A low-bias simulation scheme for the SABR stochastic volatility model

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1 Scheme specification

The low-bias SABR[1] Monte Carlo simulation scheme proposed by Bin Chen, C.W. Oosterlee and H. Van Der Weide in [1] is an alternative when a truncated Euler scheme gives rise to significant bias, even with a very large number of time steps, which is the case, for example, when $S_0 \approx 0$ or when the skewness parameter, β, is less than $\frac{1}{2}$.
This scheme can deal with the martingale property of the discrete scheme, and with the - often problematic - behavior of the CEV[2] process S_t in the vicinity of the zero boundary. The low-bias scheme is stable and exhibits a highly satisfactory convergence behavior compared to the truncated Euler scheme.
It used the fact that the conditional SABR process, given the terminal volatility level and the integrated variance, is a squared Bessel process. The scheme is based on the idea of mixing conditional distributions and a direct inversion of the noncentral chi-square distributions.

2 Model specification

The SABR stochastic volatility model is used for the pricing of fixed income instruments, interpolation of volatility surfaces and the hedging of volatility risk. Here we compute the prices relative to specifiable strike.
The Stochastic Alpha Beta Rho Stochastic Volatility is given by the following system of stochastic differential equations with constant parameters α and β:

$$\begin{aligned} dS_t &= \sigma_t S_t^{\beta} dW_{1,t} \\ d\sigma_t &= \alpha \sigma_t dW_{2,t} \\ \langle dW_{1,t}, dW_{2,t} \rangle &= \rho dt, \quad \text{with } \rho \in [-1,1] \end{aligned}$$

In [1], it is proved that the cumulative distribution of S_Δ with an absorbing boundary at $S_t = 0$ given σ_Δ and $\int_0^\Delta \sigma_s^2 ds$, for some S_0 and $0 < \beta < 1$, reads:

$$\mathbb{P}\left(S_\Delta \leq K | S_0 > 0,\ \sigma_\Delta,\ \int_0^\Delta \sigma_s^2 ds\right) = 1 - \chi^2(a;b,c) \tag{1}$$

where,

$$a = \frac{1}{\nu(\Delta)}\left(\frac{S_0^{1-\beta}}{1-\beta} + \frac{\rho}{\alpha}(\sigma_\Delta - \sigma_0)\right)^2, \qquad b = 2 - \frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)},$$

$$c = \frac{K^{2(1-\beta)}}{(1-\beta)^2\nu(\Delta)}, \qquad \nu(\Delta) = (1-\rho^2)\int_0^\Delta \sigma_s^2, ds$$

and $\chi^2(a;b,c)$ is the noncentral chi-square cumulative distribution function.
From this result we can define the absorbtion probability:

$$\mathbb{P}\left(S_\Delta = 0 | S_0 > 0,\ \sigma_\Delta,\ \int_0^\Delta \sigma_s^2 ds\right) = 1 - \chi^2(a;b,0).$$

[1] Stochastic Alpha Beta Rho.
[2] Constant Elasticity of Variance

Knowing that if U has a $\mathcal{U}[0,1]$ distribution, then $\mathbb{P}\left(S_\Delta \leq K|S_0>0,\ \sigma_\Delta,\ \int_0^\Delta \sigma_s^2 ds\right) \sim U$. Thus, one can easily handle the absorbing boundary at zero:
If $S_0 = 0$, then $S_\Delta = 0$.
Else if $U < \mathbb{P}\left(S_\Delta = 0|S_0>0,\ \sigma_\Delta,\ \int_0^\Delta \sigma_s^2 ds\right)$, then $S_\Delta = 0$.
Otherwise, we sample S_Δ by using either the **Moment-matched quadratic Gaussian approximation** or the **Direct inversion scheme**. The reason is that the quadratic Gaussian approximation is accurate only if S_0 is sufficiently large (i.e: the probability of absorption is small). If so, we invert (1) using a Newton-type method.

2.1 Moment-Matched quadratic Gaussian approximation

For large initial asset price S_0, the probability of hitting zero is almost zero (i.e $\mathbb{P}(\inf t|S_t = 0 < \Delta) \underset{\Delta\to 0}{\longrightarrow} 0$) and the distribution function approches an ordinary noncentral chi-square distribution:

$$\begin{aligned}\mathbb{P}\left(S_\Delta \leq K|S_0>0,\ \sigma_\Delta,\ \int_0^\Delta \sigma_s^2 ds\right) &= 1-\chi^2(a;b,c)\\ &= \chi^2(c;2-b,a) + \mathbb{P}(\inf t|S_t = 0 < \Delta)\\ &\approx \chi^2(c;2-b,a).\end{aligned}$$

