A Finite Volume Method for Pricing American options on two stocks.

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Introduction

The valuation of American options on two stocks, also called two-colours Rainbow options by practitioners, is an important problem in financial economics since a wide variety of contracts that are traded in the O.T.C. market involve such options (Exchange options, Best-of options). Unlike European options, American options cannot be valued by closed-form formulae, even in the Black-Scholes model, and require the use of numerical methods.

1 American Options on Two Stocks

The price at time 0 of an American option on two stocks in the Black-Scholes setting is given by

$$P_A(0, s_1, s_2) = \sup_{\tau \in \mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S^1_\tau, S^2_\tau)\right].$$

This price can be formulated, after a logarithm change of variable, in terms of the solution u to the following variational inequality (see e.g. [10]),

$$\left\{\begin{array}{l} \max\left(\psi-u, \frac{\partial u}{\partial t}+\frac{\sigma_1^2}{2}\frac{\partial^2 u}{\partial x_1^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 u}{\partial x_2^2}+\alpha_1\frac{\partial u}{\partial x_1}+\alpha_2\frac{\partial u}{\partial x_2}+\rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1 \partial x_2}-ru\right)=0,\ (t,x_1,x_2) \text{ in } [0,T[\times R^2 \\ u(T,x_1,x_2)=\psi(e^{x_1},e^{x_2}) \end{array}\right. \tag{1}$$

by $P_A(t, s_1, s_2) = u(t, \ln s_1, \ln s_2)$.

With the time change of variable $t' = T - t$ and the following geometrical transformation :

$$(x, y) \longmapsto (X, Y) = \left(x * \cos(\theta) + y * \sin(\theta), \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} (y * \cos(\theta) - x * \sin(\theta))\right)$$

with, if $\sigma_1^2-\sigma_2^2 \neq 0$

$$\begin{cases} \tan(2\theta)=\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2}, \\ \alpha=\frac{(\sigma_1^2+\sigma_2^2)\cos(2\theta)+\sigma_1^2-\sigma_2^2}{4\cos(2\theta)}, \\ \beta=\frac{(\sigma_1^2+\sigma_2^2)\cos(2\theta)+\sigma_2^2-\sigma_1^2}{4\cos(2\theta)}. \end{cases}$$

if, $\sigma_1^2-\sigma_2^2=0$,

$$\begin{cases} \theta=\frac{\pi\rho}{4|\rho|}, \\ \alpha=\frac{\sigma_1^2}{2}(1+|\rho|), \\ \beta=\frac{\sigma_1^2}{2}(1-|\rho|). \end{cases}$$

one obtains,

$$\begin{cases} \min\left(\psi-u, \frac{\partial u}{\partial t}-\alpha\Delta u-\operatorname{grad}(\vec{v}u)+ru\right)=0,\ (t,x_1,x_2) \text{ in } [0,T[\times R^2 \\ u(0,x_1,x_2)=\psi(e^{x_1},e^{x_2}) \end{cases} \tag{2}$$

with,

$$\vec{v}=((r-\lambda_1-\frac{\sigma_1^2}{2})\cos(\theta)+(r-\lambda_2-\frac{\sigma_2^2}{2})\sin(\theta), \left((r-\lambda_2-\frac{\sigma_2^2}{2})\cos(\theta)-(r-\lambda_1-\frac{\sigma_1^2}{2})\sin(\theta)\right)(\frac{\alpha}{\beta})^{\frac{1}{2}}).$$

2 The finite volume schemes

Let us consider the problem :

$$\frac{\partial u}{\partial t}(x,t)+\operatorname{div}(u(x,t)\vec{v})-\alpha\Delta u(x,t)+ru(x,t)\geq 0,\ (x,t)\in\Omega\times]0,T[\tag{3}$$

$$u(x,t)\geq\psi(x),\ (x,t)\in\Omega\times]0,T[\tag{4}$$

$$\left(\frac{\partial u}{\partial t}(x,t)+\operatorname{div}(u(x,t)\vec{v})-\alpha\Delta u(x,t)+ru(x,t)\right)(\psi(x)-u(x,t))=0,\ (x,t)\in\Omega\times]0,T[\tag{5}$$

$$u(x,0)=\psi(x),\ x\in\Omega \tag{6}$$

under the following assumptions

Assumption 1.

