have the standard errors), the trinomial tree (TT) in [B-G-K], the Simpson recursive quadrature method (SQ) (grid spacing 2000 points) in [F-R]. The WH solution in summation form (9) has been computed using $n_1 = 20$, $n_2 = m_2 = 300$. The WH solution in integral from (**??**) has been computed with the contour C running from $-\infty$ to $+\infty$ along a line parallel to the real axis with $\Im(\xi) = (\gamma(1-\alpha) + \Im(\mu_0))/2$ when $\alpha \leq 1 + \Im(\mu_0)/\gamma$, and the contour C_0 running from $-\infty$ to $+\infty$ along a line such that $\Im(\zeta) = \Im(\mu_0)/2$. The parameter s has been set equal to 10 and we have truncated the sum using $n = 10$. The Padé approximant has been chosen with P=12 (degree of the numerator) and M=18 (degree of the denominator). The inversion of the z-transform has been computed setting $\rho = 10^{-\mu/2N}$, $\mu = 8$.

l	N	WH+ZT Formula (9)	MCh	TT	SQ	MC (s.e.)
95	25	6.63156	6.6307	6.6181	6.6317	6.63204 (0.0009)
99.5	25	3.35555	3.3552	3.3122	3.3564	3.35584 (0.00068)
99.9	25	2.95073	3.0095	2.9626	3.0098	3.00918 (0.00064)
95	125	6.16864	6.1678	6.1692	6.1687	6.16879 (0.00088)
99.5	125	1.9613	1.9617	1.9624	1.9628	1.96142 (0.00053)
99.9	125	1.5123	1.5138	1.5116	1.5123	1.5105 (0.00046)

Table 3

In the Table we show the convergence of the discrete option price and its derivatives to the continuous monitoring solution. Parameters used are spot price=100, strike =100, barrier =98, r=0.10, σ=0.3, T=0.2. The WH solution (**??**) has been computed using the integral representation given in Fusai et al. [1]. The parameter ρ in the z-transform has been set equal to $\rho = 10^{-\mu/2N}$, $\mu = 8$ with N the number of monitoring dates. We compare the WH results with Monte Carlo simulations (MC) with 10^8 runs if $N < 500$ and with 10^7 runs if $N \geq 500$, and with Mersenne Twister Generator and antithetic variables according to the results reported in [B-B] (in brackets we have the standard errors). In the Monte Carlo simulations, Delta and Gamma have been computed using Malliavin calculus according to Fourniè et. al. [F-L-L-L].

N	Option Price		Option Delta Δ		Option Gamma Γ	
	WH+ZT	MC (s.e)	WH+ZT	MC (s.e)	WH+ZT	MC (s.e.)
10	4.18224	4.18282 (0.00061)	0.79480	0.79503 (0.00018)	0.04513	0.04522 (0.00007)
50	3.12633	3.12605 (0.00055)	0.96652	0.96626 (0.00032)	0.02868	0.02845 (0.00026)
80	2.93918	2.93907 (0.00054)	0.99934	0.99923 (0.00039)	0.00891	0.00920 (0.00038)
100	2.86442	2.86415 (0.00054)	1.0106	1.01080 (0.00042)	-0.00090	-0.00047 (0.00047)
120	2.80903	2.80933 (0.00053)	1.01782	1.01807 (0.00046)	-0.00805	-0.00773 (0.00055)
150	2.7474	2.74720 (0.00053)	1.02457	1.02408 (0.00051)	-0.01481	-0.01507 (0.00067)
180	2.70163	2.70165 (0.00052)	1.02874	1.02914 (0.00055)	-0.01823	-0.01798 (0.00079)
200	2.67640	2.67621 (0.00052)	1.03087	1.03125 (0.00058)	-0.01936	-0.01962 (0.00088)
220	2.65481	2.65460 (0.00052)	1.03258	1.03250 (0.00061)	-0.01986	-0.01994 (0.00096)
250	2.62725	2.62717 (0.00052)	1.03471	1.03474 (0.00065)	-0.01986	-0.02073 (0.00108)
280	2.60410	2.60373 (0.00052)	1.03651	1.03538 (0.00068)	-0.01957	-0.02048 (0.00121)
300	2.59056	2.59051 (0.00069)	1.03759	1.03667 (0.00070)	-0.01904	-0.02159 (0.00129)
500	2.50259	2.50140 (0.00161)	1.04517	1.04516 (0.00283)	-0.01747	-0.02171 (0.00667)
800	2.43832	2.43790 (0.00160)	1.05101	1.05119 (0.00354)	-0.01794	-0.02055 (0.01059)
1000	2.4125	2.41461 (0.00159)	1.05332	1.05329 (0.00394)	-0.01812	-0.01254 (0.01318)
1500	2.37209	2.37305 (0.00158)	1.05691	1.05105 (0.00480)	-0.01832	-0.01924 (0.01959)
∞	2.18861		1.073105		-0.019252	