A Forward Solution for Computing Derivatives Exposure*

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Abstract

In this paper, we derive a forward analytical formula for computing the expected exposure of financial derivatives. Under general assumptions about the underlying diffusion process, our solution consists of two terms: The first term is an intrinsic value part which is directly deduced from the term structure of the forward mark-to-market. The second term expresses the variability of the future mark-to-market and represents the time value part.

In the spirit of Dupire's equation for local volatility, our formula establishes a differential equation of the evolution of the expected exposure with respect to the observation date. Our results are twofold: First, we derive analytically an integral representation of the exposure's expectation and we show that our result is assimilated to a generalized occupation time formula. A straightforward link with local times is highlighted in dimension 1, while the multidimensional extension is based on the co-area formula. Second, we show that from a numerical perspective, our solution can be significantly efficient when compared with standard numerical methods. The accuracy and time-efficiency of the forward representation are of special interest in computing xVA valuation adjustments in a benchmarking setting.

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1 Pricing framework

We consider a standard pricing setup using a risk neutral measure denoted $\mathbb{Q}$ or any equivalent measure.

We consider a financial contract where cash-flow are paid at discrete times $(T_i)_{i=1,\dots,N} \in [0,T]$ between pricing time 0 and the maturity of the contract $T > 0$. We define $s \in [0,T]$ and $t \in [0,T]$ as respectively future valuation and marking-to-market dates. $D(s,t)$ denotes the (deterministic) discount factor, i.e. the value at s of one monetary unit received at t. Let $\Pi(t,T)$ to be the discounted cash-flows of the underlying contract exchanged between t and T. It is well known that the arbitrage-free value of the contract at time s is given by

$$m(s;t) \triangleq D(s,t)\,\mathbb{E}_s[\Pi(t,T)] \tag{1.1}$$

where $\mathbb{E}_s$ ($\mathbb{E}_0 = \mathbb{E}$) denotes the $\mathbb{Q}-$expectation based on market information up to time s. Then, the value of the expected exposure to a default occurring at time t reads

$$EE(t) \triangleq D(0,t)\,\mathbb{E}\left[(m(t,t))^+\right] \tag{1.2}$$

where only cash-flows exchanged after t are accounted.

This quantity expresses the loss on favorable mark-to-market scenarios in case of default of the counterparty. Here, we don't consider the recovery rate neither the collateral held. In addition, the exposure is calculated on the basis of *risk-free closeout*, i.e. risk-free mark-to-market at default. In presence of continuous collateralization triggered at $H > 0$, the above exposure expression becomes

$$EE(t) = D(0,t)\,\mathbb{E}\left[(m(t,t))^+ - (m(t,t) - H)^+\right] \tag{1.3}$$

Additional deterministic collateral specifications such as Independent Amounts and Initial Margins can be easily accounted for in 1.3.

In a Markov setup, we admit that $\Pi(t,T)$ depends on the realization at time t of a stochastic process $(X_s)_{s\in[0,t]}$ so that:

$$m(s;t) \equiv m(s,X_s;t) = D(s,t)\,\mathbb{E}_s[\Pi(t,T)\,|X_t] \tag{1.4}$$

X refers to a multidimensional risk factor.

Without loss of generality, we assume that discounting rates are nil, so that $D(s,t) = 1$. As a consequence, $(m(s,X_s;t))_{s\in[0,t]}$ is a local martingale.

We suppose that $X \in \mathbb{R}^d$ evolves under $\mathbb{Q}$ according to the stochastic differential equation

$$\forall u \in [0,T],\ X_u = \left(X_{i,u}\right)_{i=1\dots d} = X_0 + \int_0^u \mu(v,X_v)\,dv + \int_0^u \sigma(v,X_v)\,dW_v \tag{1.5}$$

where $W = \left(W^1,\dots,W^r\right)^T$ is a $\mathbb{R}^r-$valued standard Brownian motion, $X_0 = \left(X_{1,0},\dots,X_{d,0}\right)^T \in \mathbb{R}^d$, $\mu : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times r}$ satisfying the assumption ($\mathscr{H}_1$) as in Bally and Talay [1]:

Assumption 1.1 ($\mathscr{H}_1$)**.** *The derivatives μ and σ w.r.t. the space variable X exist at any order and are bounded. In addition, $\sigma\sigma^T$ fulfills the uniform ellipticity condition; i.e. there exits $\sigma_\infty > 0$ such that*

$$\left\|\sigma(u,x)\,\sigma(u,x)^T\right\| > \sigma_\infty \tag{1.6}$$

for all (u,x), $u \in [0,T]$ and $x \in \mathbb{R}^d$.

