Heston stochastic volatility model simulation

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1 Theory

According to Mr. Michael B. Giles and Mr. Lukasz Szpruch's paper [1], let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t>0}$ satisfying the usual conditions, let $\omega(t)$ be a D-dimensional Brownian motion defined on the probability space. We would like to estimate $E[P(x(T))]$ with

$$dx(t) = f(x(t))dt + g(x(t))d\omega(t)$$

$x(t)\ in\mathcal{R}^d, f \in C^2(\mathcal{R}^d, \mathcal{R}^d), g \in C^2(\mathcal{R}^d, \mathcal{R}^{d\times D})$ defining the tensor $h_{ijk}(x)$ as

$$h_{ijk}(x) = \frac{1}{2}\sum_{l=1}^{d} g_{lk}(x)\frac{\partial g_{ij}}{\partial x_l}(x) \qquad i = 1\cdots d \ \ and \ \ k, j = 1\cdots D$$

Then

$$\hat{X}_{i,n+1} = \hat{X}_{i,n} + f_i(\hat{X}_n)\Delta t + \sum_{j=1}^{D} g_{ij}(\hat{X}_n)\Delta\omega_{j,n} + \sum_{j,k=1}^{D} h_{ijk}(\hat{X}_n)(\Delta\omega_{j,n}\Delta\omega_{k,n} - \Omega_{jk}\Delta t - A_{jk,n})$$

and $A_{jk,n}$ is the lévy area and in antithetic case we neglect this term.

2 Heston stochastic volatility model

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t}S_t(\rho dW_2 + \sqrt{1-\rho^2}dW_1) \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_2 \end{aligned}$$

We can transform these differential equation by using differential on a vector as

$$d\begin{bmatrix} log(S_t) \\ V_t \end{bmatrix} = \begin{bmatrix} r - \frac{V_t}{2} \\ \kappa(\theta - V_t) \end{bmatrix} dt + \sqrt{V_t}\begin{bmatrix} \sqrt(1-\rho^2) & \rho \\ 0 & \sigma \end{bmatrix}\begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}$$

so in this case $i, j, k \in 1, 2$ and

$$f = \begin{bmatrix} r - \frac{V_t}{2} \\ \kappa(\theta - V_t) \end{bmatrix}$$

$$g_{11}(x) = \sqrt{x_2}\sqrt{1-\rho^2} \qquad g_{12}(x) = \sqrt{x_2}\rho \qquad g_{21}(x) = 0 \qquad g_{22}(x) = \sigma\sqrt{x_2}$$

Then we could get the first order derivative by x as:

$$\begin{array}{llll} \frac{\partial g_{11}(x)}{\partial x_1} = 0 & \frac{\partial g_{11}(x)}{\partial x_2} = \frac{\sqrt{1-\rho^2}}{2\sqrt{x_2}} & \frac{\partial g_{12}(x)}{\partial x_1} = 0 & \frac{\partial g_{12}(x)}{\partial x_2} = \frac{\rho}{2\sqrt{x_2}} \\ \frac{\partial g_{21}(x)}{\partial x_1} = 0 & \frac{\partial g_{21}(x)}{\partial x_2} = 0 & \frac{\partial g_{22}(x)}{\partial x_1} = 0 & \frac{\partial g_{22}(x)}{\partial x_2} = \frac{\sigma}{2\sqrt{x_2}} \end{array}$$

Then we begin to calculate the presentation of h:

$$
\begin{aligned}
&h_{112}(x) = \frac{1}{2}\sum_{l=1}^{2} g_{l2}(x)\frac{\partial g_{11}(x)}{\partial x_2} = \frac{1}{2}g_{22}(x)\frac{\partial g_{11}(x)}{\partial x_2} = \frac{\sigma}{4}\sqrt{1-\rho^2} \\
&h_{122}(x) = \frac{1}{2}\sum_{l=1}^{2} g_{l2}(x)\frac{\partial g_{12}(x)}{\partial x_l} = \frac{1}{2}g_{22}(x)\frac{\partial g_{12}(x)}{\partial x_2} = \frac{\sigma}{4}\rho \\
&h_{111}(x) = 0 \qquad h_{122}(x) = 0 \\
&h_{222}(x) = \frac{1}{2}\sum_{l=1}^{2} g_{l2}(x)\frac{\partial g_{22}(x)}{\partial x_l} = \frac{1}{2}g_{22}(x)\frac{\partial g_{22}(x)}{\partial x_2} = \frac{\sigma^2}{4} \\
&h_{211}(x) = 0 \qquad h_{221}(x) = 0 \qquad h_{212}(x) = 0
\end{aligned}
$$

Finally, we could get the discrete simulation formula as:

$$
\begin{aligned}
log(S_{n+1}) &= log(S_n) + (r - \frac{1}{2}V_n)\Delta t + \sqrt{V_n}\Delta\omega_{1,n} + \frac{1}{4}\sigma\Delta\omega_{1,n}\Delta\omega_{2,n} - \frac{1}{4}\sigma\rho\Delta t \\
V_{n+1} &= V_n + \kappa(\theta - V_n)\Delta t + \sigma\sqrt{V_n}\Delta\omega_{2,n} + \frac{1}{4}\sigma^2(\Delta\omega_{2,n}^2 - \Delta t)
\end{aligned}
$$

3 Numeric simulation

Suppose that $\Delta t = \frac{T}{m^l}$, choose G_1, G_3 are the Gaussian Brownian Motion for log(Price) and G_2, G_4 are the Gaussian Brownian Motion for volatility, also to make sure that they are pairwise independent.

