# Premia 22
# Closed Formula Methods

## Contents

# 1 The Black and Scholes model

Suppose the underlying asset price $(S_t, t \geq 0)$ evolves according to the Black and Scholes model with continuous yield $\delta$, that is

$$dS_s = S_s\left((r-\delta)ds + \sigma dB_s\right), \quad S_t = x$$

From now on, we denote by

- $T =$ maturity date $(Tt)$
- $\theta = T - t$
- $b = r - \delta$
- $K =$ strike price

In the following, we state the closed form solutions for the price of options whose payoff is given by $f(S_T)$, being $f$ a suitable function, i.e. for the quantity

$$F(t,x) = E\left[e^{-r\theta} f(S_T(x))\right].$$

# 2 Standard European Options

We have the general version of the Black-Sholes Formula [4] to price European options on stocks paying a continuos dividend yieald. In this section, we set:

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad d_2 = d_1 - \sigma\sqrt{\theta}$$

and $N$ as the cumulative normal distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx.$$

## 2.1 Call Options

$$\begin{array}{ll} \textsc{Payoff} & C_T = (S_T - K)_+ \\ \textsc{Price} & C(t,x;K) = xe^{-\delta\theta}N(d_1) - Ke^{-r\theta}N(d_2) \\ \textsc{Delta} & \dfrac{\partial C(t,x;K)}{\partial x} = e^{-\delta\theta}N(d_1) \end{array}$$

Routine `cf_call.c`

## 2.2 Put Options

$$\begin{array}{ll} \textsc{Payoff} & P_T = (K - S_T)_+ \\ \textsc{Price} & P(t,x;K) = Ke^{-r\theta}N(-d_2) - xe^{-\delta\theta}N(-d_1) \\ \textsc{Delta} & \dfrac{\partial P(t,x)}{\partial x} = -e^{-\delta\theta}N(-d_1) \end{array}$$

Routine `cf_put.c`

## 2.3 Call Spread Options

$$\begin{array}{ll} \textsc{Payoff} & CS_T = (S_T - K_1)_+ - (S_T - K_2)_+ \\ \textsc{Price} & CS(t,x) = C(t,x:K_1) - C(t,x;K_2) \\ \textsc{Delta} & \dfrac{\partial CS(t,x)}{\partial x} = \dfrac{\partial C(t,x:K_1)}{\partial x} - \dfrac{\partial C(t,x:K_2)}{\partial x} \end{array}$$

Routine `cf_callspread.c`

## 2.4 Digit Options

$$\begin{array}{ll} \textsc{Payoff}: & C_T = \begin{cases} K & \text{if} \quad S_T K, \\ 0 & \text{otherwise} \end{cases} \\ \textsc{Price}: & C(t,x) = Ke^{-r\theta}N(d_1) \\ \textsc{Delta}: & \dfrac{\partial C(t,x)}{\partial x} = e^{-r\theta}K\dfrac{e^{-d_2^2/2}}{\sqrt{2\pi\theta}\,\sigma x} \end{array}$$

Routine `cf_digit.c`

# 3 European Barrier Options

Barrier options are known as *knock-out* if the value of the option nullifies if the underlying asset price reaches a fixed barrier before the maturity date and *knock-in* if it does not. We use the names *up-out* and *up-in* call/put options, or *down-out* and *down-in* call/put options, to stress if the considered barrier is an upper or lower one. Reiner-Rubinstein [3] have developed formulas for pricing standard barrier options with cash rebate R.