1. *$d\in\mathbb{N}^*$,*
2. *$\Omega\subset\mathbb{R}^d$ is a bounded open polygonal,*
3. *$\psi\in H_0^1(\Omega)\cap C^2(\bar{\Omega})$,*
4. *$\psi\geq 0$ a.e on Ω,*
5. *$T>0$.*

A weak for of the problem (3)-(6) yields the following variational inequality :

$$\begin{cases} u\in L^2(0,T;H_0^1(\Omega)), \frac{\partial u}{\partial t}\in L^2(\Omega\times]0,T[), u(x,0)=\psi(\Omega),\ \text{p.p. } x\in\Omega, satisfying\ : \\ \displaystyle\int_\Omega\left(\frac{\partial u}{\partial t}(x,t)+ru(x,t)+\operatorname{div}(u(x,t)\vec{v})\right)(v(x,t)-u(x,t))+\alpha\nabla u(x,t)\nabla(v(x,t)-u(x,t))dx\geq 0 \\ \text{p.p } t\in]0,T[,\forall v\in H^1(\Omega), v\geq\psi. \end{cases} \tag{7}$$

By [12], there exits a unique solution of (7).
In order to obtain a numerical approximation of the solution of (7), let us now describe the space and time discretization of $\Omega\times]0,T[$.

Definition 1 (Admissible meshes)**.** *An admissible mesh of* Ω *is given by a set* τ *of open bounded polygonal convex subsets of* Ω *called control volumes, a family* ε *of subsets of* $\bar{\Omega}$ *contained in hyperplanes of* $\mathbb{R}^d$ *with strictly positive measure, and a family of point* $(x_K)_{K\in\tau}$ *(the "centers" of control volumes) satisfying the following properties:*

(i) *The closure of the union of all control volumes is* $\bar{\Omega}$.

(ii) *For any* $K\in\tau$*, there exits a subset* ε_K *of* ε *such that* $\partial K=\cup_{\sigma\in\varepsilon_K}\bar{\sigma}$*. Furthermore,* $\varepsilon=\cup_{K\in\tau}\varepsilon_K$.

(iii) *For any* $(K,L)\in\tau^2$ *with* $K\neq L$*, either the "lenght" (i.e. the (d-1) Lebesgue measure) of* $\bar{K}\cap\bar{L}$ *is* 0 *or* $\bar{K}\cap\bar{L}=\bar{\sigma}$ *for some* $\sigma\in\varepsilon$*. In the latter case, we shall write* $\sigma=K|L$ *and* $\varepsilon_{int}=\sigma\in\varepsilon,\exists(K|L)\in\tau^2,\sigma=K|L$*. For any* $K\in\tau$*, we shall denote by* $\mathcal{N}_K$ *the set of boundary control volumes of* K*, i.e.* $\mathcal{N}_K=\{L\in\tau,K|L\in\varepsilon_K\}$.

(iv) *The family of points* $(x_K)_{K\in\tau}$ *is such that* $x_K\in K$ *(for all* $K\in\tau$*) and, if* $\sigma=K|L$*, it is assumed that the straight line* (x_K,x_L) *is orthogonal to* σ.

For a control volume $K\in\tau$*, we will denote by* $m(K)$ *its measure and* $\varepsilon_{ext,K}$ *the subset of the edges of* K *included in the boundary* $\partial\Omega$*. If* $L\in\mathcal{N}_K$*,* $m(K|L)$ *will denote the measure of the edge between* K *and* L*,* $\tau_{K|L}$ *the "transmissibility" through* $K|L$*, defined by* $\tau_{K|L}=\frac{m(K|L)}{d(x_K,x_L)}$*. Similarly, if* $\sigma\in\varepsilon_{ext,K}$*, we will denote by* $m(\sigma)$ *its measure and* τ_σ *the "transmissibility" through* σ*, defined by* $\tau_\sigma=\frac{m(\sigma)}{d(x_K,\sigma)}$*. One denotes* $\varepsilon_{ext}=\cup_{K\in\tau}\varepsilon_{ext,K}$ *an for* $\sigma\in\varepsilon_{ext}$*, one denotes by* K_σ *the control volume* K *such that* $\sigma\in\varepsilon_{ext,K}$*. The size of the mesh* τ *is defined by*

