Dynamic Hedging of Synthetic CDO Tranches with Default contagion

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Credit Derivatives

The following method to compute the hedge of a CDO tranche is based on the article of R. Frey and J. Backhaus [1]. Compared to this paper, we consider the simplified case where there is no spread risk ($\Psi_t = \psi$).

1 The model

We fix some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$. The σ-field $\mathcal{F}_t$ represents the information available to investors at time t; all the processes introduced below will be $(\mathcal{F}_t)$-adapted. We consider a fixed portfolio of m firms, indexed by $i \in \{1, \cdots, m\}$. The $(\mathcal{F}_t)$-stopping time τ_i with values in $(0, \infty)$ represents the default time of firm i. The default state of the portfolio is thus described by the default indicator process $Y = (Y_{t,1}, \cdots, Y_{t,m})_{t \geq 0}$ where $Y_{t,i} = \mathbf{1}_{\tau_i \leq t}$; note that $Y_t \in \{0, 1\}^m$. Since we consider only models without simultaneous defaults, we can define the ordered default times $T_0 < T_1 < \cdots < T_m$ recursively by $T_0 = 0$ and for $1 \leq n \leq m$, $T_n = \min\{\tau_i : 1 \leq i \leq m, \tau_i > T_{n-1}\}$.
We assume that the default-free interest rate is deterministic and equal to $r \geq 0$; $p_0(t, T) = e^{-r(T-t)}$ denotes the price of the default-free zero-coupon bond with maturity T. The measure Q represents a risk-neutral measure. The default intensity of firm i is given by some nonnegative function $\lambda(t, Y_t)$. Since λ depends on the current portfolio state Y_t, the default intensity of a firm may change if there is a change in the default state of other firms in the portfolio, so that default contagion can be modelled explicitly.

$$\lambda(t, Y_t) = h(t, M_t) \text{ where } M_t = \sum_{i=1}^{m} Y_{t,i}$$

and

$$h(t, l) = \lambda_0 \psi + \frac{\lambda_1}{\lambda_2} \big(\exp(\lambda_2 \frac{[l - \mu(t)]^+}{m}) - 1 \big), \;\; \lambda_0 > 0, \lambda_1, \lambda_2 \geq 0$$

.

Here, μ_t is some deterministic threshold measuring the expected number of defaults up to time t. In the algorithm, we set $\mu_t := (1 - e^{-\frac{sp}{1-R} t}) m$ where sp is the single name CDS spread (assuming constant for all firms) and $1 - R$ is the loss-given-default (again constant and deterministic for all firms). (We set $sp = 26bp$ and $R = 40\%$). The term $\lambda_0 \psi$ gives the linear dependency on the factor process, λ_0 determines the credit quality of firms in the portfolio. λ_1 models the strength of the default contagion for a 'normal' number of defaults. λ_2 gives the tendency of the model to generate default cascades. We set $\psi = 0.005$, $\lambda_0 = 0.8591$, $\lambda_1 = 0.18803$ and $\lambda_2 = 22.125$ (calibrated to the index-level and to observed tranche spreads of the iTraxx with a maturity of 5 years).

2 Credit Derivatives

In this section we present the payments, the market value and the gain process of CDSs and CDOs, useful to study the dynamic hedging of portfolio credit derivatives. We consider the case where an investor tries to hedge a protection-seller position in a CDO tranche by taking a protection buyer position in the CDSs or in the CDS index underlying the transaction, so we model the cash flows of the CDO tranche from the viewpoint of a protection seller and the cash flows of CDSs from the viewpoint of a protection buyer. T denotes the maturity, the nominal of eah CDS is normalized to one, we set $\delta = 1 - R$ the default payment. The spread payments of all credit derivatives are scheduled at N payment dates $0 < z_1 < \cdots < z_N = T$. We define $z_0 := 0$ and $\Delta z_n := z_n - z_{n-1}$. For $t \geq 0$,

$$n_t^z := |\{i = 1, \cdots, N : z_i \leq t\}|;$$

so that n_t^z is the number of spread payment dates up to time t.

2.1 Single-name CDSs

The market value of a protection-buyer position in a CDS on firm k with fixed spread s^k is given by the difference between the value of the default payment and the value of the future premium payments (regular and accrued). Hence we get

$$\begin{aligned} V_{t,k}^{CDS} = & \mathbf{1}_{\tau_k > t} \mathbb{E}_Q \Big(\delta p_0(t, \tau_k) \mathbf{1}_{\tau_k \leq T} \\ & - s^k \sum_{n=n_t^z+1}^{N} (\Delta z_n p_0(t, z_n) \mathbf{1}_{\tau_k > z_n} + (\tau_k - z_{n-1}) p_0(t, \tau_k) \mathbf{1}_{\{z_{n-1} < \tau_k \leq z_n\}}) | \mathcal{F}_t \Big). \end{aligned}$$

