An efficient tree method for the discretization of the CEV model

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Abstract

We describe a robust and stable lattice method which permits to obtain very accurate American and European option prices under the CEV model.

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1 Model specification

This brief note presents a robust tree method for pricing American and European options under the CEV local volatility model. More precisely, we consider the CEV diffusion $(S_t)_{t\geq 0}$ described (under the risk neutral probability) by

$$dS(t) = (r-q)S(t)dt + \sigma S(t)^{\beta} dW_t, \quad S(0) = S_0 > 0, \tag{1}$$

where $\sigma > 0$, $\beta \in (0,1)$, $r \geq 0$ is the instantaneous risk free interest rate, $q \geq 0$ is the continous dividend. We underline that if $\beta \in (0, 1/2)$ we choose the unic solution S that once reached zero it remains zero for all subsequent times. In order to discretize the CEV dynamics, we construct a tree by using the technique introduced by Nelson and Ramaswamy in [4] and developed in [1] in the case of the CIR process.

2 The tree

Hereafter, we set $T > 0$ a given horizon (the option maturity time) and for a fixed number N of discretization intervals of $[0, T]$, $h = T/N$ will denote the common length.
The simple binomial tree is built through a transformation of $(S_t)_{t\geq 0}$ which gives a unitary diffusion coefficient. According to the Itô's formula, the transformation is

$$\psi_\beta(s) := \frac{s^{1-\beta}}{\sigma(1-\beta)}, \qquad s \geq 0\,.$$

ψ_β is twice differentiable in $(0,+\infty)$ and is invertible in $[0,+\infty)$, with inverse

$$g_\beta(y) = (\sigma(1-\beta)y)^{\frac{1}{1-\beta}}\,, \quad y \geq 0\,.$$

Setting $Y_t = \psi_\beta(S_t)$, the Itô's formula would give

$$dY_t = (\text{drift})dt + dW_t.$$

The term "dW_t" gives the foremost contribution to the local movement of Y_t and the standard random walk (binomial recombining tree) for the Brownian motion at times ih, $i = 0, 1, \ldots, N$, lives on the lattice

$$(2k-i)\sqrt{h}, \quad k = 0, 1, \ldots, i.$$

So, we construct the tree associated to $Y_t = \psi_\beta(S_t)$ as follows:

$$y_{i,k} := \psi_\beta(S_0) + (2k-i)\sqrt{h}\,, \quad i = 0, \ldots, N, k = 0, \ldots, i.$$

This is the binomial recombining lattice we start from. Actually, here the application of the Itô's formula is just heuristic – it cannot be really applied (S can reach 0 and ψ_β is not twice differentiable at 0) but we don't care because we will never use the diffusion Y: we just start from the lattice above. In fact, since at least heuristically one has $S_t = g_\beta(Y_t)$, the tree for $(S_t)_{t\in[0,T]}$ is built by the inverse positions $g_\beta(y_{i,k})$ on the nodes (i,k) such that $y_{i,k} \geq 0$. So, we define

$$\begin{aligned} &s_{i,k} := \left(S_0^{1-\beta} + \sigma(1-\beta)(2k-i)\sqrt{h}\right)^{\frac{1}{1-\beta}} \mathbf{1}_{\{S_0^{1-\beta}+\sigma(1-\beta)(2k-i)\sqrt{h}\geq 0\}}, \\ &i = 0, \ldots, N, \quad k = 0, \ldots, i. \end{aligned} \tag{2}$$

where $\mathbf{1}$ denotes the indicator function. We stress that i represents the discretized time whereas k gives the position at time step i. Note moreover that the above lattice is binomial recombining, since the $y_{i,k}$'s are. Observe finally that, for i large, the tree may collapse at 0.