$$size(\tau)=\max_{K\in\tau} diam(K), \tag{8}$$

and a geometrical factor, linked with the regularity of the mesh, is defined by

$$reg(\tau)=\max_{K\in\tau}(card\varepsilon_K,\max_{\sigma\in\varepsilon_K}\frac{diam(K)}{d(x_K,\sigma)}). \tag{9}$$

Definition 2 (Time discretization of $(0,T)$)**.** *A time discretization of* $(0,T)$ *is given by an integer value* N *and by an increasing sequence of real values* $(t^n)_{n\in[0,N+1]}$ *with* $t^0=0$ *and* $t^{N+1}=T$*. The time step is uniform and defined by* $\delta t=t^{n+1}-t^n$*, for* $n\in[0,N]$.

Definition 3 (Space-time discretization of $\Omega\times(0,T)$)**.** *A finite volume discretization* $\mathcal{D}$ *of* $\Omega\times(0,T)$ *is a family* $\mathcal{D}=(\tau,\varepsilon,(x_K)_{K\in\tau},N,(t^n)_{n\in[0,N]})$*, where* $\tau,\varepsilon,(x_K)_{K\in\tau}$ *is an admissible mesh of* Ω *in the sense of Definition 1 and* $N,(t^n)_{n\in[0,N+1]}$ *is a time discretization of* $(0,T)$ *in the sense od Definition 2. For a given mesh* $\mathcal{D}$*, one defines :*

$$size(\mathcal{D})=\max(size(\tau),\delta t),$$
$$reg(\mathcal{D})=reg(\tau).$$

Let us now introduce the space of piecewise constant functions associated with an admissible mesh and some "discrete H_0^1" norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.

Definition 4. *Let* Ω *be an open boundedpolygonal subset of* $\mathbb{R}^d$*, and* τ *an admissible mesh. Define* $X(\tau)$ *as the set of functions from* Ω *to* $\mathbb{R}$ *which are constant over each control volume of the mesh.*

Definition 5 (Discrete norms)**.** *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, and τ an admissible finite volume mesh in the sense of Definition 1. For $u, v \in X(\tau)$ we define a scalar product by*

$$[u,v]_{1,\tau} = \sum_{\sigma\in\varepsilon} T_{sigma} D_\sigma u D_\sigma v = \sum_{\substack{\sigma\in\varepsilon_{int}\\ \sigma=K|L}} T_{KL}(u_L-u_K)(v_L-v_K) + \sum_{\sigma\in\varepsilon_{ext}} T_{K\sigma}u_{K\sigma}v_{K\sigma} \tag{10}$$

where, for any $\sigma\in\varepsilon, T_\sigma = \frac{m(\sigma)}{d_\sigma}$ and

$$D_\sigma u = |u_K u_L| \text{ if } \sigma\in\varepsilon_{int}, \sigma = K|L,$$

$$D_\sigma u = |u_K| \text{ if } \sigma\in\varepsilon_{K,ext},$$

where u_K denotes the value taken by u on the control volume K and the sets $\varepsilon, \varepsilon_{int}, \varepsilon_{ext}, \varepsilon_{K,ext}$ are definied in definition 1. We note $\|\|_{1,\tau}$ the discrete H_0^1 norm associated.