$(\mathscr{H}_1)$ implies the existence of a smooth transition density $q(s,x) \equiv q(s,x:0,X_0)$ and the existence of exponential bounds for q and its derivatives w.r.t. $s>0$, x and X_0.

To perform our analysis, we introduce smoothness requirements on the payoff function Π :

Assumption 1.2 $(\mathscr{H}_2)$**.** *We assume that the value function $(s,x) \mapsto m(s,x;t)$ is $\mathscr{C}^{1,2}\left([0,t[\times\mathbb{R}^d \to \mathbb{R}\right)$. In addition, the first and second partial derivative of m w.r.t. x are bounded.*

2 The forward exposure representation for $X \in \mathbb{R}$

Theorem 2.1. *Given $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $\mathscr{H}_3(1)$, the expected exposure at time t satisfies the following equation*

$$EE(t) = (m(0,t))^+ + \frac{1}{2}\int_0^t \sum_{i=1}^{n(s,t)} \sigma^2(s,l_i(s,t)) \left|\partial_x m(s,x;t)|_{x=l_i(s,t)}\right| q(s,l_i(s,t))\,ds \tag{2.1}$$

Theorem 2.1 presents a convenient decomposition of the expected exposure $EE(t)$. The first term $(m(0,t))^+ = (\mathbb{E}[\Pi(t,T)])^+$ corresponds the positive part of the forward mark-to-market which can be assimilated to the intrinsic value of the exposure. The second term depends on the volatility function σ and represents consequently the time value part of the exposure.

Noticing that $(m(0,t))^+$ remains constant between two successive coupon payment dates. An incremental reformulation of the expected exposure is then possible. It is given in the following corollary.

Corollary 2.2 (Incremental exposure)**.** *Given Theorem 2.1, one has $\forall i \in [[1\ldots N]]$,*

$$\begin{aligned} EE(T_{i-1}) &= (m(0,T_{i-1}))^+ \\ &+ \frac{1}{2}\int_0^{T_{i-1}} \sum_{j=1}^{n(s,T_{i-1})} \sigma^2\left(s,l_j(s,T_{i-1})\right) \left|\partial_x m(s,x,T_{i-1})|_{x=l_j(s,T_{i-1})}\right| q\left(s,l_j(s,T_{i-1})\right) ds \end{aligned} \tag{2.2}$$

and $\forall t \in [T_{i-1},T_i[$

$$EE(t) = EE(T_{i-1}) + \frac{1}{2}\int_{T_{i-1}}^t \sum_{i=1}^{n(s,t)} \sigma^2(s,l_i(s,t)) \left|\partial_x m(s,x;t)|_{x=l_i(s,t)}\right| q(s,l_i(s,t))\,ds \tag{2.3}$$

Remark 2.3. Different choices of the state variable X, and implicitly q, are possible for pricing a contingent claim. In the purpose of using our formula (2.1), an optimal choice of X should satisfy $(\mathscr{H}_3(1))$ as well as an explicit calculation of $l_i(t)$ roots.

2.1 Valuation examples in dimension 1 - Equity forward contract:

We consider an Equity forward contract for which the expected exposure in given by the prices of European call options. We refer by S the price of an Equity asset and by X its logarithm. We assume that X has the following dynamics

$$\begin{cases} dX_s &= -\frac{1}{2}\sigma(s,X_s)^2\,ds + \sigma(s,X_s)\,dW_s \\ X_0 &= x \end{cases} \tag{2.4}$$

where σ the local volatility function. The price of the forward contract is given at time s by

$$m(s,x;t) = \mathbb{E}\left[e^{X_T^{s,x}} - K\right] = e^x - K$$

and the expected exposure by the call option price

$$EE(t) = \mathbb{E}\left[m(t,t_s,t)^+\right] = \mathbb{E}\left[\left(e^{X_t} - K\right)^+\right]$$

$EE(t)$ is given explicitly in case of the classical Black model or some special cases of local volatility models such as the *Constant Elasticity of Variance (CEV)*. For any local volatility function σ, $EE(t)$ is given explicitly using our forward representation with:

- $n(t,s) = 1$
- $l_1(s,t) = \ln(K)$ i.e. $(m(s,l_1(s,t),t) = 0)$
- $\partial_x m(s,x;t)|_{x=l_1(s,t)} = e^{l_1(s,t)} = K$
- $q(s,x)$ is the density function X_s, either given explicitly or constructed numerically.