The jump rule is now specified in order that the first local moment best fit the one computed on the CEV dynamics (1) (see the forthcoming formulas (6)).
The down and up jump positions are defined as follows: starting from node (i,k), whose position on the tree is $s_{i,k}$, we set the down-jump node $(i+1, k_d(i,k))$ and the up-jump node $(i+1, k_u(i,k))$ as

$$k_d(i,k) = \max\{k^* \,:\, 0 \leq k^* \leq k \text{ and } s_{i,k} + (r-q)s_{i,k}h \geq s_{i+1,k^*}\}, \tag{3}$$

$$k_u(i,k) = \min\{k^* \,:\, k+1 \leq k^* \leq i+1 \text{ and } s_{i,k} + (r-q)s_{i,k}h \leq s_{i+1,k^*}\}, \tag{4}$$

with the understanding $k_d(i,k) = 0$ and/or $k_u(i,k) = i+1$ if the associated set is empty. This setting is called the *multiple jump approach*. This is because, differently from the celebrated Cox-Ross-Rubinstein binomial discretization [2] of the Black and Scholes model, the jumps are not necessarily the closest down/up node at time $i+1$: here, in principle, one can have $k_d(i,k) < k$ and $k_u(i,k) > k+1$. This is done in

order to account for non-constant drift and diffusion coefficients, which might bring the process "far away".
The probability that the discrete process jumps to $(i+1, k_u(i,k))$ starting from (i,k) is set as

$$p_{i,k} = 0 \vee \frac{(r-q)s_{i,k}h + s_{i,k} - s_{i+1,k_d(i,k)}}{s_{i+1,k_u(i,k)} - s_{i+1,k_d(i,k)}} \wedge 1\,, \tag{5}$$

and then, the jump to $(i+1, k_d(i,k))$ happens with probability $1 - p_{i,k}$.

We set $(S_i^h)_{i=0,\dots,N}$ the Markov chain running on the above lattice structure, that is:

- $S_0^h = S_0$;
- at time ih, the state space for S_i^h is given by $\{s_{i,k} \,:\, k = 0, \dots, i\}$;
- given the current position $s_{i,k}$ at time ih, the transition law at time $(i+1)h$ is defined as

$$\Pi_h(s_{i,k}; dx) := p_{i,k}\delta_{\{s_{i+1,k_u(i,k)}\}}(dx) + (1-p_{i,k})\delta_{\{s_{i+1,k_d(i,k)}\}}(dx),$$

  $\delta_{\{a\}}$ denoting the Dirac mass in $a \in \mathbb{R}$.

The main fact of the above construction is that, for h small, it allows for an accurate approximation of the first and second local moment computed on the CEV diffusion (1): for every $i = 0, \dots, N-1$ and $k = 0, \dots, i$, one can prove that

$$\begin{aligned} \int (x - s_{i,k})\, \Pi_h(s_{i,k}; dx) &= (r-q)s_{i,k}\, h + o(h), \\ \int (x - s_{i,k})^2\, \Pi_h(s_{i,k}; dx) &= \sigma s_{i,k}^{\beta}\, h + o(h). \end{aligned} \tag{6}$$

In fact, the following result holds.

Theorem 1. *Let $(\bar{S}_t^h)_{t\in[0,T]}$ denote the continuous-time process defined through the linear interpolation (in time) of the chain $(S_i^h)_{i=0,\dots,N}$:*

$$\bar{S}_t^h = S_i^h + \frac{t-ih}{h}\big(S_{i+1}^h - S_i^h\big), \quad t \in [ih, (i+1)h],\ i = 0, \dots, N-1. \tag{7}$$

Then, if $\beta \in [1/2, 1)$, the process $(\bar{S}_t^h)_{t\in[0,T]}$ converges in law in the path space $C([0,T]; [0,+\infty))$ to the CEV diffusion process $(S_t)_{t\in[0,T]}$ in (1).

The proof can be developed following Nelson and Ramaswamy [4] or the classical approach by Stroock and Varadhan [5]. A detailed proof can be found in [3].

3 The dynamic programming algorithm

We are interested in pricing European and American options with maturity $T > 0$ written on the CEV model (1) for the underlying asset price. We denote by Ψ the function giving the payoff. For example, for vanilla options one has

$$\begin{aligned} &\text{call option:} \quad \Psi(x) = (x-K)_+, \\ &\text{put option:} \quad \Psi(x) = (K-x)_+, \end{aligned}$$

K denoting the strike price.

