Optimal Execution Under Jump Models for Uncertain Price Impact

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ABSTRACT

In this paper, we tackle the execution cost problem.
In fact, the price of underlying assets subject to transactions are not known in advance. They vary according to multiple parameters such as volatility and interest rates.
The aim of this article is to focus on the implicit price impacts of large trades on execution costs.
As a result, we build three optimal execution strategies under compound jump processes for price impacts.

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1 INTRODUCTION

The execution cost problem consists on minimizing the cost or difference between the value of an ideal deal and its real implementation. This extra cost could come from two possible sources, explicit cost such as commissions and broker fees, and implicit costs which are mainly due to the price impact of large trades made by investors. To decrease this price impact, trades are usually split into small fragments traded over a period of time. Such procedure is called an execution strategy. To achieve this, one should meticulously model the price impacts of large trades along with the market price dynamics that the classical brownian motion market price model fails to capture.

The paper is organised as follows:

First, we start by modeling the price impact of large trades which we split into permanent price impact and temporary price impact. The permanent price impact comes from imbalance between supply and demand and the information transmitted on the market. The temporary price impact on the other hand reflects the liquidity cost on additional price an investor pays for immediate execution of the trades.

Secondly, we model the market price dynamics under an additive or multiplicative jump diffusion models based on compound Poisson Processes.
Finally, we develop closed-form expressions for naive, additive and multiplicative optimal execution strategies and their costs under the previous market price dynamics accounting for permanent and temporary price impacts.

2 EXECUTION STRATEGIES

In order to build optimal execution strategies, we should present the general framework and the mathematical formalism of the problem.
We assume that an investor plans to liquidate his holding of an asset during N periods in the time interval.
Let $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ where $\tau = t_k - t_{k-1} = \frac{T}{N}$ for $k = 1, 2, .., N$. The investors position at time $t_k = k\tau$ is denoted by x_k. The investors initial position is $x_0 = S_0$ shares in number of units and the final position is $x_N = 0$. The difference between positions at two consecutive times t_{k-1} and t_k is denoted by $n_k = x_{k-1} - x_k; \quad k = 1, 2, .., N$.
A negative n_k implies that the asset is bought between t_{k-1} and t_k.
We refer to a sequence $\{n_k\}_{k=1}^N$ satisfying $\sum_{k=1}^N n_k = S_0$ as an execution strategy.

3 PRICE DYNAMICS UNDER ADDITIVE AND MULTIPLICATIVE JUMP PROCESSES

3.1 Jump Processes

We assume that the arrival time of large trades from other institutions as well as their impact are unkown to the investor. These uncertain arrivals are modeled using a Poisson process with constant arrival rates. The uncertain impact of these arrivals is assumed to follow a known distribution. Combining both, we model the uncertain price impact of uncertain trades from other institution by a compound Poisson process.

To further refine the model, we distinguish buys from sells, we assume that they are independant and follow a Poisson Process with deterministic arrival rates.

Let $\{X_t \,:\, t \in [0,T]\}$ be a Poisson process in the execution horizon $[0,T]$ with a constant arrival rate $\lambda_x \geqslant 0$ that models uncertain arrivals of sell trades.
Let $\{Y_t \,:\, t \in [0,T]\}$ be a Poisson process with a constant arrival rate $\lambda_y \geqslant 0$ representing the arrivals of buy trades.

Combining both processes we model the uncertain permanent price impact of trades by other institution in $[t_{k-1}, t_k]$ as:

$$J(k) = \sum_{l=1}^{Y_{t_k}-Y_{t_{k-1}}} \chi_l(k) - \sum_{l=1}^{X_{t_k}-X_{t_{k-1}}} \Pi_l(k) \tag{3.1}$$

where $\chi_l(k)$ is a random variable with a known distribution.
$\{\chi_l(k)\}$ are independently distributed with mean $\mu_x(k)$ and standard deviation $\sigma_x(k)$. It captures the permanent price impact of the l-th buy trade in the period $[t_{k-1}, t_k]$.