We set here:

- $A = \phi x e^{-\delta\theta} N(\phi x_1) - \phi K e^{-r\theta} N(\phi x_1 - \phi\sigma\sqrt{\theta})$
- $B = \phi x e^{-\delta\theta} N(\phi x_2) - \phi K e^{-r\theta} N(\phi x_2 - \phi\sigma\sqrt{\theta})$
- $C = \phi x (\frac{L}{x})^{2(\mu+1)} e^{-\delta\theta} N(\eta y_1) - \phi K e^{-r\theta} (\frac{L}{x})^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{\theta})$
- $D = \phi x (\frac{L}{x})^{2(\mu+1)} e^{-\delta\theta} N(\eta y_2) - \phi K e^{-r\theta} (\frac{L}{x})^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{\theta})$
- $E = R e^{-r\theta} \left[ N(\eta x_2 - \eta\sigma\sqrt{\theta}) - (\frac{L}{x})^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{\theta}) \right]$
- $F = R(\frac{L}{x})^{\mu+\lambda} N(\eta z) + (\frac{L}{x})^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{\theta}))$

and

- $dA = \phi e^{-\delta\theta} N(\phi x_1)$
- $dB = \phi e^{-\delta\theta} N(\phi x_2) + e^{-\delta\theta} \frac{n(x_2)}{\sigma\sqrt{\theta}} (1 - \frac{k}{l})$
- $dC = \frac{1}{x}\phi 2\mu \left[\frac{L}{x}\right)^{2\mu} \left( x e^{-\delta\theta} \frac{L^2}{x^2} N(\eta y_1) - K e^{-r\theta} N(\eta y_1 - \eta\sigma\sqrt{\theta}) - \right.$
  $\left. \phi \left(\frac{L}{x}\right)^{2(\mu+1)} e^{-b\theta} N(\eta y_1) \right]$
- $dD = -2\mu\frac{\phi}{x} \left[\frac{L}{x}\right)^{2\mu} \left( s e^{-\delta\theta} \frac{L^2}{x^2} N(\eta y_2) - K e^{-r\theta} N(\eta y_2) \times \right.$
  $\times N \left( \eta y_2 - \eta\sigma\sqrt{\theta} - \phi \left(\frac{L}{x}\right)^{2\mu+2} e^{-\delta\theta} \right) N(\eta y_2) - \phi\eta \left(\frac{L}{x}\right)^{2\mu+2} e^{-\delta\theta} \right] \times$
  $\times \frac{n(y_2)}{\sigma\sqrt{\theta}} \left(1 - \frac{k}{l}\right)$
- $dE = 2\frac{R}{x} e^{-r\theta} (\frac{L}{x})^{2\mu} \left[ N(\eta y_2 - \eta\sigma\sqrt{\theta})\mu + \eta \frac{n(\eta y_2 - \sigma\sqrt{\theta})}{\sigma\sqrt{\theta}} \right]$
- $dF = -\frac{R}{x}(\frac{L}{x})^{\mu+\lambda} \left[ (\mu+\lambda) N(\eta z) + (\mu-\lambda)(\frac{L}{x})^{2\lambda} \right] - 2\eta R (\frac{L}{x})^{\mu+\lambda} \frac{n(z)}{x\sigma\sqrt{\theta}}$

where

- $x_1 = \dfrac{\log(x/K)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta}$ $\qquad x_2 = \dfrac{\log(x/L)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta}$
- $y_1 = \dfrac{\log(L^2/xK)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta}$ $\qquad y_2 = \dfrac{\log(L/x)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta}$
- $z = \dfrac{\log(L/x)}{\sigma\sqrt{\theta}} + \lambda\sigma\sqrt{\theta}$
- $\mu = \dfrac{b - \sigma^2/2}{\sigma^2}$ $\qquad \lambda = \sqrt{\mu^2 + \dfrac{2r}{\sigma^2}}$

and $\eta, \phi$ are suitable numbers belonging to $\{-1, 1\}$ (see next sections for details).