European Case

The price at time 0 of the European option with maturity T and payoff $\Psi(S_T)$ is given by

$$U_0 = \mathbb{E}\Big[e^{-rT}\Psi(S_T)].$$

The numerical computation of U_0 is done via the Markov chain $(S_i^h)_{i=0,\dots,N}$, through the approximated price

$$\tilde{U}_0 = \mathbb{E}\Big[e^{-rT}\Psi(S_N^h)\Big]\,.$$

By using the backward approach, $\tilde{U}_0 = \tilde{u}_0(S_0)$, where $\tilde{u}_0$ is the solution of the following simple dynamic programming algorithm:

$$\begin{cases} \tilde{u}_N(s) = \Psi(s), \quad s \in \{s_{N,k} \,:\, k = 0, 1, \dots, N\}\,; \\ \tilde{u}_i(s) = e^{-rh}\mathbb{E}(\tilde{u}_{i+1}(S_{i+1}^h) \mid S_i^h = s), \quad s \in \{s_{i,k} \,:\, k = 0, 1, \dots, i\}. \end{cases}$$

By inserting the specifications (3), (4) and (5) of the tree, one can write

$$\begin{cases} \tilde{u}_N(s_{N,k}) = \Psi(s_{N,k}), \qquad k \in \{0, 1, \dots, N\}\,; \\ \tilde{u}_i(s_{i,k}) = e^{-rh}\Big[\tilde{u}_{i+1}(s_{i+1,k_u(i,k)})p_{i,k} + \tilde{u}_{i+1}(s_{i+1,k_d(i,k)})(1 - p_{i,k})\Big], \quad k \in \{0, 1, \dots, i\}. \end{cases}$$

American Case

The price at time 0 of the American option with maturity T and payoff $(\Psi(S_t))_{t\in[0,T]}$ is given by

$$U_0 = \sup_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}\Big[e^{-r\tau}\Psi(S_\tau)\Big]\,,$$

where $\mathcal{T}_{t,T}$ is the set of all the stopping times taking values in $[t, T]$. The price U_0 is then approximated by means of the price computed on the Markov chain, that is,

$$\tilde{U}_0 = \sup_{\tau\in\tilde{\mathcal{T}}_{0,N}}\mathbb{E}\Big[e^{-r\tau}\Psi(S_\tau^h)\Big]\,,$$

where $\tilde{\mathcal{T}}_{j,N}$ is the set of the stopping times taking values in $\{j, \dots, N\}$. By using the standard theory of optimal stopping, one gets $\tilde{U}_0 = \tilde{u}_0(S_0)$, where $\tilde{u}$ is the solution of the following dynamic programming algorithm:

$$\begin{cases} \tilde{u}_N(s) = \Psi(s), \quad s \in \{s_{N,k} \,:\, k = 0, 1, \dots, N\}\,; \\ \tilde{u}_i(s) = \max\Big(\Psi(s)\,,\, e^{-rh}\mathbb{E}(\tilde{u}_{i+1}(S_{i+1}^h) \mid S_i^h = s)\Big), \quad s \in \{s_{i,k} \,:\, k = 0, 1, \dots, i\}. \end{cases}$$

By inserting (3), (4) and (5), one has

$$\begin{cases} \tilde{u}_N(s_{N,k}) = \Psi(s_{N,k}), \qquad k \in \{0, 1, \dots, N\}\,; \\ \tilde{u}_i(s_{i,k}) = \max\Big(\Psi(s_{i,k})\,,\, e^{-rh}\Big[\tilde{u}_{i+1}(s_{i+1,k_u(i,k)})p_{i,k} + \tilde{u}_{i+1}(s_{i+1,k_d(i,k)})(1 - p_{i,k})\Big]\Big), \\ \qquad\qquad k \in \{0, 1, \dots, i\}. \end{cases}$$

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