The schemes :

Let $\mathcal{D}$ be a finite volume discretization of $\Omega\times(0,T)$. Let us now define an implicit upwind finite volume scheme, the discrete unknowns are $u = (u_K^{n+1})_{K\in\,\tau, n\,\in\,[0,T]}$ and $\tilde{u} = (\tilde{u}_K^{n+1})_{K\in\,\tau, n\,\in\,[0,T]}$ and verify :

$$u_K^0 = \psi(x_K) = \psi_K, \tag{11}$$

$$u_K^{n+1} = \max\left(\tilde{u}_K^{n+1}, u_K^0\right), \tag{12}$$

$$m_K\left(\tilde{u}_K^{n+1} - u_K^n\right) + \Delta t \sum_{\sigma\in\varepsilon_K} v_{K,\sigma}u_{\sigma,+}^{n+1} + \alpha\Delta t[u^{n+1}, 1_K]_{1,\tau} + r\Delta t m_K u_K^{n+1} = 0, \tag{13}$$

with,

$$\vec{v}\in\mathbb{R}^d,$$

$$v_{K,\sigma} = -\int_\sigma \vec{v}.\vec{n}_{K\sigma} d\gamma(x) = -\vec{v}.\vec{n}_{K\sigma} m(\sigma).$$

$$u_{\sigma,+}^n = \begin{cases} u_K^n & \text{si } \sigma\in\varepsilon_{int}, \sigma = K|L, v_{K,\sigma}\geq 0\\ u_L^n & \text{si } \sigma\in\varepsilon_{int}, \sigma = K|L, v_{K,\sigma} < 0\\ u_K^n & \text{si } \sigma\in\varepsilon_{K,ext}, v_{K,\sigma}\geq 0\\ 0 & \text{si } \sigma\in\varepsilon_{K,ext}, v_{K,\sigma} < 0\end{cases}$$

Remark 1. *we can also consider a implicit central finite volume scheme :*

$$u_K^0 = \psi(x_K) = \psi_K, \tag{14}$$

$$u_K^{n+1} = \max\left(\tilde{u}_K^{n+1}, u_K^0\right), \tag{15}$$

$$m_K\left(\tilde{u}_K^{n+1} - u_K^n\right) + \Delta t\left[\sum_{\substack{\sigma\in\varepsilon_{K,int}\\ \sigma=K|L}} v_{KL}\frac{u_K^{n+1}+u_L^{n+1}}{2} + \sum_{\sigma\in\varepsilon_{K,ext}} v_{K\sigma}\frac{u_{K\sigma}^{n+1}}{2}\right] + \alpha\Delta t[u^{n+1}, 1_K]_{1,\tau} + r\Delta t m_K u_K^{n+1} = 0. \tag{16}$$

Definition 6 (Approximate solution)**.** *Let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of definition 3. The approximate solution (C^∞ in time on $\Omega\times(0,T)$)of (3) − (6) associated to the discretization $\mathcal{D}$ is defined almost everywhere in $\Omega\times(0,T)$ by :*

$$u_{\mathcal{D}}(x,t) = \frac{t-n\Delta t}{\Delta t}u_K^{n+1} + \frac{(n+1)\Delta t - t}{\Delta t}u_K^n, \forall (x,t)\in K\times[n\Delta t, (n+1)\Delta t], \forall n = 0\ldots N, \forall K\in\tau.$$

Thanks to this Definition, one gets almost everywhere in $\Omega \times (0,T)$:

$$\frac{\partial u_{\mathcal{D}}(x,t)}{\partial t} = \frac{u_K^{n+1} - u_K^n}{\Delta t}, \forall t \in [n\Delta t, (n+1)\Delta t], \forall x \in K, \forall n = 0 \ldots N, \forall K \in \tau.$$

3 Existence of the solution and stability results for the implicit schemes

Lemma 1. *Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 3. If $(u_K^n)_{\substack{K\in\tau\\ n\in\mathbb{N}}}$ is a solution of the implicit upwind finite volume scheme (11) - (13), then there exists a sequence $(\theta_K^n)_{\substack{n=0\ldots N\\ K\in\tau}} \in [0,1]$ such that :*

$$u_K^{n+1} - u_K^n = \theta_K^n \left(\tilde{u}_K^{n+1} - u_K^n\right), \forall K \in \tau, \forall n = 0 \ldots N.$$