Let

$$M_{t,T} = \sup_{t \le \tau \le T} S_\tau \qquad \text{and} \qquad m_{t,T} = \inf_{t \le \tau \le T} S_\tau$$

## 3.1 Down-Out Call Options

$$\textsc{Payoff} \qquad C_T = \begin{cases} (S_T - K)_+ & \text{if} \quad m_{t,T} L, \\ R & \text{otherwise} \end{cases}$$

$$\textsc{Price} \qquad C(t,x) = \begin{cases} A - C + F & \text{if} \quad KL \\ B - D + F & \text{otherwise} \end{cases}$$

$$\textsc{Delta} \qquad \frac{\partial C(t,x)}{\partial x} = \begin{cases} dA - dC + dF & \text{if} \quad KL \\ dB - dD + dF & \text{otherwise} \end{cases}$$

and take

$$\eta = 1 \qquad \phi = 1$$

`Routine cf_calldownout.c`

## 3.2 Down-Out Put Options

Payoff
$$P_T = \begin{cases} (K - S_T)_+ & \text{if} \quad m_{t,T} L, \\ R & \text{otherwise} \end{cases}$$

Price
$$P(t,x) = \begin{cases} A - B + C - D + F & \text{if} \quad K L, \\ F & \text{otherwise} \end{cases}$$

Delta
$$\frac{\partial P(t,x)}{\partial x} = \begin{cases} dA - dB + dC - dD + dF & \text{if} \quad K L, \\ dF & \text{otherwise} \end{cases}$$

and take
$$\eta = 1 \qquad \phi = -1$$

Routine `cf_putdownout.c`

## 3.3 Up-Out Call Options

Payoff
$$C_T = \begin{cases} (S_T - K)_+ & \text{if} \quad M_{t,T} < L, \\ R & \text{otherwise.} \end{cases}$$

Price
$$C(t,x) = \begin{cases} F & \text{if} \quad K L, \\ A - B + C - D + F & \text{otherwise} \end{cases}$$

Delta
$$\frac{\partial C(t,x)}{\partial x} = \begin{cases} dF & \text{if} \quad K L, \\ dA - dB + dC - dD + dF & \text{otherwise} \end{cases}$$

and take
$$\eta = -1 \qquad \phi = 1$$

Routine `cf_callupout.c`

## 3.4 Up-Out Put Options

PAYOFF $$P_T = \begin{cases} (K - S_T)_+ & \text{if} \quad M_{t,T} < L, \\ R & \text{otherwise.} \end{cases}$$

PRICE $$P(t,x) = \begin{cases} B - D + F & \text{if} \quad KL, \\ A - C + F & \text{otherwise} \end{cases}$$

DELTA $$\frac{\partial P(t,x)}{\partial x} = \begin{cases} dB - dD + dF & \text{if} \quad KL, \\ dA - dC + dF & \text{otherwise} \end{cases}$$

and take

$$\eta = -1 \qquad \phi = -1$$

Routine cf_putupout.c

## 3.5 Down-In Call Options

PAYOFF $$C_T = \begin{cases} R & \text{if} \quad m_{t,T}L, \\ (S_T - K)_+ & \text{otherwise.} \end{cases}$$

PRICE $$C(t,x) = \begin{cases} C + E & \text{if} \quad KL, \\ A - B + D + E & \text{otherwise} \end{cases}$$

DELTA $$\frac{\partial P(t,x)}{\partial x} = \begin{cases} dC + dE & \text{if} \quad KL, \\ dA - dB + dD + dE & \text{otherwise} \end{cases}$$

and take

$$\eta = 1 \qquad \phi = 1$$

Routine cf_calldownin.c

## 3.6 Down-In Put Options

PAYOFF
$$P_T = \begin{cases} R & \text{if} \quad m_{t,T} L, \\ (K - S_T)_+ & \text{otherwise.} \end{cases}$$

PRICE
$$P(t,x) = \begin{cases} B - C + D + E & \text{if} \quad K L, \\ A + E & \text{otherwise} \end{cases}$$