Similarly $\Pi_l(k)$ represents the permanent price impact of the l-th sell trade in the period $[t_{k-1}, t_k]$, with mean $\mu_y(k)$ and standard deviation $\sigma_y(k)$.

3.1.1 *Additive Jump Process*

Using the equation (3.1) the additive jump process becomes:

$$J^a(k) = \sum_{l=1}^{Y_{t_k}-Y_{t_{k-1}}} \chi_l^a(k) - \sum_{l=1}^{X_{t_k}-X_{t_{k-1}}} \Pi_l^a(k) \tag{3.1.1}$$

We note the expected value $\mathbb{E}[J^a(k)]$ and the variance $\mathrm{Var}[J^a(k)]$ as $E_J^a(k)$ and $V_J^a(k)$ respectively.

For normally distributed jump sizes $\Pi_l^a(k)$ and $\chi_l^a(k)$ with means $\mu_x^a(k)$ and $\mu_y^a(k)$ and standard deviations and $\sigma_x^a(k)$ and $\sigma_y^a(k)$ respectively.

$$\begin{aligned} E_J^a(k) &= \tau\lambda_y \mathbb{E}[X_l^a(k)] - \tau\lambda_x \mathbb{E}[\Pi_l^a(k)] \\ &= \tau\big(\lambda_y \mu_y^a(k) - \lambda_x \mu_x^a(k)\big) \end{aligned}$$

$$\begin{aligned} V_J^a(k) &= \tau\lambda_x\big(\mathrm{Var}(\Pi_l^a(k)) + (\mathbb{E}[\Pi_l^a(k)])^2\big) + \tau\lambda_y\big(\mathrm{Var}(\chi_l^a(k)) + (\mathbb{E}[\chi_l^a(k)])^2\big) \\ &= \tau\lambda_x\big((\sigma_x^a(k))^2 + (\mu_x^a(k))^2\big) + \tau\lambda_y\big((\sigma_y^a(k))^2 + (\mu_y^a(k))^2\big) \end{aligned}$$

3.1.2 *Multiplicative Jump Process*

As stated in the general equation (1.1) the multiplicative jump process has the form below:

$$J^m(k) = \sum_{l=1}^{Y_{t_k}-Y_{t_{k-1}}} (\chi_l^m(k) - 1) - \sum_{l=1}^{Y_{t_k}-Y_{t_{k-1}}} (\Pi_l^m(k) - 1) \tag{3.1.2.1}$$

A log-Normally distributed jump amplitudes $X_l^m(k), \Pi_l^m(k)$, are assumed for this implementation for which an explicit expected value and variance could be written:

$$E_J^m(k) = \tau\lambda_y\big(\exp(\mu_y^m(k) + \frac{(\sigma_y^m(k))^2}{2}) - 1\big) - \tau\lambda_x\big(\exp(\mu_x^m(k) + \frac{(\sigma_x^m(k))^2}{2}) - 1\big)$$

$$\begin{aligned} V_J^m(k) = \tau\lambda_x&\big(\mathrm{Var}[\Pi_l^m(k)] + (\mathbb{E}[\Pi_l^m(k)] - 1)^2\big) \\ &+ \tau\lambda_y\big(\mathrm{Var}[\Pi_l^m(k)] + (\mathbb{E}[\Pi_l^m(k)] - 1)^2\big) \end{aligned}$$

3.2 Price dynamics under jump models

The general case for price dynamics assumed in this paper is:

$$p_k = F_{k-1}(p_{k-1}) - \tau g(\frac{n_k}{\tau}); \qquad k = 1, 2, ..N-1 \qquad (3.2.1)$$

where $F_{k-1}(p_{k-1})$ denotes the market price at time t_k when the investor does not trade $[t_{k-1}, t_k]$ and $g()$ is a deterministic function of the trading rate that represents the permanent price impact of the investor's trade.

