# Importance Sampling and the Statistical Romberg method

Kaouther HAJJI

March 3, 2020

This work is based on a preprint which is part of my thesis.

## 1 Input parameters

- Number of discretizations $n$.
- Generator type.
- The heston model parameters: the spot $S_0$, the current variance $V_0$, the annual interest rate $r$, the volatility of the variance $\sigma$, the long-run variance $\bar{v}$, the mean reversion $\kappa$ and $\rho$.
- Maturity $T$.
- Strike $K$.

## 2 Output parameters

- Price.
- Length of the Price confidence interval.
- Price confidence interval: [Price inf, Pricesup].
- Delta.
- Length of the Delta confidence interval.
- Delta confidence interval: [Delta inf, Delta sup].
- CPU time.

Premia 22

## 3 Description

### 3.1 Heston model

We consider the Heston stochastic Volatility model (1993):

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t = \kappa(\bar{v} - V_t)dt + \sigma\sqrt{V_t} dW_t^2 \end{cases}$$

Where $W^1$ and $W^2$ are two correlated brownian motions with $< W^1, W^2 >= \rho t$. $\kappa$, $\sigma$ and $\bar{v}$ are strictly positive constants.

- $k$ is the rate at which $V_t$ reverts to $\bar{v}$

- $\bar{v}$ is the long run average price variance

- $\sigma$ is the volatility of the variance

- $V_0$ and $S_0$ are constant.

Our aim is to use the importance sampling method in order to reduce the variance when computing the price of an european call option (resp. european put option), with strike $K$, under the heston model. The payoff of the option is $\psi(S_T) = (S_t - K)_+$ (resp. $\psi(S_T) = (K - S_T)_+$). Then, the price is $e^{-rT}\mathbb{E}\left[\psi(S_T)\right]$. After a density transformation, given by Girsanov theorem, the price will be defined by:

$$e^{-rT}\mathbb{E}\left[g(\theta, S_T^\theta)\right] = e^{-rT}\mathbb{E}\left[\psi(S_T^\theta) \quad e^{-\theta.W_T - \frac{1}{2}|\theta|^2 T}\right], \qquad \theta \in \mathbb{R}^2.$$

To approximate $S_T$, we consider the step $\frac{T}{n}$ and we discretize the stochastic process using the Euler scheme:

$$\begin{cases} S_{t_{i+1}} = S_{t_i}\left(1 + r\frac{T}{n} + \sqrt{V_{t_i}\frac{T}{n}}Z_i^1\right) \\ V_{t_{i+1}} = V_{t_i} + \kappa(\bar{v} - V_{t_i})\frac{T}{n} + \sigma\sqrt{V_{t_i}\frac{T}{n}}Z_i^2. \end{cases}$$

with $(Z_i^1, Z_i^2)$ are standard normal random variables with correlation $\rho$. Using the Cholesky decomposition, $Z_i^1 = \phi_i^1$ and $Z_i^1 = \rho\phi_i^1 + \sqrt{1-\rho^2}\phi_i^2$ where $\phi_i^1$ and $\phi_i^2$ are independent standard random variables. Hence, the price of the considered option has the following expression:

$$e^{-rT}\mathbb{E}\left[g(\theta, S_T^{n,\theta})\right] = e^{-rT}\mathbb{E}\left[\psi(S_T^{n,\theta}) \quad e^{-\theta.W_T - \frac{1}{2}|\theta|^2 T}\right], \qquad \theta \in \mathbb{R}^2.$$

To calculate the price of the considered option, we use the Statistical Romberg instead of the classical Monte Carlo method. Hence, the price is computed as follows (with $N_1 = n^2$ and $N_2 = n^{\frac{3}{2}}$):

$$\frac{1}{N_1}\sum_{i=1}^{N_1} g(\theta_n, \hat{S}_{T,i+1}^{\sqrt{n},\theta_n}) + \frac{1}{N_2}\sum_{i=1}^{N_2} g(\theta_n, S_{T,i+1}^{n,\theta_n}) - g(\theta_n, S_{T,i+1}^{\sqrt{n},\theta_n}).$$