$$
\begin{aligned}
log(S_(t^f_{n+\frac{1}{2}})) &= log(S^f_{t_n}) + (r - \frac{V^f_n}{2})\frac{\Delta t}{2} + \sqrt{V^f_n}\sqrt{\frac{\Delta t}{2}} \times (\sqrt{1-\rho^2}G_1 + \rho G_2) \\
&+ \frac{\sigma}{4}\frac{\Delta t}{2} \times (\sqrt{1-\rho^2}G_1 + \rho G_2)G_2 - \frac{\sigma\rho}{4}\frac{\Delta t}{2} \\
V^f_{t_{n+\frac{1}{2}}} &= V^f_n + \kappa(\theta - V^f_n)\frac{\Delta t}{2} + \sigma\sqrt{V^f_n}\sqrt{\frac{\Delta t}{2}}G_2 + \frac{\sigma^2}{4}(\frac{\Delta t}{2}G_2^2 - \frac{\Delta t}{2}) \\
log(S_(t^f_{n+1})) &= log(S^f_{t_{n+\frac{1}{2}}}) + (r - \frac{V^f_{n+\frac{1}{2}}}{2})\frac{\Delta t}{2} + \sqrt{V^f_{n+\frac{1}{2}}}\sqrt{\frac{\Delta t}{2}} \times (\sqrt{1-\rho^2}G_3 + \rho G_4) \\
&+ \frac{\sigma}{4}\frac{\Delta t}{2} \times (\sqrt{1-\rho^2}G_3 + \rho G_4)G_4 - \frac{\sigma\rho}{4}\frac{\Delta t}{2} \\
V_{t^f_{n+1}} &= V^f_{n+\frac{1}{2}} + \kappa(\theta - V^f_{n+\frac{1}{2}})\frac{\Delta t}{2} + \sigma\sqrt{V^f_{n+\frac{1}{2}}}\sqrt{\frac{\Delta t}{2}}G_4 + \frac{\sigma^2}{4}(\frac{\Delta t}{2}G_4^2 - \frac{\Delta t}{2})
\end{aligned}
$$

For the same reason, we could use G_1, G_2, G_3, G_4 to generate the antithetic path simulation which use G_3, G_4 to simulate the first half and G_1, G_2 for the second half.

$$\begin{aligned}
log(S_(t^a_{n+\frac{1}{2}})) &= log(S^a_{t_n}) + (r - \frac{V^a_n}{2})\frac{\Delta t}{2} + \sqrt{V^a_n}\sqrt{\frac{\Delta t}{2}} \times (\sqrt{1-\rho^2}G_3 + \rho G_4) \\
&+ \frac{\sigma}{4}\frac{\Delta t}{2} \times (\sqrt{1-\rho^2}G_3 + \rho G_4)G_4 - \frac{\sigma\rho}{4}\frac{\Delta t}{2} \\
V^a_{t_{n+\frac{1}{2}}} &= V^a_n + \kappa(\theta - V^a_n)\frac{\Delta t}{2} + \sigma\sqrt{V^a_n}\sqrt{\frac{\Delta t}{2}}G_4 + \frac{\sigma^2}{4}(\frac{\Delta t}{2}G_4^2 - \frac{\Delta t}{2}) \\
log(S_(t^a_{n+1})) &= log(S^a_{t_{n+\frac{1}{2}}}) + (r - \frac{V^a_{n+\frac{1}{2}}}{2})\frac{\Delta t}{2} + \sqrt{V^a_{n+\frac{1}{2}}}\sqrt{\frac{\Delta t}{2}} \times (\sqrt{1-\rho^2}G_1 + \rho G_2) \\
&+ \frac{\sigma}{4}\frac{\Delta t}{2} \times (\sqrt{1-\rho^2}G_1 + \rho G_2)G_2 - \frac{\sigma\rho}{4}\frac{\Delta t}{2} \\
V_{t^a_{n+1}} &= V^a_{n+\frac{1}{2}} + \kappa(\theta - V^a_{n+\frac{1}{2}})\frac{\Delta t}{2} + \sigma\sqrt{V^a_{n+\frac{1}{2}}}\sqrt{\frac{\Delta t}{2}}G_2 + \frac{\sigma^2}{4}(\frac{\Delta t}{2}G_2^2 - \frac{\Delta t}{2})
\end{aligned}$$

For coarse path simulation, we decide to use the same number of step and we have $\Delta\omega_n = \delta\omega_n + \delta\omega_{n+\frac{1}{2}}$

$$\begin{aligned}
log(S_(t^c_{n+1})) &= log(S^c_{t_n}) + (r - \frac{V^c_n}{2})\Delta t + \sqrt{V^c_n}\sqrt{\frac{\Delta t}{2}} \times [\sqrt{1-\rho^2}(G_1 + G_3) + \rho(G_2 + G_4)] \\
&+ \frac{\sigma}{4}\frac{\Delta t}{2} \times [\sqrt{1-\rho^2}(G_1 + G_3) + \rho(G_2 + G_4)](G_2 + G_4) - \frac{\sigma\rho}{4}\Delta t \\
V_{t^c_{n+1}} &= V^c_n + \kappa(\theta - V^c_{n+\frac{1}{2}})\Delta t + \sigma\sqrt{V^c_n}\sqrt{\frac{\Delta t}{2}}(G_2 + G_4) + \frac{\sigma^2}{4}[\frac{\Delta t}{2}(G_2 + G_4)^2 - \Delta t]
\end{aligned}$$

References

[1] Michael B.Giles and Lukasz Szpruch. Antithetic multilevel monte carlo estimation for multi-dimensional sdes without levy area simulation. *The Annuals of Applied Probability*, 24(4):1585–1620, 2014. 1