3 The forward exposure representation for $X \in \mathbb{R}^d$, $d \geq 2$

We propose a generalization of Theorem 2.1 to the case where X is an $\mathbb{R}^d-$valued process. We use the *co-area formula* ([2, 3]).

Our main result expresses the expected exposure in terms of the Lebesgue measure on the surface $\left\{x \in \mathbb{R}^d | m(s,x;t) = 0\right\}$. This surface is explicitly characterized in case of $d = 2$ in view of numerical applications.

3.1 General case

we need to introduce the following conditions, which is a stronger analog to $(\mathcal{H}_3(1))$:

Assumption 3.1 $(\mathcal{H}_3(d))$**.** *We assume that* $|\nabla_x m(s,x;t)| \neq 0$ *on* $\Upsilon(z)$ *where* $z \in [-\varepsilon, \varepsilon]$.

We now give the multidimensional extension of our main result. We follow a straightforward derivation that is only based on the co-area formula a

Theorem 3.2. *Assuming* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3(d))$, *the expected exposure satisfies the following equation*

$$EE(t) = (m(0,t))^+ + \frac{1}{2}\int_0^t ds \int_{\Upsilon(0)} \frac{g(s,x;t)}{|\nabla_x m(s,x;t)|} q(s,x)\, d\upsilon_{\Upsilon(0)}(x) \tag{3.1}$$

where

$$g(s,X_s;t) = \sum_{i,j=1}^{d} \sum_{k=1}^{r} \nabla_{x_i} m(s,X_s;t)\, \sigma_{i,k}(s,X_s)\, \nabla_{x_j} m(s,X_s;t)\, \sigma_{j,k}(s,X_s) \tag{3.2}$$

3.2 Case $X \in \mathbb{R}^2$

Given the times(s,t), $\Upsilon(0)$ corresponds to the set of points (x_0, y_0) where $m(s,(x_0,y_0);t) = 0$ and $|\nabla_{(x,y)} m(s,(x_0,y_0);t)| \neq 0$. The case $d = 2$ can be made more explicit by the means of the *implicit function theorem.*

Proposition 3.3 (Implicit function theorem)**.** *Let* $(x_0,y_0) \in \Upsilon(0)$. *We admit that* $\partial_y m(s,(x_0,y_0);t) \neq 0$. *There exists a function* ψ *defined on an open interval* W *containing* x_0*, and an open set* V *in* $W \times \mathbb{R}$*, such that*

$$\forall (x,y) \in V,\ (x,y) \in \Upsilon(0) \Leftrightarrow \psi(x) = y$$

The condition $\partial_y m(s,(x_0,y_0);t) \neq 0$ is not restrictive thanks to $(\mathcal{H}_3(d))$. In fact, one has to consider an explicit function $x = \varphi(y)$ if $\partial_y m(s,(x_0,y_0);t) = 0$ and hence $\partial_x m(s,(x_0,y_0);t) \neq 0$.

The implicit function theorem allows to derive an explicit representation of the surface integral Υ in the co-area formula. A simple situation consists in having $V = \Upsilon(0)$ and unicity of the implicit function ψ. It follows then that the Lebesgue surface measure $dv_{\Upsilon(0)}(x)$ is induced by the Lebesgue real measure dx. Equation (3.1) becomes

$$EE(t) = (m(0,t))^+ + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} \frac{g(s,(x,\psi(x));t)}{|\nabla_x m(s,(x,\psi(x));t)|} q(s,(x,\psi(x)))\, dx$$

The existence of ψ is guaranteed by the implicit function theorem for any $(x_0,y_0) \in \Upsilon(0)$. We denote the set of implicit functions characterizing $\Upsilon(0)$ by Ψ. In the case where Ψ is a denumerable and is explicit, one has

$$EE(t) = (m(0,t))^+ + \frac{1}{2}\sum_{\psi\in\Psi}\int_0^t ds \int_{\mathbb{R}} \frac{g(s,(x,\psi(x));t)}{|\nabla_x m(s,(x,\psi(x));t)|} q(s,(x,\psi(x)))\, dx \tag{3.3}$$

As illustrated in the sequel, this expression can be used in numerical computations.

It is worth mentioning that, thanks to the implicit function theorem, the expected exposure formula (3.1) can be run at the cost of numerical integration in $\mathbb{R}$. The same argument can be adapted to $d \geq 3$, and shows that a numerical integration in $\mathbb{R}^{d-1}$ is required (instead of $\mathbb{R}^d$).