DELTA
$$\frac{\partial P(t,x)}{\partial x} = \begin{cases} dB - dC + dD + dE & \text{if} \quad K L, \\ dA + dE & \text{otherwise} \end{cases}$$

and take

$$\eta = 1 \qquad \phi = -1$$

Routine cf_putdownin.c

## 3.7 Up-In Call Options

PAYOFF
$$C_T = \begin{cases} R & \text{if} \quad M_{t,T} < L, \\ (S_T - K)_+ & \text{otherwise.} \end{cases}$$

PRICE
$$P(t,x) = \begin{cases} A + E & \text{if} \quad K L, \\ B - C + D + E & \text{otherwise} \end{cases}$$

DELTA
$$\frac{\partial C(t,x)}{\partial x} = \begin{cases} dA + dE & \text{if} \quad K L, \\ dB - dC + dD + dE & \text{otherwise} \end{cases}$$

and take

$$\eta = -1 \qquad \phi = 1$$

Routine cf_callupin.c

### 3.8 Up-In Put Options

$$\text{PAYOFF} \qquad P_T = \begin{cases} R & \text{if} \quad M_{t,T} < L, \\ (K - S_T)_+ & \text{otherwise.} \end{cases}$$

$$\text{PRICE} \qquad P(t,x) = \begin{cases} A - B + D + E & \text{if} \quad KL, \\ C + E & \text{otherwise} \end{cases}$$

$$\text{DELTA} \qquad \frac{\partial C(t,x)}{\partial x} = \begin{cases} dA - dB + dD + dE & \text{if} \quad KL, \\ dC + dE & \text{otherwise} \end{cases}$$

and take

$$\eta = -1 \qquad \phi = -1$$

`Routine cf_putupin.c`

# 4 Double Barrier European Option

A double barrier option is knock-in or knock-out if the underlying price reaches or not a lower and/or an upper boundary prior to expiration. The exact value for double barrier call/put knock-out options is given by the Ikeda-Kunitomo formula [5], which allows to compute exactly the price when the boundaries suitably depend on the time variable $t$. More precisely, set

$$U(s) = Ue^{\delta_1 s} \qquad L(s) = Le^{\delta_2 s}$$

where the constants $U, L, \delta_1, \delta_2 in$ are such that $L(s) < U(s)$, for every $s \in [t, T]$. The functions $U(s)$ and $L(s)$ play the role of *upper* and *lower* barrier respectively. $\delta_1$ and $\delta_2$ determine the curvature and the case of $\delta_1 = 0$ and $\delta_2 = 0$ corresponds to two flat boundaries.

The numerical studies suggest that in most cases it suffices to calculate the leading five terms of the series giving the price of the knock-out and knock-in double barrier call options.

Let $\tau$ stand for the first time at which the underlying asset price $S$ reaches at least one barrier, i.e.

$$\tau = \inf\{st\,;\, S_s \leq L(s) \text{ or } S_s \geq U(s)\}.$$

We define the following coefficients:

- $\mu_1 = 2\dfrac{b - \delta_2 - n(\delta_1 - \delta_2)}{\sigma^2} + 1$
- $\mu_2 = 2\dfrac{n(\delta_1 - \delta_2)}{\sigma^2}$
- $\mu_3 = 2\dfrac{b - \delta_2 + n(\delta_1 - \delta_2)}{\sigma^2} + 1$

## 4.1 Knock-Out Call Options

PAYOFF $\quad C_T = \begin{cases} (S_T - K)_+ & \text{if} \quad \tau T \\ 0 & \text{otherwise} \end{cases}$