The variance of the estimated price under the statistical romberg method is given by:

$$\mathbb{E}\left[\left(\psi^2(S_T^{n,\theta}) + (\nabla\psi(S_T^{n,\theta}))^2.(U_T^{n,\theta})^2\right) \quad e^{-2\theta.W_T - |\theta|^2 T}\right]$$

## 3.2 Importance sampling

The optimal $\theta$ is given by:

$$\theta_n^* = \underset{\theta \in \mathbb{R}^2}{\arg\min} \quad \mathbb{E}\left[\left(\psi^2(S_T^{n,\theta}) + (\nabla\psi(S_T^{n,\theta}))^2.(U_T^{n,\theta})^2\right) \quad e^{-2\theta.W_T - |\theta|^2 T}\right].$$

To obtain the optimal $\theta$, the idea is to make use a constrained Robbins Monro algorithm (named "Chen algorithm") which is decribed as follows.

Let $(\mathcal{K}_i)_{i\in\mathbb{N}}$ denote an increasing sequence of compact sets satisfying $\cup_{i=0}^{\infty} \mathcal{K}_i = \mathbb{R}^d$ and $\mathcal{K}_i \subsetneq \mathring{\mathcal{K}}_{i+1}, \forall i \in \mathbb{N}$. For $\theta_0^n \in \mathcal{K}_0$, $\alpha_0^n = 0$ and a gain sequence $(\gamma_i)_{i\in\mathbb{N}}$ (a positive deterministic sequence decreasing to 0 which satisfies $\sum_{i=1}^{\infty}\gamma_i = \infty$ and $\sum_{i=1}^{\infty}\gamma_i^2 < \infty$), we define the sequence $(\theta_i^n, \alpha_i^n)_{i\in\mathbb{N}}$ recursively by

$$\begin{cases} \textbf{if} \quad \left|\theta_i^n - \gamma_{i+1}H(\theta_i^n, S_{T,i+1}^n, U_{T,i+1}^n, W_{T,i+1})\right| \in \mathcal{K}_{\alpha_i^n}, \ \textbf{then} \\ \qquad \theta_{i+1}^n = \theta_i^n - \gamma_{i+1}H(\theta_i^n, S_{T,i+1}^n, U_{T,i+1}^n, W_{T,i+1}), \text{ and } \alpha_{i+1}^n = \alpha_i^n \\ \textbf{else} \quad \theta_{i+1}^n = \theta_0^n \text{ and } \alpha_{i+1}^n = \alpha_i^n + 1, \end{cases} \tag{1}$$

where

$$H(\theta_i^n, S_{T,i+1}^n, W_{T,i}) = (\theta_i^n T - W_{T,i+1}) \quad \left(\psi^2(S_{T,i+1}^n) + (\nabla\psi(S_T^n))^2.(U_T^n)^2\right) \quad e^{-\theta_i^n.W_{T,i+1} + \frac{1}{2}|\theta_i^n|^2 T}.$$

### 3.3 Calculus of the Statistical Romberg price

By rebalancing the optimal $\theta_n^*$, we compute the price of the considered option using the Statistical Romberg method with importance sampling (SR + IS) (with $N_1 = n^2$ and $N_2 = n^{\frac{3}{2}}$):

$$\frac{1}{N_1}\sum_{i=1}^{N_1} g(\theta_n^*, \hat{S}_{T,i+1}^{\sqrt{n},\theta_n^*}) + \frac{1}{N_2}\sum_{i=1}^{N_2} g(\theta_n^*, S_{T,i+1}^{n,\theta_n^*}) - g(\theta_n^*, S_{T,i+1}^{\sqrt{n},\theta_n^*}).$$

We compute the length of a 95% confidence interval and the CPU time of our method. In comparison with the classical Monte Carlo with importance sampling method and for a given number of discretizations $n > 400$, the CPU time reduction exceeds a factor of 3.