Lemma 2 (Existence and uniqueness)**.** *Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 3. Then there exists a unique solution $(u_K^n)_{\substack{K\in\tau\\ n\in\mathbb{N}}}$ to the system of equations (11) - (13).*

Proposition 1 (L^∞ and L^2 estimate)**.** *Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 3 and let $(u_K^n)_{\substack{K\in\tau\\ n\in[0,N+1]}}$ be the unique solution of the scheme (11) - (13). Then,*

$$|u_K^n| \leq \|\psi\|_{L^\infty(\Omega)}, \forall K \in \tau, \forall n \in [0, N+1],$$

and

$$\frac{1}{2}\sum_{K\in\tau} m_K (u_K^{l+1})^2 + \sum_{K\in\tau} m_K \sum_{n=0}^{l} (u_K^{n+1} - u_K^n)^2 + \alpha \sum_{n=0}^{l} \Delta t [u^{n+1}, u^{n+1}]_{1,\tau} \leq$$
$$\|\psi\|^2_{L^\infty(\Omega)} m(\Omega) + \alpha T [u^0, u^0]_{1,\tau}, \forall l \leq N$$

4 Estimate

Corollary 1. *Under Assumptions 1, let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be sequence of discretization of $\Omega \times (0,T)$ in the sense of Definition 3 such that $\Delta t_m \underbrace{\rightarrow}_{m\to+\infty} 0, size(\tau_m) \underbrace{\rightarrow}_{m\to+\infty} 0$, $\zeta \in \mathbb{R}$ such that $\zeta \geq reg(D_m)$ $\forall m \in \mathbb{N}$,and let $(u_K^n)_{\substack{K\in\tau\\ n\in[0,N+1]}}$ be the unique solution of the scheme (11) - (13). Then, there exists $U \in L^2\left(\Omega\times]0,T[\right)$, and a subsequence noted $(u_{D_m})_{m\in\mathbb{N}}$ such that $u_{D_m} \underbrace{\rightharpoonup}_{m\to+\infty} U$ for the weak topology of $L^2\left(\Omega\times]0,T[\right)$.*

Proposition 2. *Under Assumptions 1, let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be sequence of discretization of $\Omega \times (0,T)$ in the sense of Definition 3, $\zeta \in \mathbb{R}$ such that $\zeta \geq reg(D)$,and let $(u_K^n)_{\substack{K\in\tau\\ n\in[0,N+1]}}$ be the unique solution of the scheme (11) - (13). Then, there exists $C > 0$ only depending on ψ, α, Ω, T, $\vec{v}$, r, such that :*

$$\alpha[u^{N+1}, u^{N+1}]_{1,\tau} + \sum_{n=0}^{N} \Delta t \sum_{K\in\tau} m_K \left(\frac{u_K^{n+1} - u_K^n}{\Delta t}\right)^2 \leq C(u^0, \psi, \alpha, \Omega, T, \vec{v}, r).$$

Corollary 2. *Under Assumptions 1, let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be sequence of discretization of $\Omega\times(0,T)$ in the sense of Definition 3 such that size(τ_m) $\underbrace{\rightarrow}_{m\to+\infty}$ 0, $\zeta\in\mathbb{R}$ such that $\zeta\geq$ reg(D),and let $(u_K^n)_{\substack{K\in\mathcal{T}\\ n\in[0,N+1]}}$ be the unique solution of the scheme (11) - (13). Then, the set $\{\frac{\partial u_D}{\partial t}\}_{N,\mathcal{T}}$ is borned in $L^2(\Omega\times]0,T[)$ and so, there exists $Z\in L^2(\Omega\times]0,T[)$ such that, op to a subsequence, $\{\frac{\partial u_{D_m}}{\partial t}\}_{m\in\mathbb{N}}$ tends to Z in the weak topology of $L^2(\Omega\times]0,T[)$ as $m\to+\infty$.*

