Fourier transform algorithms for pricing discretely monitored variance products and volatility derivatives under Lévy processes processes

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1. Lévy processes: basic facts

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1})\nu(dy), \tag{1}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2}$$

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = S_0 e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\Im\xi \in (-1, 0)$ and continuous continuation into the closed strip $\Im\xi \in [-1, 0]$.

Example 1. If Lévy measure of a jump diffusion process is given by normal distribution:

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left(-\frac{(x-\gamma)^2}{2\delta^2}\right)dx,$$

then we obtain Merton model. The parameter λ characterizes the intensity of jumps. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\left(1 - \exp\left(-\frac{\delta^2\xi^2}{2} + i\gamma\xi\right)\right), \tag{3}$$

where $\sigma, \delta, \lambda \geq 0$, $\mu, \gamma \in \mathbf{R}$.

Example 2. The characteristic exponent of a pure jump KoBoL process (a.k.a. CGMY model) of order $\nu \in (0,2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{4}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Note that Boyarchenko and Levendorskiĭ (2000, 2002) consider a more general version with $c_\pm$ instead of c, as well as the case $\nu = 1$ and cases of different exponents $\nu_\pm$. If $\nu \geq 1$ or $\mu = 0$, then the order of the KoBoL process equals to the order of the infinitesimal generator as PDO, but if $\nu < 1$ and $\mu \neq 0$, then the order of the process is ν, and the order of the PDO $-L = \psi(D)$ is 1.

Example 3. If Lévy density is given by exponential functions on negative and positive axis:

$$F(dy) = \mathbf{1}_{(-\infty;0)}(y)c_+\lambda_+e^{\lambda_+ y}dy + \mathbf{1}_{(0;+\infty)}(y)c_-(-\lambda_-)e^{\lambda_- y},$$

where $c_\pm \geq 0$ and $\lambda_- < -1 < 0 < \lambda_+$, then we obtain Kou model (see Kou (2002)). The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{5}$$

2. Variance products

Let S_t denote the value of a stock or a stock index at time t, and $r \geq 0$ be the riskless rate. We assume that the variance product starts at time zero and ends at time T. Let $0 = t_0 < t_1 < \ldots < t_M = T$ be the monitoring dates for the discretely sampled variance and T be the maturity date. Assume that $S_t = \exp(X_t)$ is modeled by some Lévy process X_t, then the annualized realised variance of returns over the time interval $[0;T]$ is determined by

$$V(T,M) = \frac{1}{T}\sum_{n=0}^{M}\left[\ln\frac{S_{t_n}}{S_{t_{n-1}}}\right]^2. \tag{6}$$

Typically, one sets $T = \frac{M}{252}$. If it is needed to express the realized variance in annual volatility points one should multiply the right hand side of (6) by 10000.

Now consider derivatives written on variance $V(T,M)$.

A variance swap is an instrument which allows investors to trade future realized (or historical) volatility against current implied volatility.

A *variance swap* with maturity T and strike K^2 pays the holder

$$VarS(K,T) = V(T,M) - K^2.$$

The quantity

$$m_T = \mathbf{E}[V(T,M)]$$

is called "fair strike of a variance swap". For the variance swap with strike price K^2 and maturity T in the future its fair price at the time 0 is

$$m_T - K^2.$$

A *variance call* and a *variance put* with strike K^2 and time to expiry T pays the holder at time T

$$VarCall(K,T) = \Big(V(T,M) - K^2\Big)^+$$

and

$$VarPut(K,T) = \Big(K^2 - V(T,M)\Big)^+,$$

respectively. Typically, the input parameter K is given in volatility points.

In the most well-known model-free approaches such as the CBOE method, the price of variance derivative may be approximated by some portfolio (the so called replicating portfolio) of some amount of underlying and derivatives on it. One may find the construction of replicating portfolio e.g. in Bossu et al. (2005) and Buehler (2009). The alternative approach developed in Fukasawa et. al. (2011) is based on the implied volatility integration. Both methods begin with the assumption that S_t follows some diffusion process of the type:

$$\frac{dS_t}{S_t} = \mu(t, S_t, ...)dt + \sigma(t, S_t, ...)dW_t$$

where W_t is a Wiener process, and the drift μ and the volatility σ are unknown coefficients (either deterministic or stochastic).

There are many empirical studies on American and European financial markets (see e.g. Cont and Tankov (2004) and the bibliograpy therein) supporting the fact that pure non-Gaussian Lévy models are more adequate than diffusion models. In the case of time changed Lévy models, a formula for the variance swap price can be found in Carr et al. (2012).

