Pricing and hedging American-style options: a simple simulation-based approach, by Y.Wang and R.Caflisch

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1 Discrete optimal stopping problem

Let us consider a discrete non-negative random process $(H_i)_{0\leq i\leq k}$ adapted to a filtration $(\mathcal{F}_i)_{0\leq i\leq k}$. We suppose that this process is a function of another underlying process X_i: $H_i = h_i(X_i)$ for some function h_i.

We are interested in the problem of finding a stopping time that maximizes the $\mathcal{F}_i$-conditional expectation $\mathbb{E}_i[H_\tau]$ over stopping time τ taking value in $\{i, ..., k\}$

$$\sup_{\tau\in\{i,...,k\}} \mathbb{E}_i[H_\tau] \tag{1}$$

We call an *optimal stopping time* for this problem an $\mathcal{F}_i$-stopping time τ_i^* such that

$$\mathbb{E}_i[H_{\tau_i^*}] = \sup_{\tau\in\{i,...,k\}} \mathbb{E}_i[H_\tau]$$

We define the process $Q_i^* := \mathbb{E}_i[H_{\tau_i^*}]$. The process $(Q_i^*)_{0\leq i\leq k}$ is called the *Snell envelope* process of $(H_i)_{0\leq i\leq k}$. It can be constructed using the well known *backward dynamic programming* algorithm. In fact, by definition we have $Q_k^* = H_k$

$$\begin{cases} Q_k^* &= H_k \\ Q_i^* &= \max\left(H_i,\ \mathbb{E}_i\left[Q_{i+1}^*\right]\right)\ \text{, for } 0\leq i\leq k-1 \end{cases} \tag{2}$$

Then τ_i^* can be represented by

$$\begin{aligned} \tau_i^* &= \inf\left\{j, i\leq j\leq k : H_j \geq \mathbb{E}_j[Q_{j+1}^*]\right\} \\ &= \inf\left\{j, i\leq j\leq k : H_j \geq \mathbb{E}_j[H_{\tau_{j+1}^*}]\right\} \end{aligned}$$

or

$$\begin{cases} \tau_k^* &= k \\ \tau_i^* &= i\mathbb{1}_A + \tau_{i+1}^*\mathbb{1}_{A^c} \end{cases} \tag{3}$$

where $A = \{H_i \geq \mathbb{E}_{i+1}[H_{\tau_{i+1}^*}]\}$

2 Regression method to price American option: Longstaff-Schwartz algorithm

The optimal stopping time family $(\tau_i^*)_i$ (or equivalently the price process (Q_i^*)) can be estimated via plain Monte Carlo simulation of the backward dynamic program (3). But this raises the problem of how to estimates the conditional expectation $\mathbb{E}_{i+1}[H_{\tau_{i+1}^*}]$. Using Monte Carlo method to estimate this latter implies nested simulations, which makes the algorithm potentially very slow.

In fact, to estimate $Q_i^* = \max\left(H_i,\ \mathbb{E}_i\left[Q_{i+1}^*\right]\right)$, we need the conditional expectation $\mathbb{E}_i\left[Q_{i+1}^*\right]$. We estimate this latter by sampling conditional on the state at time i, hence the nested simulations.

One possible way to circumvent this problem is the well known Longstaff-Schwartz algorithm. We note the price given by this algorithm by Q_i^{LS}.

The idea is to estimate the conditional expectations $\mathbb{E}_i\left[Q_{i+1}^{LS}\right]$ using some regression methods, without any nested simulation. In fact, in the case of a Markovian setting, the conditional expectation is a function of the underlying process X_i

$$\mathbb{E}_i\left[Q_{i+1}^{LS}\right] = \phi_i(X_i)$$

The function ϕ_i is then approximated by an orthogonal projection $\langle\alpha_i, g_i\rangle$. We then choose α_i as the one that minimize the second order moment

$$\begin{cases} \min \ \mathbb{E}\left[\left(Q_{i+1}^{LS} - \langle \alpha_i, g_i \rangle (X_i)\right)^2\right] \\ \alpha_i \in \mathbb{R}^d \end{cases}$$