As in [?] we determine the values of the relevant parameters by moment matching; if $Y \sim \chi(c;2-b,a)$, ($\mathbb{E}[Y] = 2-b+a$ and $var(Y) = 2(2-b+2a)$), then Y is computed by :

$$Y = d(e+Z)^2, \quad Z \sim \mathcal{N}(0,1).$$

where,

$$e^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1}-1}, \qquad d = \frac{\mathbb{E}[Y]}{1+e^2}, \qquad \psi = \frac{var(Y)}{(\mathbb{E}[Y])^2}.$$

We then compute $c = \frac{S_\Delta^{2(1-\beta)}}{(1-\beta)^2\nu(\Delta)}$ (where we have set $K = S_\Delta$ in (1)) by the quadratic normal approximation :

$$\frac{S_\Delta^{2(1-\beta)}}{(1-\beta)^2\nu(\Delta)} = d(e+Z)^2 \quad \Longrightarrow \quad S_\Delta = \left((1-\beta)^2\nu(\Delta)d(e+Z)^2\right)^{\frac{1}{2(1-\beta)}}$$

2.2 Direct inversion scheme

If S_0 is small, we invert (1) (where we have set $K = S_\Delta$) using the Newton-Raphson algorithm : we determinine the root c^* of the equation $H(a,b,c) = 1-\chi(a;b,c) - U = 0$:

$$c_{n+1} = c_n - \frac{H(a,b,c_n)}{\frac{\partial H(a,b,c_n)}{\partial c_n}}$$

where,

$$\frac{\partial H(a,b,c_n)}{\partial c_n} = -\frac{1}{2}(\frac{c_n}{a})^{\frac{b-2}{4}} exp(-\frac{a+c_n}{2}) I_{|\frac{b-2}{2}|}(\sqrt{ac_n})$$

and I the Bessel function of the first kind. We then apply the inverse coordinate transform to recover the random numbers in asset price space:

$$S_\Delta = \left((1-\beta)^2\nu(\Delta)c^*\right)^{\frac{1}{2(1-\beta)}}.$$

2.3 Computation of the volatility and the integrated variance

For further details on this topic, the interested reader is referred to [?].
The volatility at time step Δ reads :

$$\sigma_\Delta = \sigma_0 \exp\left(\alpha W_{2,\Delta} - \frac{1}{2}\alpha^2\Delta\right)$$

where,

$$dW_{1,t} = \rho dW_{2,t} + \sqrt{1-\rho^2}dZ_t, \qquad Z \sim \mathcal{N}(0,1), \qquad Z \perp\!\!\!\perp W_{2,t}$$

For the integrated variance $A_\Delta = \int_0^\Delta \sigma_s^2 ds$, we use the log-normal distribution:

$$\begin{aligned}
m &= \mathbb{E}(A_\Delta) \approx \sigma_0^2\Delta(1+\alpha W_{2,\Delta} + \frac{1}{3}\alpha^2(2W_{2,\Delta}^2 - \frac{\Delta}{2}) + \frac{1}{3}\alpha^3(W_{2,\Delta}^3 - W_{2,\Delta}\Delta) + \frac{1}{5}\alpha^4(\frac{2}{3}W_{2,\Delta}^4 - \frac{3}{2}W_{2,\Delta}^2\Delta + 2\Delta^2)),\\
v &= var(A_\Delta) \approx \frac{1}{3}\sigma_0^4\alpha^3\Delta^3,\\
\mu &= \log(m) - \frac{1}{2}\log(1+\frac{v}{m^2}),\\
\sigma^2 &= \log(1+\frac{v}{m^2}),\\
A_\Delta &= \exp(\sigma N^{-1}(u)+\mu), \qquad u \sim \mathcal{U}(0,1)
\end{aligned}$$

where N^{-1} is the normal-inverse Gaussian distribution.

2.4 Premia code

Finally, the scheme is compute by the Premia code source **sabr_low_bias_scheme**:

*int sabr_low_bias_scheme(const double S0, const double K, const double T, const double alpha, const double sigma0, const int N, const double dt, const double rho, const double beta, double *price, double *up, double *down, double *delta).*

Input :

- S0: the spot price of the underlying
- K: the strike price of the option
- T: the maturity of the option
- sigma0: the volatility of the underlying at time t=0
- N: the number of Monte Carlo iterations
- n : the number of the time intervals
- rho : correlation factor

Output :

- price: the price of the asset
- up: upper confidence bounds
- down: lower confidence bounds

References

[1] B.CHEN C.W.Oosterlee J.A.M.VAN DER WEIDE. Efficient unbiased simulation scheme for the sabr stochastic volatility model. *International Journal of Theoretical and Applied Finance.*, 15(2), 2012. 1