PRICE

$$
\begin{aligned}
C(t,x) = xe^{-\delta\theta} \sum_{n=-\infty}^{+\infty} & \left[ \left(\frac{U^n}{L^n}\right)^{\mu_1} \left(\frac{L}{x}\right)^{\mu_2} \left(N(d_1^+) - N(d_2^+)\right) \right. \\
& \left. - \left(\frac{L^{n+1}}{xU^n}\right)^{\mu_3} \left(N(d_3^+) - N(d_4^+)\right) \right] \\
-Ke^{-r\theta} \sum_{n=-\infty}^{+\infty} & \left[ \left(\frac{U^n}{L^n}\right)^{\mu_1 - 2} \left(\frac{L}{x}\right)^{\mu_2} \left(N(d_1^-) - N(d_2^-)\right) \right. \\
& \left. - \left(\frac{L^{n+1}}{xU^n}\right)^{\mu_3 - 2} \left(N(d_3^-) - N(d_4^-)\right) \right]
\end{aligned}
$$

where $F = Ue^{\delta_1\theta}$ and

$$
d_1^{\pm} = \frac{\log(xU^{2n}/KL^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad d_2^{\pm} = \frac{\log(xU^{2n}/FL^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}
$$

$$
d_3^{\pm} = \frac{\log(L^{2n+2}/KxU^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad d_4^{\pm} = \frac{\log(L^{2n+2}/FxU^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}
$$

Routine `cf_callout_kunitomoikeda.c`

## 4.2 Knock-Out Put Options

PAYOFF $\quad P_T = \begin{cases} (K - S_T)_+ & \text{if} \quad \tau T \\ 0 & \text{otherwise} \end{cases}$

PRICE

$$P(t,x) = Ke^{-r\theta} \sum_{n=-\infty}^{+\infty} \left[ \left(\frac{U^n}{L^n}\right)^{\mu_1-2} \left(\frac{L}{x}\right)^{\mu_2} \left(N(y_1^-) - N(y_2^-)\right) - \left(\frac{L^{n+1}}{xU^n}\right)^{\mu_3-2} \left(N(y_3^-) - N(y_4^-)\right) \right]$$

$$-xe^{-\delta\theta} \sum_{n=-\infty}^{+\infty} \left[ \left(\frac{U^n}{L^n}\right)^{\mu_1} \left(\frac{L}{x}\right)^{\mu_2} \left(N(y_1^+) - N(y_2^+)\right) - \left(\frac{L^{n+1}}{xU^n}\right)^{\mu_3} \left(N(y_3^+) - N(y_4^+)\right) \right]$$

where $E = Le^{\delta_2\theta}$ and

$$y_1^{\pm} = \frac{\log(xU^{2n}/EL^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad y_2^{\pm} = \frac{\log(xU^{2n}/KL^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}$$

$$y_3^{\pm} = \frac{\log(L^{2n+2}/ExU^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad y_4^{\pm} = \frac{\log(L^{2n+2}/KxU^{2n}) + \left(b \pm \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}$$

Routine cf_putout_kunitomoikeda.c

## 4.3 Knock-In Call Options

The double barrier knock-in call option is priced via the no-arbitrage relationship between knock-out and knock-in option:
European “IN” + European “OUT” = European Standard

Routine cf_callin_kunitomoikeda.c

## 4.4 Knock-In Put Options

The double barrier knock-in call option is priced via the no-arbitrage relationship between knock-out and knock-in option:
European “IN” + European “OUT” = European Standard

Routine `cf_putin_kunitomoikeda.c`

# 5 Lookback European Options

Floating strike lookback options can be priced using Goldman-Sosin-Gatto formula [2] while fixed strike lookback options can be priced using Conze-Viswanathen formula[7].

We set, as $0 \leq u \leq v \leq T$,

$$M_{u,v} = \sup_{u \leq \tau \leq v} S_\tau \qquad \text{and} \qquad m_{u,v} = \inf_{u \leq \tau \leq v} S_\tau$$

and

- $$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad d_2 = d_1 - \sigma\sqrt{\theta}$$
- $$e_1 = \frac{\log\left(\frac{x}{M_{0,t}}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad e_2 = e_1 - \sigma\sqrt{\theta}$$
- $$f_1 = \frac{\log\left(\frac{x}{M_{0,t}}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \qquad f_2 = f_1 - \sigma\sqrt{\theta}$$

## 5.1 Fixed Lookback Call Options

$$\text{Payoff} \qquad C_T = (M_{t,T} - K)_+$$

Both price and delta depend on $K$ and $M_{0,t}$.