In addition to the permanent impact investor's trade induces a temporary price impact on the execution price $\widetilde{p}_k$ given by:

$$\widetilde{p}_k = \widetilde{p}_{k-1} - h(\frac{n_k}{\tau}); \qquad k = 1, 2, ..N-1 \qquad (3.2.2)$$

where $h()$ is also a deterministic function representing the temporary impact function.

For the permanent impact function $g()$ and the tempporary impact function $h()$ the use of linear functions is frequent.

$$g(v) = Gv; \qquad h(v) = Hv$$

where $v = \frac{n}{\tau}$ is the trading role.

3.2.1 *Additive Price Dynamics*

For the additive price model an additive jump is used as stated in section (1.2). The change in market price comes from a Brownian increment and a jump $J^a(k)$.

$$F_{k-1}(p_{k-1}) = p_{k-1} + \tau^{\frac{1}{2}}\Sigma^a Z_k + \tau\alpha_0^a + J^a(k) \qquad (3.2.1.1)$$

where
$\tau\alpha_0^a$: is the expected price due to small trades
Z_k : is a standard normal random variable
σ^a : is the volatility of the asset price change.

The total market price change is split into two components:
- a small trades component captured by : $\tau\alpha_0 + \tau^{\frac{1}{2}}\Sigma^a Z_k$
- a permanent price impact of large trades captured by $J^a(k)$

Using the general price form (2.1) we obtain the price dynamics for an additive jump model as follows:

$$p_k = p_{k-1} + \tau^{\frac{1}{2}}\Sigma^a Z_k + \tau\alpha_0^a + J^a(k) - \tau g(\frac{n_k}{\tau})$$

$$\text{where } J^a(k) = \sum_{l=1}^{Y_{t_k} - Y_{t_{k-1}}} \chi_l^a(k) - \sum_{l=1}^{X_{t_k} - X_{t_{k-1}}} \Pi_l^a(k); \qquad k = 1, 2, ..N-1$$

3.2.2 *Multiplicative Price Dynamics*

For a multiplicative price model, we consider a multiplicative jump as stated in (3.1.2.1). The change in the market price corresponds to:

$$F_{k-1}(p_{k-1}) = p_{k-1}\big(1 + \tau\alpha_0^m + \tau^{\frac{1}{2}}\Sigma^m Z_k + J^m(k)\big) \qquad (3.2.2.1)$$

In this case, the term $\tau\alpha_0^m$ can be interpreted as the expected return due to small trades. Using (3.2.1), we obtain the price dynamics for a multiplicative jump model as:

$$p_k = p_{k-1}\big(1 + \tau\alpha_0^m + \tau^{\frac{1}{2}}\Sigma^m Z_k + J^m(k)\big) - \tau g(\frac{n_k}{\tau})$$
$$\text{where } J^m(k) = \sum_{l=1}^{Y_{t_k}-Y_{t_{k-1}}} (\chi_l^m(k) - 1) - \sum_{l=1}^{X_{t_k}-X_{t_{k-1}}} (\Pi_l^m(k) - 1)$$
$$k = 1, 2, ..N - 1$$

4 OPTIMAL EXECUTION STRATEGIES

The general formulation of the cost execution problem was discussed in section (2). An optimal execution strategy is one that minimises the overall cost of execution which implies solving a constrained optimization problem.

Given an execution strategy $\{n_k\}_{k=1}^N$ The total amount recieved at the end of the time horizon T is $\sum_{k=1}^N p_k^N$. The difference between this quantity and the value of an ideal benchmark trade is the execution cost. The benchmark is commonly taken as the value of the portfolio at the arival price p_0.
Hence, the execution cost associated with the strategy $\{n_k\}_{k=1}^N$ is defined as
$p_0S_0 - \sum_{k=1}^N n_k p_k^N$.
Our aim is to minimize the expected execution cost.