Corollary 3 (Space-translate and Time-Translate estimate)**.** *Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega\times(0,T)$ in the sense of Definition 3, $\zeta\in\mathbb{R}$ such that $\zeta\geq$ reg(D) and let u_D the approximate solution in the sense of Definition 6 be prolonged by zero on $\mathbb{R}^{d+1}\setminus\Omega\times]0,T[$. Then there exists C_2 only depending on T,ψ,α,τ,d and C_3 only depending on $T,\psi,\alpha,d,r,\vec{v}$ such that :*

$$\|u_D(.+\eta,.)-u_D(.,.)\|^2_{L^2(\mathbb{R}^{d+1})}\leq C_2|\eta|\left(|\eta|+4\text{size}(\tau)\right),\ \forall\eta\in\mathbb{R}^d.$$

and

$$\|u_D(.,.+\lambda)-u_D(.,.)\|^2_{L^2(\mathbb{R}^{d+1})}\leq\lambda C_3,\ \forall\lambda\in]0,T[.$$

With this preceding estimates, one can apply the Riesz-Frechet-Kolmogorov compactness criterion.

5 Compactness

Corollary 4. *Under Assumptions 1, let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be sequence of discretization of $\Omega\times(0,T)$ in the sense of Definition 3 such that Δt_m $\underbrace{\rightarrow}_{m\to+\infty}$ 0, size(τ_m) $\underbrace{\rightarrow}_{m\to+\infty}$ 0, $\zeta\in\mathbb{R}$ such that $\zeta\geq$ reg(D_m) $\forall m\in\mathbb{N}$,and let $(u_K^n)_{\substack{K\in\mathcal{T}\\ n\in[0,N+1]}}$ be the unique solution of the scheme (11) - (13). Then, up to a subsequence,*

$$u_{D_m}\underbrace{\rightarrow}_{m\to+\infty}U\ \textit{dans}\ L^2(\Omega\times]0,T[).$$

where U is defined by the Corollary 1 and checks $U\in L^2\left(0,T;H^1(\Omega)\right)$, $U(t,.)=0$ a.e. on $\partial\Omega$, a.e.t $\in]0,T[$, and $\frac{\partial U}{\partial t}=Z$ a.e. on $]0,T[\times\Omega$.

6 Convergence

Proposition 3. *Under Assumptions 1, let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be sequence of discretization of $\Omega\times(0,T)$ in the sense of Definition 3 such that Δt_m $\underbrace{\rightarrow}_{m\to+\infty}$ 0, size(τ_m) $\underbrace{\rightarrow}_{m\to+\infty}$ 0, $\zeta\in\mathbb{R}$ such that $\zeta\geq$ reg(D_m) $\forall m\in\mathbb{N}$,and let $(u_K^n)_{\substack{K\in\mathcal{T}\\ n\in[0,N+1]}}$ be the unique solution of the scheme (11) - (13). Let $(u_{D_m})_{m\in\mathbb{N}}$ the sequence of approximate solution in the sense of the Definition* **??**, *and let U the limit of a subsequence $(u_{D_m})_{m\in\mathbb{N}}$ thanks to Corollary 4. Then, for all function $w\in L^2\left(0,T;H_0^1(\Omega)\right)$, such that $w(t,.)\geq\psi$ a.e. on Ω, the following inequality holds :*

$$\int_\Omega\frac{\partial U}{\partial t}(x,t)(w(x,t)-U(x,t))+\alpha\nabla U(x,t)\nabla(w(x,t)-U(x,t))+$$

$$rU(x,t)(w(x,t)-U(x,t))+(w(x,t)-U(x,t))\,div(U(x,t)\vec{v})dx\geq 0\ p.p\ t\in]0,T[.$$

7 Numerical Results

7.1 Localization and Stability

We define three localizations.