- IF $K > M_{0,t}$ THEN

$$\text{Price} \qquad C(t,x) = xe^{-\delta\theta}N(d_1) - Ke^{-r\theta}N(d_2) + xe^{-r\theta}\frac{\sigma^2}{2b}\left[-\left(\frac{x}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(d_1 - \frac{2b}{\sigma}\sqrt{\theta}\right) + e^{b\theta}N(d_1)\right]$$

$$\text{Delta} \qquad \frac{\partial C(t,x)}{\partial x} = e^{-\delta\theta}N(d_1)\left(1 + \frac{\sigma^2}{2b}\right) + e^{-\delta\theta}\frac{n(d_1)}{\sigma\sqrt{\theta}} - e^{-r\theta}\frac{K}{x}\frac{n(d_2)}{\sigma\sqrt{\theta}} + e^{-r\theta}\left(\frac{x}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(d_1 - \tfrac{2b}{\sigma}\sqrt{t}\right)\left(1 - \frac{\sigma^2}{2b}\right)$$

- IF $K \leq M_{0,t}$ THEN

PRICE
$$C(t,x) = e^{-r\theta}(M_{0,t} - K) + xe^{-\delta\theta}N(e_1) - M_{0,t}e^{-r\theta}N(e_2)$$
$$+xe^{-r\theta}\frac{\sigma^2}{2b}\left[-\left(\frac{x}{M_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \frac{2b}{\sigma}\sqrt{\theta}\right) + e^{b\theta}N(e_1)\right]$$

DELTA
$$\frac{\partial C(t,x)}{\partial x} = e^{-\delta\theta}N(e_1)(1 + \frac{\sigma^2}{2b}) + e^{-\delta\theta}\frac{n(e_1)}{\sigma\sqrt{\theta}} - e^{-r\theta}\frac{M_{0,t}}{x}\frac{n(d_2)}{\sigma\sqrt{\theta}}$$
$$+e^{-r\theta}\left(\frac{x}{M_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \tfrac{2b}{\sigma}\sqrt{t}\right)\left(1 - \frac{\sigma^2}{2b}\right)$$

Routine cf_fixed_calllookback.c

## 5.2 Fixed Lookback Put Options

PAYOFF $\quad P_T = (K - m_{t,T})_+$

Both price and delta depend on $K$ and $m_{0,t}$.

- IF $K < m_{0,t}$ THEN

PRICE
$$P(t,x) = Ke^{-r\theta}N(-d_2) - xe^{-\delta\theta}N(-d_1)$$
$$+xe^{-r\theta}\frac{\sigma^2}{2b}\left[\left(\frac{x}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(-d_1 + \frac{2b}{\sigma}\sqrt{\theta}\right) - e^{b\theta}N(-d_1)\right]$$

DELTA
$$\frac{\partial P(t,x)}{\partial x} = e^{-\delta\theta}N(d_1)\left(1 + \frac{\sigma^2}{2b}\right) + e^{-\delta\theta}\frac{n(d_1)}{\sigma\sqrt{\theta}} - e^{-r\theta}\frac{K}{x}\frac{n(d_2)}{\sigma\sqrt{\theta}}$$
$$+e^{-r\theta}\left(\frac{x}{K}\right)^{-\frac{2b}{\sigma^2}} N\left(-d_1 + \tfrac{2b}{\sigma}\sqrt{t}\right)\left(\tfrac{\sigma^2}{2b} - 1\right) - e^{-\delta\theta}\left(\frac{\sigma^2}{2b} - 1\right)$$