A generic form of the problem could be written as below:

$$\min_{x_1,\dots x_N \in \mathbb{R}^N} \mathbb{E}[p_0S_0 - \sum_{k=1}^N n_k\widetilde{p}_k] + c\rho(p_0S_0 - \sum_{k=1}^N n_k\widetilde{p}_k) \tag{4.1}$$
$$s.t. \quad \sum_{k=1}^N n_k = S_0$$

where $\rho()$ is a risk measure of the execution cost and $c \geqslant 0$ is a risk aversion parameter. The inequality constraints $n_k \geqslant 0$ can be included in (4.1) to rule at buying in the strategy.

Finally, in our implementation, only optimal risk neutral execution strategy with purshasing allowed is considered:

$$\min_{x_1,\dots x_N \in \mathbb{R}^N} \mathbb{E}[p_0S_0 - \sum_{k=1}^N n_k\widetilde{p}_k] \tag{4.2}$$
$$s.t. \quad \sum_{k=1}^N n_k = S_0$$

For the previous problem, stochastic dynamic programming should be used to solve it.

The main ingredients of the solution rely upon the following method:
Let the optimal value formation at t_{k-1} corresponding to the problem (4.2) be:

$$V_k^*(p_{k-1}, x_{k-1}) = \min_{x_1,\dots x_N \in \mathbb{R}^N} \mathbb{E}[p_0S_0 - \sum_{j=k}^N n_j\widetilde{p}_j(p_{k-1}, x_{k-1})]$$
$$s.t. \quad \sum_{j=k}^N n_j = x_{k-1}$$

Where x_{k-1} is the current asset holding, and p_{k-1} is the current market price.

For $k = N$, $n_N^* = x_{N-1}$ since there is no choice but to execute the entire remaining order x_{N-1}. Hence, the optimal value function for the last period becomes :

$$\begin{aligned}
&V_N^*(p_{N-1}, x_{N-1}) = \min_{n_N} \mathbb{E}[p_0 S_0 - n_N \widetilde{p}_N(p_{N-1}, x_{N-1})] \\
&= p_0 S_0 - x_{N-1}(p_{N-1} - h(\frac{x_{N-1}}{\tau})) \qquad (4.3) \\
&s.t. \;\; x_{N-1} - x_N = 0
\end{aligned}$$

For the linear temporary price impact function $h(v) = Hv$, we have,

$$V_N^*(p_{k-1}, x_{k-1}) = p_0 S_0 - x_{N-1} P_{N-1} + \frac{1}{2} x_{N-1} \frac{2H}{\tau} x_{N-1} \qquad (4.4)$$

Once n_{k+1}^* and $V_{k+1}^*(p_k, X_k)$ have been determined, the optimal execution n_k^* and the optimal value function $V_k^*(p_{k-1}, X_{k-1})$ can be determined from the Bellman's principle of optimality which relates recusively backwards in time the optimal value function in period k to the optimal value function in period k+1.

Based on the general method presented previously, we can define three different strategies associated with different model assumptions which are the naive strategy (Zero Expected Market Price Charge), the additive strategy and the multiplicative strategy.

The following combined impact parameter $\theta = \frac{2H}{\tau} - G$ will be used in the following section.

4.1 Naive Strategy (Zero Expected Market Price Change)

The naive strategy consists on liquidating at each period $[t_k, t_{k+1}]$; $k = 1, 2, \ldots, N$ the same quantity of the asset:

$n_k = \frac{S_0}{N} \qquad k = 1, 2, \ldots, N$

The naive strategy can be applied under the following assumptions:
- The expected market price change is zero $(\tau\alpha_0 + E[J] = 0)$
equivalent to $\mathbb{E}[F_{k-1}(p_{k-1})|p_k] = p_{k-1}$; $k = 1, 2, \ldots, N-1$
- The impact functions $g()$ and $h()$ are deterministic and depend only on the trading rate $\frac{n_k}{\tau}$.

Under such assumptions the general solution of the problem (4.2) using the Bellman method (4.3) and (4.4) could be written for each iteration k as:

$V_k^*(p_{k-1}, x_{k-1}) = p_0 S_0 - p_{k-1} x_{k-1} + \frac{1}{2} x_{k-1} \big(\frac{\theta}{N-k+1} + G\big) x_{k-1}$

which yields the total execution cost $\quad V_1^*(p_0, x_0) = \frac{1}{2}\big(\frac{\theta}{N} + G\big) S_0^2$.