The gain process has dynamics

$$\mathrm{d}G_{t,k}^{CDS} = -s^k (\Delta z_{n_t^z})(1 - Y_{t,k}) \mathrm{d}n_t^z + (\delta - s^k (t - z_n^t)) \mathrm{d}Y_{t,k} + \mathrm{d}V_{t,k}^{CDS}. \tag{1}$$

2.2 CDO Tranches

A synthetic CDO tranche on the reference portfolio is characterized by a fixed lower and upper bound attachment points $0 \leq l < u \leq 1$. The tranche consists of a default payment leg and a premium payment leg. We denote by L_t the cumulative portfolio loss up to time t: $L_t := \sum_{i=1}^{m} \delta Y_{t,i}$. The cumulative tranche loss $L_t^{[l,u]}$ is defined by

$$L_t^{[l,u]} := (L_t - ml)_+ - (L_t - mu)_+,$$

and denote the remaining notional of the tranche by $N_t^{[l,u]} := m(u - l) - L_t^{[l,u]}$. At a default time $T_k \leq T$ there is a default payment of size $\Delta L_{T_k}^{[l,u]} := L_{T_k}^{[l,u]} - L_{T_k^-}^{[l,u]}$. The market value of a protection seller position equals

$$\begin{aligned} V_t^{[l,u]} = \mathbb{E}_Q \Bigg(& - \int_t^T p_0(t, s) dL_s^{[l,u]} + s^{[l,u]} \sum_{n=n_t^z+1}^{N} \{ p_0(t, z_n)(\Delta z_n) N_{z_n}^{[l,u]} \\ & + \sum_{k=1}^{m} p_0(t, T_k)(T_k - z_{n-1}) \Delta L_{T_k}^{[l,u]} \mathbf{1}_{\{z_{n-1} < T_k \leq z_n\}} \} | \mathcal{F}_t \Bigg). \end{aligned}$$

In the Markov chain model, $V^{[l,u]} = v^{[l,u]}(t, Y_t)$ where $v^{[l,u]}$ is some function from $[0,T] \times \{0,1\}^m \to \mathbb{R}$. The gain process $G_t^{[l,u]}$ has dynamics

$$\mathrm{d}G_t^{[l,u]} = s^{[l,u]}(\Delta z_{n_t^z}) N_t^{[l,u]} \mathrm{d}n_t^z + s^{[l,u]}(t - z_{n_t^z}) dL_t^{[l,u]} - dL_t^{[l,u]} + \mathrm{d}V_t^{[l,u]}. \tag{2}$$

3 Sensitivity-based hedging with default contagion

The default delta of a CDO tranche w.r.t. name i ($\Delta_{t,i}^{def}$) gives the number of CDSs of firm i one has to hold at time t in order to immunize the portfolio against the change-in-value occuring in the hypothetical scenario where name i defaults at time t. In the presence of default contagion, the default of firm i impacts the market value $V_{t,j}^{CDS}, j \neq i$. In a homogeneous portfolio (the case we consider) Δ^{def} is identical for all firms, we get

$$\Delta_t^{def} = -\frac{\Delta G_t^{[l,u]}|_{\tau_i = t}}{\Delta G_t^{Ind}|_{\tau_i = t}}, \tag{3}$$

where $\Delta G_t^{Ind}|_{\tau_i=t}$ represents the the change in the gain process of the CDS index, i.e.

$$\Delta G_t^{Ind}|_{\tau_i=t} = (m - M_t - 1)\Delta V^{CDS}|_{\tau_i=t} - V_t^{CDS} + \delta - s^{CDS}(t - z_{n_t^z}),$$

and $\Delta V_t^{[l,u]}|_{\tau_i=t} = v^{[l,u]}(t, Y_{t^-}^i) - v^{[l,u]}(t, Y_{t^-})$.

4 Practical computation of default and payment legs of CDSs and CDOs

4.1 Default leg of CDO

We aim at computing $\mathbb{E}_Q(-\int_t^T p_0(t,s) dL_s^{[l,u]} | \mathcal{F}_t)$. An integration by parts gives

$$\mathbb{E}_Q[p_0(t,T) L_T^{[l,u]} | \mathcal{F}_t] - L_t^{[l,u]} + r \int_t^T e^{-r(s-t)} \mathbb{E}_t[L_s^{[l,u]}] ds,$$

and $\mathbb{E}[L_s^{[l,u]} | \mathcal{F}_t] = \mathbb{E}[v(L_s) | \mathcal{F}_t] = \mathbb{E}[v(\delta M_s) | \mathcal{F}_t] = \sum_{k=1}^m v(\delta k) \mathbb{P}(M_s = k | \mathcal{F}_t)$. We refer to the next section for the computation of $\mathbb{P}(M_s = k | \mathcal{F}_t)$.