3.3 A valuation examples in dimension 2 - FX Swap

We consider an FX swap under the joint Garman-Kolhagen/Hull-White model. The swap corresponds to receiving a float rate in the domestic currency and paying a fixed rate in the foreign currency. We denote by P and $\tilde{P}$ the zero-coupon bond price functions in respectively the domestic and foreign currencies, and $\left(X_t^{f/d}\right)$ the price of 1 monetary unit of the foreign currency expressed in the domestic currency.

The mark-to-market function m reads

$$\begin{aligned} m(s,t) &= \sum_{i|T_i\geq t}^{N} \mathbb{E}_s\left[P(s,T_i)\tau_i\left(L(T_{i-1},T_i)-X_{T_i}K\right)\right] \\ &= \sum_{i|T_i\geq t}^{N} \mathbb{E}_s\left[P(s,T_{i-1})-P(s,T_i)\right] - \sum_{i|T_i\geq t}^{N} \tau_i K \mathbb{E}_s\left[P(s,T_i)X_{T_i}^{f/d}\right] \\ &= P\left(s,T_{\beta(t)-1}\right) - P(s,T_N) - X_s^{f/d}\sum_{i|T_i\geq t}^{N}\tau_i K\tilde{P}(s,T_i) \end{aligned}$$

where $\beta(t) = \inf\{i \in [1..N] \,|\, T_i \geq t\}$.

As for the IR swap, we consider the 1-factor Hull&White short rate model that states that

$$\begin{cases} r_u &= \phi(u) + \sigma\int_t^u e^{-a(u-v)}dW_v^r \\ \tilde{r}_u &= r_u + s(u) \end{cases}$$

where a and σ are respectively the mean-reversion and the volatility parameters and s is a deterministic spread function specified initially. As a consequence, the prices P and $\tilde{P}$ are explicit functions of r

$$\begin{cases} P(s,T) &\triangleq P(s,r_s,T) = \mathbb{E}_s\left[e^{-\int_s^T r_u du}\right] = A(s,T)\exp\left(-B(s,T)r_s\right) \\ \tilde{P}(s,T) &\triangleq \tilde{P}(s,\tilde{r}_s,T) = \mathbb{E}_s\left[e^{-\int_s^T \tilde{r}_u du}\right] = \tilde{A}(s,T)\exp\left(-\tilde{B}(s,T)r_s\right) \end{cases} \tag{3.4}$$

Finally, the exchange rate is driven by a log-normal dynamics such that

$$\frac{dX_u^{f/d}}{X_u^{f/d}} = (r_u - \tilde{r}_u)\,du + \sigma^X dW_u^X$$

The Brownian motions W^r and W^X are correlation, i.e. $d\left\langle W^r, W^X\right\rangle_t = \rho dt$.

In order to compute the expected exposure $EE(t)$, we specify quantities that are involved in the forward representation (2.1). For $s \leq t$,

- $\partial_r m\left(s,r_s,X_s^{f/d};t\right) = -B\left(s,T_{\beta(t)-1}\right)P\left(s,T_{\beta(t)-1}\right) + B(s,T_N)P(s,T_N) + X_t^{f/d}\sum_{i|T_i\geq t}^{N}\tau_i K\tilde{B}(s,T_i)\tilde{P}(s,T_i)$
- $\partial_X m\left(s,r_s,X_s^{f/d};t\right) = -\sum_{i|T_i\geq t}^{N}\tau_i K\tilde{P}(s,T_i)$
- $\sigma(s,r_s) = \sigma$ and q is the transition density function of the Gaussian distribution.

In particular $|\nabla_{(r,X)}m| \neq 0$ since $\partial_X m\left(s,r_s,X_s^{f/d};t\right) \neq 0$.

Finally, the set $\Upsilon(0)$ is entirely specified using the implicit function theorem. In fact, one can easily find that $m\left(s,r_s,X_s^{f/d},t\right) = 0$ imposes that

$$X_s^{f/d} = \frac{P\left(s,T_{\beta(t)-1}\right) - P(s,T_N)}{\sum_{i|T_i\geq t}^{N}\tau_i K\tilde{P}(s,T_i)} = \psi_t(r_s)$$

and the implicit theorem function ψ_t is given explicitly in terms of A, $\tilde{A}$, B and $\tilde{B}$.

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