It's though important to mention that the price dynamics don't really affect this strategy as it doesn't really depend on the market price model.

4.2 Additive strategy

An additive strategy is associated with an additive jump model (3.1.1) and the corresponding price dynamic (3.2.1.1) the optimal strategy for the problem (4.2) using the general Bellman method defined by (4.3) and (4.4), for the additive strategy $n^* = \{n_k^*\}_{k=1}^N$ is uniquely defined as:

$$n_k^* = \frac{-(b_{k+1} - E_J^a(k+1) + E_J^a(k) + (\theta - A_{k+1})x_{k-1}^*)}{A_{k+1}}, \qquad k = 1, 2, \ldots, N-1$$

$$n_N^* = S_0 - \sum_{k=1}^{N-1} n_k^*$$

$$\text{where } x_0^* = S_0 \text{ and } x_k^* = x_{k-1}^* - n_k^* \qquad k = 1, 2, \ldots, N-2$$

$$A_k = A_{k+1} - \frac{(A_{k+1} - \theta)^2}{A_{k+1}} \qquad k = 1, 2, \ldots, N-2$$

$$A_N = 2\theta > 0$$

$$b_k = b_{k+1} - \frac{(\theta - A_{k+1})(b_{k+1} - E_J^a(k+1) + E_J(k))}{A_{k+1}}$$

$$- E_J^a(k+1) + 2E_J^a(k) + \tau\alpha_0^a$$

$$k = 1, 2, \ldots, N-2$$

$$b_N = E_J^a(N) + \tau\alpha_0^a$$

$$c_k = c_{k+1} - \frac{1}{2}\frac{(b_{k+1} - E_J^a(k+1) + E_J^a(k))^2}{A_{k+1}}$$

$$k = 1, 2, \ldots, N-2$$

$$c_N = 0$$

The execution cost at each iteration k is

$V_k^*(P_{k-1}, Xk-1)p_0S_0 - \frac{1}{2}x_{k-1}^2(\theta - A_k - G) - (p_{k-1} + b_k - E_J^a(1) - \tau\alpha_0^a)x_{k-1} - c_k$

From which we could deduce that the total execution cost

$V_1^*(p_0, x_0) = p_0S_0 - \frac{1}{2}S_0^2(\theta - A_1 - G) + (p_0 + b_1 - E_J^a(1) - \tau\alpha_0^a) - c_1$

where $E_J^a(k) = E_J^a$ constant for each k.

Much simpler equations could be derived.

The strategy can be written as below :

$n_k^* = \frac{S_0}{N} - \frac{N+1-2k}{2\theta}(E_J^a + \tau\alpha_0^a); \qquad k = 1, 2, \ldots, N$

The cost at each iteration:
$V_k^*(p_{k-1}, x_{k-1}) = p_0 S_0 - \frac{1}{2}x_{k-1}^2(\theta - A_k - G) - (p_{k-1} + b_k - E_J^a - \tau\alpha_0^a)x_{k-1} - c_k$

The total execution cost
$V_1^*(p_{k-1}, x_{k-1}) = p_0 S_0 - \frac{1}{2}S_0^2(\theta - A_1 - G) + (p_0 + b_1 - E_J^a - \tau\alpha_0^a) - c_1$

4.3 Multiplicative Strategy

The multiplicative strategy derives its name from the multiplicative dynamics (3.2.2.1) associated to the multiplicative jump model (3.1.2.1).