4.2 Payment leg of CDO

The first term gives

$$\mathbb{E}[\sum_{n=n_t^z+1}^{N} p_0(t, z_n) \Delta z_n N_{z_n}^{[a,b]} | \mathcal{F}_t] = \sum_{n=n_t^z+1}^{N} p_0(t, z_n) \Delta z_n (m(u-l) - \mathbb{E}[L_{z_n}^{[a,b]} | \mathcal{F}_t]).$$

The second one is

$$\mathbb{E}[\sum_n^N \sum_{k=1}^m p_0(t, T_k)(T_k - z_{n-1}) \Delta L_{T_k}^{[a,b]} 1_{\{z_{n-1} < T_k \leq z_n\}} | \mathcal{F}_t] = \sum_n^N \sum_{k=1}^m (\Delta L_{T_k}^{[a,b]})) \int_{z_{n-1}}^{z_n} p_0(t,s)(s - z_{n-1}) d\mathbb{P}(T_k \leq s | \mathcal{F}_t).$$

An integration by part formula leads to

$$\int_{z_{n-1}}^{z_n} p_0(t,s)(s-z_{n-1})d\mathbb{P}(T_k \le s|\mathcal{F}_t) = p_0(t,z_n)(\Delta z_n)\mathbb{P}(T_k \le z_n|\mathcal{F}_t)$$
$$- \int_{z_{n-1}}^{z_n} \mathbb{P}(T_k \le s|\mathcal{F}_t)p_0(t,s)\{1-r(s-z_{n-1})\}ds$$

We refer to the next section for the computation of $\mathbb{P}(T_k \le s|\mathcal{F}_t)$.

4.3 Default leg of CDS

We aim at computing $\delta\mathbb{E}_Q(p_0(t,\tau_k)\mathbf{1}_{\tau_k\le T}|\mathcal{F}_t)$, which is $\delta\int_t^T p_0(t,s)\mathbb{P}(\tau \in ds|\mathcal{F}_t)$ where $\mathbb{P}(\tau_k \in ds|\mathcal{F}_t) = \frac{1}{m}\sum_{j=M_t}^{m-1} h(s,j)\mathbb{P}(M_s = j|\mathcal{F}_t)(m-j)$.

4.4 Payment leg of CDS

We aim at computing $\mathbb{E}_Q\left(\sum_{n=n_t^z+1}^{N}\left(\Delta z_n p_0(t,z_n)\mathbf{1}_{\tau_k>z_n} + (\tau_k - z_{n-1})p_0(t,\tau_k)\mathbf{1}_{\{z_{n-1}<\tau_k\le z_n\}}\right)|\mathcal{F}_t\right)$. To compute this term we need $\mathbb{P}(\tau_k > z_n|\mathcal{F}_t) = 1 - \int_t^{z_n}\mathbb{P}(\tau_k \in ds|\mathcal{F}_t)$, which boils down to have $\mathbb{P}(M_s = j|\mathcal{F}_t)$ for $j = M_t, \cdots, m$.

4.5 Computation of the conditional law of M

In this section we explain how to compute $\mathbb{P}(M_s = k|\mathcal{F}_t)$, for $k = M_t, \cdots, m$. This will give us the condtional law of T_k, since

$$\mathbb{P}(T_k \le s|\mathcal{F}_t) = \mathbb{P}(M_s \ge k|\mathcal{F}_t = 1 - \mathbb{P}(M_s < k|\mathcal{F}_t) = 1 - \sum_{i=M_t}^{k-1}\mathbb{P}(M_s = i|M_t).$$

We compute the transition probability $\mathbb{P}(M_s = i|M_t)$ by using the Kolmogorov equation. The generator of the Markov process $(\mathbf{M}_t)_{(t\ge0)}$ is given by

$$G_{[s]}^M f(l) = (m-l)h(s,l)(f(l+1)-f(l)) \tag{4}$$

Assume that at time $s \ge 0$ there is $\mathbf{M}_s$ defaults, the function $t \mapsto p(s,M_s,t,k) = \mathbb{P}(\mathbf{M}_t = k|\mathbf{M}_s)$ satisfies for all $0 \le k \le m$,

$$\frac{\partial p(s,M_s,t,.)}{\partial t} = (G_{[t]}^M)^* p(s,M_s,t,.). \tag{5}$$

We get

$$\frac{\partial p(s,M_s,t,Ms)}{\partial t} = -(m-M_s)h(t,M_s)p(s,M_s,t,M_s) \tag{6}$$

and for $M_s + 1 \le k \le m$:

$$\frac{\partial p(s,M_s,t,k)}{\partial t} = (m-k+1)h(t,k-1)p(s,M_s,t,k-1) - (m-k)h(t,k)p(s,M_s,t,k), \tag{7}$$

and the initial condition is $p(s,M_s,s,.) = \mathbf{1}_{M_s}(.)$.
We solve these equations by computing the exponential of the matrix associated to the generator G^M.

References

[1] RÃ$\frac{1}{4}$diger Frey and Jochen Backhaus. Dynamic hedging of synthetic cdo tranches with spread risk and default contagion. 2007. 1