- IF $K \geq m_{0,t}$ THEN

PRICE
$$P(t,x) = e^{-r\theta}(K - m_{0,t}) - xe^{-\delta\theta}N(-f_1) + m_{0,t}e^{-r\theta}N(-f_2)$$
$$+xe^{-r\theta}\frac{\sigma^2}{2b}\left[\left(\frac{x}{m_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \frac{2b}{\sigma}\sqrt{\theta}\right) - e^{b\theta}N(-f_1)\right]$$

DELTA
$$\frac{\partial P(t,x)}{\partial x} = e^{-\delta\theta}\left(1 + \frac{\sigma^2}{2b}\right)(N(f_1) - 1) + e^{-\delta\theta}\frac{n(f_1)}{\sigma\sqrt{\theta}} - e^{-r\theta}\frac{M_{0,t}}{x}\frac{n(d_2)}{\sigma\sqrt{\theta}}$$
$$+e^{-r\theta}\left(\frac{x}{M_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \tfrac{2b}{\sigma}\sqrt{t}\right)\left(\frac{\sigma^2}{2b} - 1\right)$$

Routine cf_fixed_putlookback.c

### 5.3 Floating Lookback Call Options

PAYOFF $C_T = S_T - m_{t,T}$

PRICE
$$C(t,x) = xe^{-\delta\theta}N(f_1) - m_{0,t}e^{-r\theta}N(f_2)$$
$$+xe^{-r\theta}\tfrac{\sigma^2}{2b}\left[\left(\frac{x}{m_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \tfrac{2b}{\sigma}\sqrt{\theta}\right) - e^{b\theta}N(-f_1)\right]$$

DELTA
$$\frac{\partial C(t,x)}{\partial x} = e^{-\delta\theta}N(f_1)\left(1+\frac{\sigma^2}{2b}\right) + e^{-\delta\theta}\frac{n(a_1)}{\sigma\sqrt{\theta}}$$
$$-e^{-r\theta}\frac{\sigma^2}{2b} + e^{-r\theta}\left(\frac{x}{m_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \frac{2b}{\sigma}\sqrt{t}\right)\left(\frac{\sigma^2}{2b} - 1\right)$$

Routine cf_floating_calllookback.c

### 5.4 Floating Lookback Put Options

PAYOFF $P_T = M_{t,T} - S_T$

PRICE
$$P(t,x) = M_{0,t}e^{-r\theta}N(-e_2) - xe^{-\delta\theta}N(-e_1)$$
$$+xe^{-r\theta}\tfrac{\sigma^2}{2b}\left[-\left(\frac{x}{M_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \tfrac{2b}{\sigma}\sqrt{\theta}\right) + e^{b\theta}N(b_1)\right]$$

DELTA
$$\frac{\partial P(t,x)}{\partial x} = e^{-\delta\theta}N(e_1)\left(1+\frac{\sigma^2}{2b}\right) + e^{-r\theta}\left(\frac{x}{M_{0,t}}\right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \frac{2b}{\sigma}\sqrt{t}\right)\left(1 - \frac{\sigma^2}{2b}\right)]$$
$$-e^{-r\theta}\left(\frac{x}{M_{0,t}}\right)\left(\frac{n(b_1)}{\sigma\sqrt{\theta}} - 1\right) + e^{-\delta\theta}\left(\frac{n(b_1)}{\sigma\sqrt{\theta}} - 1\right)$$

Routine cf_floating_putlookback.c

# 6 Standard $2D$ European Options

Consider the pair of processes $S_t = (S_t^1, S_t^2)$ solution to

$$\begin{cases} dS_t^1 = S_t^1((r-\delta_1)dt + \sigma_1 dW_t^1), \; S_0^1 = x^1 \\ dS_t^2 = S_t^2((r-\delta_2)dt + \sigma_2 dW_t^2), \; S_0^2 = x^2. \end{cases}$$

where $(W_t^1, t \geq 0)$ and $(W_t^2, t \geq 0)$ denote two real–valued Brownian motions with istantaneous correlation $\rho$. The price of option with payoff $f$ is:

$$F(t, x^1, x^2) = E\left[e^{-rT} f\left(S_T^1(x^1), S_T^2(, x^2)\right)\right].$$

Here, closed formulas due to Johnson and Stulz are presented [1],[6].