Under this assumption the optimal strategy is given by:

$n_k = \frac{1}{D_{k+1}}(1 - (B_{k+1} + 2A_{k+1}G)L_k)p_{k-1} - \frac{1}{D_{k+1}}(2C_{k+1} + B_{k+1})x_{k-1}$
$n_N = S_0 - \sum_{k=1}^{N-1} n_k$

And the associated optimal cost at each iteration k:

$$V_k^*(p_{k-1}, x_{k-1}) = p_0 S_0 - A_k p_{k-1}^2 - B_k x_{k-1} p_{k-1} - c_k x_{k-1}^2$$

where :

$$\begin{aligned} A_N &= 0 \\ A_{k-1} &= A_k \Phi_{k-1} + A_k L_{k-1}^2 + \frac{1}{2D_k}(1 - L_{k-1}(2A_k G + B_k))^2 \end{aligned}$$

$$\begin{aligned} B_N &= 1 \\ B_{k-1} &= B_k L_{k-1} + \frac{1}{D_k}(1 - L_{k-1}(B_k G + 2A_k G))(2C_k + GB_k) \end{aligned}$$

$$\begin{aligned} C_N &= \frac{-H}{\tau} \\ C_{k-1} &= C_k + \frac{1}{D_k}(2C_k + GB_k) \end{aligned}$$

and

$$\begin{aligned} L_{k-1} &= 1 + \tau\alpha_0 + E_J^m(k-1); && k = 2, .., N-1 \\ Q_{k-1} &= \tau(\sigma^m)^2 + V_J^m(k-1) \\ D_k &= -2A_k G^2 + \frac{2H}{\tau} - 2B_k G - 2C_k; && k = 1, .., N \end{aligned}$$

The total cost of execution obtained with the Bellman principle is given by:

$V_1^*(p_0, X_0) = p_0 S_0 - p_0 A_1 p_0 - p_0 B_1 x_0 - x_0^2 C_1$

5 TRUE EXPECTED PRICE UNDER MULTIPLICATIVE JUMP MODEL

A closed formula for the expected price under the multiplicative jump model could be derived.

We recall that the price under the multiplicative jump model is:
$p_k = p_{k-1}(1 + \tau\alpha_0^m + \tau^{\frac{1}{2}}\Sigma^m Z_k + J^m(k)) - \tau g(\frac{n_k}{\tau} \quad ; k = 1, 2, .., N$

The expected price under such model can be written as

$\mathbb{E}[p_k] = p_0 L^k - G \sum i = 1^k n_i L^{k-i}$

where
$L = (1 + \tau\alpha_0^m + E_J^m)$

and $\{n_i\}i = 1^N$ the corresponding strategy.

6 THE EXPECTED EXECUTION COST UNDER MULTIPLICATIVE JUMP MODEL FOR THE NAIVE AND ADDITIVE STRATEGY

6.1 True expected execution cost for the naive strategy

The true expected execution cost of the naive strategy equal :

$\mathbb{E}[p_0 S_0 - \sum_{k=1}^{N} \widetilde{p}_k \widetilde{n}_k] = S_0 p_0 + S_0^2 \frac{H}{N\tau} + \frac{S_0}{N} \sum_{k=1}^{N} L^{k-1} (\frac{N-k}{N} G S_0 - p_0)$

6.2 True expected cost for the additive strategy

The true expected execution cost for the additive execution strategy equals:

$$\begin{aligned}\mathbb{E}[p_0 S_0 - \sum_{k=1}^{N} p_k n_k] = p_0 S_0 &+ \frac{H S_0^2}{N\tau} + \frac{S_0}{N} \sum_{k=1}^{N} L^{k-1} (\frac{N-k}{N} G S_0 - p_0) \\ &+ \sum_{k=1}^{N-1} \frac{E_J^a + \tau\alpha_0}{\theta} (\frac{N-2k-1}{2} L^k p_0) \\ &+ \frac{E_J^a + \tau\alpha_0}{\theta^2} \Big[\frac{N(N^2-1)}{12\tau} \\ &+ \frac{1}{12} \sum_{k=1}^{N-1} (N-k)(N^2 - 1 - 2k(k+N)) G L^{k-1}) \Big]\end{aligned}$$

where $L = 1 + \tau\alpha_0^m + E_J^m$.

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