An Efficient Monte Carlo for Discrete Variance Contracts

Hervé Azevedo Chaves

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1 Model specification

Nicolas Merener and Leonardo Vicchi proposed in [1] an efficient Monte Carlo method for the valuation of a financial contract with payoff dependent on discretely realized variance. Here we focus on the implementation of the method with the Heston model. Let's denote

- T: the maturity of the contract.
- S_t: the underlying at time $t \geq 0$.
- V_t: the stochastic volatility of S_t at time $t \geq 0$.
- N: the number of days deals in the contract.
- M: the number of Monte Carlo iterations.
- r: the rate of return of the asset.
- d: the dividend.
- κ: the rate of which V_t reverts to theta.
- θ: long variance (or long run average price).
- σ: volatility of the volatility.
- ρ: correlation factor between Z and W.
- K: the strike.
- $B(0,T) = e^{-rT}$: the discount factor.
- Let $x \mapsto g(x)$ be the function of a payoff (Variance Swap: $g(x) = \left(10,000\ x - K^2\right)$).
- Let $(Z_t, t \geq 0)$ and $(W_t, t \geq 0)$ two independants Brownian Motion.

The Heston model reads:

$$\begin{aligned} dS_t &= (r-d)\, S_t dt + \sqrt{V_t}\ S_t\ dZ_t \\ dV_t &= \kappa\,(\theta - V_t)\, dt + \sigma\sqrt{V_t}\left(\rho dZ_t + \sqrt{1-\rho^2}\ dW_t\right). \end{aligned}$$

Let's denote by $\Delta = \frac{T}{N}$ the discretization step over a deterministic time grid $t_0 < t_1 < \cdots < t_N$ on $[0,T]$ with $t_i = i\Delta,\ \forall i\ \in \{0, \cdots, N\}$. The Euler Full Truncation Scheme reads:

$$\begin{aligned} \widehat{S}_{i+1} &= \widehat{S}_i e^{\Delta\left(r-d-\widehat{V}_i/2\right)+\sqrt{\widehat{V}_i}\sqrt{\Delta}Z_{i+1}} \\ \tilde{V}_{i+1} &= \tilde{V}_i + \kappa\left(\theta - \tilde{V}_i^+\right)\Delta + \sigma\sqrt{\tilde{V}_i^+}\sqrt{\Delta}\left(\rho Z_{i+1} + \sqrt{1-\rho^2}\ W_{i+1}\right) \\ \widehat{V}_{i+1} &= \tilde{V}_{i+1}^+. \end{aligned}$$

2 The algorithm

The price of a discrete realized variance contracts under such model is computed as the expectation of a deterministic function of discretely recorded variance defined as

$$RV\left(Z,W\right)=\sum_{i=0}^{N-1}\left(\log\frac{S_{i+1}}{S_i}\right)^2.$$

Their method combines deterministic integration with random sampling and Monte Carlo simulation for the pricing of such contracts:

$$\begin{aligned}C&=B(0,T)\mathbb{E}\left[g(RV\left(Z,W\right))\right]\\&=B(0,T)\mathbb{E}\left[\mathbb{E}\left[g(RV\left(Z,W\right))\mid\left(||Z||_2^2,\langle v,W\rangle\right)\right]\right]\\&=B(0,T)\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}\left[g(RV\left(Z,W\right))\mid\left(||Z||_2^2,\langle v,W\rangle\right)=(z,w)\right]\ f_{||Z||_2^2}(z)\ f_{\langle v,W\rangle}(w)\ dz\ dw\end{aligned}$$

With

- $||Z||_2^2=\sum_{i=0}^{N-1}Z_i^2$, the Euclidiean norm of Z, being a χ^2 distribution of degree N.
- $||V||_1=\sum_{i=0}^{N-1}V_i$, the one norm of V.
- $v=\nabla_W||V||_1$ evaluated at $\{Z=0,W=0\}$, the optimal integration direction.
- $\langle v,W\rangle=\sum_{i=0}^{N-1}v_iW_i$, the scalar product of v and W, being a standard normal random variable.
- $f_{\langle v,W\rangle}(w)=\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}$, the probability density function of the standard normal distribution.
- $f_{||Z||_2^2}(z)=\frac{1}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})}z^{\frac{N}{2}-1}e^{-\frac{z}{2}}$, the probability density function of the χ^2 of degree N distribution.

We now combine deterministic integration over the values of the components of $\left(||Z||_2^2,\langle v,W\rangle\right)$ with Monte Carlo sampling conditional on $\left(||Z||_2^2,\langle v,W\rangle\right)$ with a proportional allocation of paths. Namely $M'=Mf_{||Z||_2^2}(z)f_{\langle v,W\rangle}(w)$:

$$\mathbb{E}\left[g(RV\left(Z,W\right))\mid\left(||Z||_2^2,\langle v,W\rangle\right)=(z,w)\right]\approx\frac{1}{M'}\sum_{k=1}^{M'}g(RV\left(Z^{(k)}_{|\ ||Z||_2^2=z},W^{(k)}_{|\ \langle v,W\rangle=w}\right)),$$

with $Z^{(k)}_{|\ ||Z||_2^2=z}$ sampled as:

$$Z^{(k)}_{|\ ||Z||_2^2=z}=\sqrt{z}\frac{\xi}{||\xi||_2},\qquad\xi\sim\mathcal{N}(0,1).$$

and $W^{(k)}_{|\ \langle v,W\rangle=w}$ sampled as:

$$W^{(k)}_{|\ \langle v,W\rangle=w}=v_kw+\left(1-v_k^2\right)\Lambda,\qquad\Lambda\sim\mathcal{N}(0,1).$$

The estimator $\widehat{C}$ of C now reads:

$$\widehat{C}=B(0,T)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{M'}\sum_{k=1}^{M'}g(RV\left(Z^{(k)}_{|\ ||Z||_2^2=z},W^{(k)}_{|\ \langle v,W\rangle=w}\right))\ f_{||Z||_2^2}(z)\ f_{\langle v,W\rangle}(w)\ dz\ dw,$$

(noticing that $M'=Mf_{||Z||_2^2}(z)f_{\langle v,W\rangle}(w)$),

$$=B(0,T)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{M}\sum_{k=1}^{M'}g(RV\left(Z^{(k)}_{|\ ||Z||_2^2=z},W^{(k)}_{|\ \langle v,W\rangle=w}\right))\ dz\ dw$$

We now approximate the two integrals with 40 equally spaced steps on the real line, ranging from -4 to 4 for the standard normal distribution: $w_i=-4+i\Delta w$ with $\Delta w=\dfrac{8}{40}$, $\forall i\in\{0,\dots,40\}$. And 30 equally spaced steps on the positive real half line, ranging from 0 to $15\sqrt{N}$ for the χ^2 distribution: $z_j = j\Delta z$ with $\Delta z = \dfrac{15\sqrt{N}}{30}$, $\forall j \in \{0,\dots,30\}$:

$$C \approx B(0,T)\frac{\Delta z \Delta w}{M}\sum_{i=0}^{40}\sum_{j=0}^{30}\sum_{k=1}^{\lfloor M f_{||Z||_2^2}(z_j) f_{\langle v,W\rangle}(w_i)\rfloor} g(RV\left(Z^{(k)}_{|\ ||Z||_2^2=z}, W^{(k)}_{|\ \langle v,W\rangle=w}\right)).$$

References

[1] N.Merener L.Vicchi An efficient Monte Carlo method for discrete variance contracts. *The Journal of Computational Finance.*, 16 (4), 2013.

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