Using Monte Carlo method, this minimization problem can be rewritten, for a set of M simulated paths $\{w_1, w_2, ..., w_M\}$:

$$\begin{cases} \min \ \sum_m \left[Q_{i+1}^{LS}(w_m) - \langle \alpha_i, g_i \rangle (X_i(w_m))\right]^2 \\ \alpha_i \in \mathbb{R}^d \end{cases}$$

Then Q_i^{LS} is estimated by $Q_i^{LS}(w_m) = \max\left(h_i(X_i(w_m)),\ \langle \alpha_i, g_i \rangle (X_i(w_m))\right)$.
The stopping time given by this method is

$$\tau_i^{LS} = \inf\{j\ ,\ i \leq j \leq k :\ h_j(X_j) \geq \langle \alpha_j, g_j \rangle (X_j)\}$$

for $1 \leq i \leq k$. At initial time $i = 0$, the continuation value is just the plain expectation $\mathbb{E}_0\left[Q_1^{LS}\right]$, so we don't need regression to evaluate it. The option price at initial time is then given by $Q_0^{LS} = \max\left(H_0,\ \mathbb{E}_0\left[Q_1^{LS}\right]\right)$.

However, this method gives only an approximation of the price, and no information about the sensitivities of American option, also called the greeks. The method proposed by [Wang, Caflisch 2009], based on the Least Squares approximation of conditional expectations by [Longstaff, Schwartz 2001], gives an estimation of these sensitivities.

3 Modified Longstaff-Schwartz algorithm to estimate the greeks

The approach proposed in [Wang, Caflisch 2009] is based on the least squares regression method in Longstaff-Schwartz algorithm (LS). We will note this algorithm (MLSM) as in [Wang, Caflisch 2009]. The key idea is to use a random set of initial stock price and approximate the option price function at initial date by an additional regression. This means that instead of having X_0 as deterministic, we consider that it has some predefined distribution.

Hence, by using the Longstaff-Schwartz algorithm, we get the option price at initial time 0 on this random set of initial stock price, then by an additional regression we can have an option price functional approximation of the form $Q_0^{MLSM}(X) = \langle \alpha_0, g_0 \rangle (X)$. Therefore, to obtain option price and its the sensitivities to X, we just evaluate the function $\langle \alpha_0, g_0 \rangle$ at the actual initial stock price X_0 and use the explicit derivative of this basis function ie $\dfrac{\partial\langle\alpha_0, g_0\rangle}{\partial X}$.

We can get for instance the option price, the delta and the gamma by

$$
\begin{aligned}
Q_0^{MLSM}(X_0) &= \langle\alpha_0, g_0\rangle(X_0) \\
\Delta_0^{MLSM}(X_0) &= \frac{\partial Q_0^{MLSM}}{\partial X}(X_0) &= \frac{\mathrm{d}\langle\alpha_0, g_0\rangle}{\mathrm{d}X}(X_0) \\
\Gamma_0^{MLSM}(X_0) &= \frac{\partial^2 Q_0^{MLSM}}{\partial X^2}(X_0) &= \frac{\mathrm{d}^2\langle\alpha_0, g_0\rangle}{\mathrm{d}X^2}(X_0)
\end{aligned}
$$

As noted in [Wang, Caflisch 2009], this algorithm can be interpreted as starting the stock price process from a date before initial time 0.

Numerical implementation

- Taking into account the previous remark, we generate the random set of initial stock price from an initial distribution of the form

$$X(w, t=0) = X_0 e^{-\alpha\sigma^2 + \sigma\sqrt{\alpha}w}\ ,\ w \sim \mathcal{N}(0,1)$$

  where σ, T are the corresponding stock volatility and option maturity. α should be used to adjust the variance of the distribution. We note that the mean of $X(w, t=0)$ is X_0.

- For the regression step, we used the least squares algorithm implemented in the Premia Numerical Library (PNL) via the function *pnl_basis_fit_ls*. It permits to use canonical, Hermite and Tchebychev polynomials as regression basis and provides also the derivatives value.

References

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[Longstaff, Schwartz 2001] Longstaff, F. A., and Schwartz, E. S. (2001) Valuing American options by simulation: a simple least-square approach *Review of Financial Studies*, 14, 113–147. 3