Lemma 3 (Localization). *We consider,*

$$P_{Aloc}(0,s_1,s_2) = \sup_{\tau\in\mathcal{T}_{0,T}} E\left[e^{-r\tau\wedge T^{0,s}_{loc}}\psi(S^1_{\tau\wedge T^{0,s}_{loc}}, S^2_{\tau\wedge T^{0,s}_{loc}})\right],$$

with $T^{0,s}_{loc} = inf\{t>0, |S^1_t - s^1| < loc, |S^2_t - s^2| < loc\}$.
Then, if $\psi(s^1,s^2) = (K - \min(s^1,s^2))^+$,

$$|P_A(0,s_1,s_2) - P_{Aloc}(0,s_1,s_2)| \le 4K\left[2 - N(\frac{loc-a1}{\sqrt{T}\sigma_1}) - N(\frac{loc-a2}{\sqrt{T}\sigma_2})\right]$$

if $\psi(s^1,s^2) = (\max(s^1,s^2) - K)^+$,

$$|P_A(0,s_1,s_2) - P_{Aloc}(0,s_1,s_2)| \le$$

$$8\left[\exp\left(s^1 + T(|r-\lambda_1| + \frac{\sigma_1^2}{2})\right) + \exp\left(s^2 + T(|r-\lambda_2| + \frac{\sigma_2^2}{2})\right)\right]\sqrt{(2 - N(\frac{loc-a1}{\sqrt{T}\sigma_1}) - N(\frac{loc-a2}{\sqrt{T}\sigma_2})}$$

if $\psi(s^1,s^2) = (s^1 - \mu s^2)^+$,

$$|P_A(0,s_1,s_2) - P_{Aloc}(0,s_1,s_2)| \le 8\exp\left(s^1 + T(|r-\lambda_1| + \frac{\sigma_1^2}{2})\right)\sqrt{(2 - N(\frac{loc-a1}{\sqrt{T}\sigma_1}) - N(\frac{loc-a2}{\sqrt{T}\sigma_2})}$$

where, $ai = T|r_i - \lambda_i - \frac{\sigma_i^2}{2}|$ *and* N *is the repartition function of a standard normal distribution.*

One uses the following polynomial approximation for the repartition function of a standard normal distribution :

$$N(x) \approx 1 - \frac{1}{\sqrt{2\Pi}}\exp\left(-\frac{x^2}{2}\right)(b_1t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5), \text{ si } x > 0,$$

with

$$\begin{aligned} p &= 0.2316419\\ b_1 &= 0.319381530\\ b_2 &= -0.356563782\\ b_3 &= 1.781477937\\ b_4 &= -1.821255978\\ b_5 &= 1.330274429\\ t &= \tfrac{1}{1+px}. \end{aligned}$$

Explicit central finit volume scheme :

$$u^0_K = \psi_K, \tag{17}$$

$$u^{n+1}_K = \max\left(\tilde{u}^{n+1}_K, u^0_K\right), \tag{18}$$

$$m_K\left(\tilde{u}^{n+1}_K - u^n_K\right) + \Delta t\sum_{L\in N_K}\left(\frac{u^n_K + u^n_L}{2}\right)\underbrace{\int_{K|L} -\vec{v}.\vec{n}_{KL}d\gamma(x)}_{v_{KL}} + \Delta t\underbrace{\sum_{\sigma\in\varepsilon_{K,ext}}\frac{u^n_K}{2}\int_\sigma -\vec{v}.\vec{n}_{K\sigma}d\gamma(x) +}_{k_{K\sigma}}$$

$$\alpha\Delta t[u^n, 1_K]_{1,\tau} + r\Delta t m_K u^n_K = 0. \tag{19}$$

Explicit upwind finite volume scheme

$$u_K^0 = \psi_K, \tag{20}$$

$$u_K^{n+1} = \max\left(\tilde{u}_K^{n+1}, u_K^0\right), \tag{21}$$

$$m_K\left(\tilde{u}_K^{n+1} - u_K^n\right) + \Delta t \sum_{\sigma\in\varepsilon_K} v_{K\sigma} u_{\sigma,+}^n + \alpha\Delta t[u^n, 1_K]_{1,\mathcal{T}} + r\Delta t m_K u_K^n = 0. \tag{22}$$

with,

$$\vec{v} = ((r-\lambda_1-\frac{\sigma_1^2}{2})\cos(\theta)+(r-\lambda_2-\frac{\sigma_2^2}{2})\sin(\theta), \left((r-\lambda_2-\frac{\sigma_2^2}{2})\cos(\theta)-(r-\lambda_1-\frac{\sigma_1^2}{2})\sin(\theta)\right)(\frac{\alpha}{\beta})^{\frac{1}{2}}).$$