We set

- $d = \dfrac{\log \frac{x_1}{x_2} + \left(\delta_2 - \delta_1 + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{theta}}$
- $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$
- $d_i = \dfrac{\log\left(\frac{x_i}{K}\right) + \left(r - \delta_i + \frac{\sigma_i^2}{2}\right)\theta}{\sigma_1\sqrt{\theta}}, \quad i = 1, 2$
- $\rho_1 = \dfrac{\sigma_1 - \rho\sigma_2}{\sigma}$
- $\rho_2 = \dfrac{\sigma_2 - \rho\sigma_1}{\sigma}$

and $M$ as the cumulative bivariate normal distribution function:

$$M(a, b; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} dxdy.$$

## 6.1 Call On the Maximum

PAYOFF $\quad C_T = (\max(S_T^1, S_T^2) - K)_+$

PRICE
$$C(t, x_1, x_2) = x_1 e^{-\delta_1\theta} M(d_1, d; \rho_1) + x_2 e^{-\delta_2\theta} M(d_2, -d + \sigma\sqrt{\theta}; \rho_2)$$
$$-Ke^{-r\theta}\left(1 - M(-d_1 + \sigma_1\sqrt{\theta}, -d_2 + \sigma_2\sqrt{\theta}; \rho)\right)$$

DELTA
$$\frac{\partial C(t, x_1, x_2)}{\partial x_1} = e^{-\delta_1\theta} M(d_1, d; \rho_1)$$
$$\frac{\partial C(t, x_1, x_2)}{\partial x_2} = e^{-\delta_2\theta} M(d_2, -d + \sigma\sqrt{\theta}; \rho_2)$$

Routine cf_callmax.c

## 6.2 Put On the Minimum

PAYOFF $P_T = (K - \min(S_T^1, S_T^2))_+$

PRICE $P(t, x_1, x_2) = Ke^{-r\theta} - c_0 + c_1$

DELTA $\dfrac{\partial P(t, x_1, x_2)}{\partial x_1} = e^{-\delta_1\theta}(1 - N(d)) + e^{-\delta_1\theta}M(d_1, -d; -\rho_1)$

$\dfrac{\partial P(t, x_1, x_2)}{\partial x_2} = e^{-\delta_2\theta}N(d - \sigma) + e^{-\delta_2\theta}M(d_2, d - \sigma\sqrt{\theta}; -\rho_2)$

where

- $c_0 = x_1 e^{-\delta_1\theta}(1 - N(d)) + x_2 e^{-\delta_2\theta}N(d - \sigma\sqrt{\theta})$
- $c_1 = x_1 e^{-\delta_1\theta}M(d_1, -d, -\rho_1) + x_2 e^{-\delta_2\theta}M(d_2, d - \sigma\sqrt{\theta}; -\rho_2)$
  $-Ke^{-r\theta}M(d_1 - \sigma_1\sqrt{\theta}, d_2 - \sigma_2\sqrt{\theta}; \rho)$

Routine cf_putmin.c

## 6.3 Exchange Options

PAYOFF $E_T = (S_T^1 - \lambda S_T^2))_+$

PRICE $E(t, x_1, x_2) = x_1 e^{-\delta_1\theta}N(\widehat{d}_1) - \lambda x_2 e^{-\delta_2\theta}N(\widehat{d}_2)$

where

$$\widehat{d}_1 = \frac{\log\left(\frac{x_1}{\lambda x_2}\right) + \left(\delta_2 - \delta_1 + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}, \qquad \widehat{d}_2 = \hat{d}_1 - \sigma\sqrt{\theta}$$

Routine cf_exchange.c

## 6.4 Best Of Option

The payoff is $B_T = (\max(S_T^1 - K_1, S_T^2 - K_2)))_+$.

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