Lemma 4 (L^∞ stability)**.** *Let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of definition 3.*
Let $(u_K^n)_{K,n}$ be the unique solution of the explicit central scheme (17) - (19).
If, $\Delta t \leq \frac{m_K}{m_K\frac{r}{2}+\frac{\alpha}{2}\sum_{\sigma\in\varepsilon_{K,ext}} T_{K\sigma}+\sum_{L\in N_K}\left(\alpha T_{KL}+\frac{|v_{KL}|}{4}\right)}, \forall K\in\tau$ and if $T_{K\sigma}\geq\frac{1}{2\alpha}\left|\int_\sigma -\vec{v}.\vec{n}_{K\sigma}d\gamma(x)\right|, \forall K\in\tau, \forall\sigma\in\varepsilon_K$, then :

$$\|u_{\mathcal{T}}^{n+1}\|_{L^\infty} \leq \|u_{\mathcal{T}}^0\|_{L^\infty}, \forall n = 0\ldots N.$$

Let $(u_K^n)_{K,n}$ be the unique solution of the explicit upwind scheme (20) - (22).
If $\Delta t \leq \frac{m_K}{rm_K+(\sum_{\sigma\in\varepsilon_K}(v_{K\sigma})^+ +\alpha T_{K\sigma})}, \forall K\in\tau$, then :

$$\|u_{\mathcal{T}}^{n+1}\|_{L^\infty} \leq \|u_{\mathcal{T}}^0\|_{L^\infty}, \forall n = 0\ldots N.$$

7.2 Pratical implementation and results

We choose to evaluate the American Put option on the minimum of two underlying assets with payoff $\psi(S^1,S^2) = \left(K-\min(S^1,S^2)\right)^+$. We assume that the initial values of the stock prices are $s^1 = 100, s^2 = 100$, the volatility $\sigma_1 = 0.2, \sigma_2 = 0.2$, the interest rate $r = \log(1.05)$, the continuous dividend rates $\delta_1 = 0, \delta_2 = 0$, the exercice price $K = 100$, the maturity $T = 1$ and the correlation $\rho = 0$. We take as the "true" reference price, the one issued of the multinomial BEG tree-method with 3000 step and compare it wich the following algrithm :

1. the explicit DPEXP algorithm
2. the DPADI algorithm
3. the BEG algorithm
4. the explicit finite volume algorithm
5. the explicit finite volume algorithm with a smaller time step

For the last algorithm, we multiply by 0.6 the time step obtained by the stability condition.
All computation was perfomed on a PC Pentium IV 2.4 GH computer with 512 Mb of RAM.
The "centers" of control volumes are defined as follow :

$$x_K = x[i*N+j] = (v1-loc+i*h, v2-loc+j*g), \forall(i,j)\in[0,N]^2$$

$N \times M$	DP-EXP	DP-ADI	BEG	FV-EXP	FV-EXPdt*0.6	TRUE
100×100	10.3065;1s	10.2947;1s	10.2974;1s	10.3196;1s	10.3081;1s	10.3080
200×200	10.3054;6s	10.3031;2s	10.3030;1s	10.3108;3s	10.3095;4s	10.3080
300×300	10.3073;32s	10.3050;7s	10.3048;1s	10.3098;11s	10.3084;19s	10.3080
400×400	10.3082;100s	10.3058;18s	10.3057;2s	10.3085;35s	10.3082;59s	10.3080

Table 1: American Put option on the minimum of two underlying assets

where the space steps are defined by $h = \frac{2*loc}{N}$ and $g = \frac{2*loc*\sqrt{\frac{\alpha}{\beta}}}{N}$. The control volume K is the rectangle centered in x_K with the measure $m(K) = hg$.
It appears that the numerical FV-EXP method are finally faster than DP-EXP and slower than DP-ADI or BEG.

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