W. Zheng and Y.K. Kwok, (2014) developed fast Fourier transform algorithms for pricing and hedging discretely sampled variance products and volatility derivatives under additive processes including Lévy models. The algorithms uses Fourier time stepping procedure combined with updating rules suggested in Windcliff et al. (2006) to solve a correspondent partial integro-differential equation.

3. Fourier time stepping algorithm for pricing variance products

We shortly describe a numerical framework of W. Zheng and Y.K. Kwok, (2014) to value variance contracts. To value a general variance product, one introduces two additional state variables. Let P and Z denote the logarithm of asset price on the previous monitoring date and the running average of the squared returns accumulated up to the current time, respectively. These variables stay constant between two consecutive monitoring dates and change only at the discrete volatility sampling times, t_m, with updating according to the following rules.

$$P_{t_m^+} = X_{t_m}, \quad Z_{t_m^+} = Z_{t_m^-} + \frac{R_m^2 - Z_{t_m^-}}{m}, \tag{7}$$

where $R_m = \ln(S_{t_m}/S_{t_{m-1}})$, t_m^- and t_m^+ represent the instants immediately before and after the mth monitoring date t_m, $m = 1, 2, \ldots, M$. We set $P_0 = X_0$ and $Z_0 = 0$.

The time-t value of the variance product can be regarded as a function of the logarithm of the underlying asset price X_t and time t. Between two consecutive monitoring dates, the price function $U = U(X; t; P; Z)$ is a function of X and t while the state variables P and Z are treated as parameters.

According to the martingale pricing theory, the time stepping calculations between consecutive monitoring dates look as follows

$$U(X_{t_{m-1}}; t_{m-1}^+; P; Z) = e^{-r(t_m - t_{m-1})} \mathbf{E}_{t_{m-1}^+}[U(X_{t_m}; t_m^-; P; Z)] \tag{8}$$

The efficient implementation of (8) can be reached by means of the Fast Fourier transform algorithm.

The time stepping calculations are initiated at the instant right before maturity T^- by the following formula

$$U(X_T; T^-; P_{T^-}; Z_{T^-}) = G\left(\frac{1}{T}\left[(M-1)Z_{T^-} + (X_T - P_{T^-})^2\right]\right), \tag{9}$$

where G is some specified terminal payoff function. We implemented into Premia put and call options cases with $G(V) = (K^2 - V)^+$ and $G(V) = (V - K^2)^+$, respectively.

Since there is no cash flow to the holder of the option on variance across a monitoring date, the option price should remain the same at time right before and after any volatility sampling time t_m. The jump condition is defined by

$$U(X; t_m^-; P_{t_m^-}; Z_{t_m^-}) = U(X; t_m^+; P_{t_m^+}; Z_{t_m^+}). \tag{10}$$

4. Fast Fourier Transform algorithm in time stepping procedure

For $m = M-1, M-2, \ldots, 1$ the price $f(x, t_m) = U(x; t_m^+; P; Z)$ in (8) can be found as the price of the European option with the terminal payoff $f(X_{t_{m+1}}, t_{m+1}) = U(X_{t_{m+1}}; t_{m+1}^-; P; Z)$ and the expiry date t_{m+1}:

$$f(x, t_m) = \mathbf{E}[e^{-r\Delta\tau_m} f(X_{t_{m+1}}) \mid X_{t_m} = x], \ \Delta\tau_m = (t_m - t_{m-1}). \tag{11}$$

If an explicit formula for the probability density $p_{\Delta\tau_m}$ of $X_{\Delta\tau_m}$ under EMM is known (e.g. GBM or NIG model), we can use it to write (11) in the form

$$f(x, t_m) = e^{-r\Delta\tau_m} \int_{-\infty}^{+\infty} p_{\Delta\tau_m}(y) f(x + y, t_{m+1}) dy, x > 0. \tag{12}$$

In the general case, $p_{\Delta\tau_m}$ can be expressed in terms of the characteristic exponent $\psi(\xi)$, by using the Fourier transform

$$p_{\Delta\tau_m}(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi - \Delta\tau_m \psi(\xi)} d\xi. \tag{13}$$

Now, we can rewrite (12) by using (13), and we obtain

$$f(x,t_m) = e^{-r\Delta\tau_m}(2\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x+y,t_{m+1})\exp[-iy\xi - \Delta\tau_m\psi(\xi)]d\xi dy.$$

We change a variable $z = x+y$. If $f(x,t_{m+1})$ is absolutely integrable, we can change the order of the integration:

$$\begin{aligned} f(x,t_m) &= e^{-r\Delta\tau_m}(2\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(z,t_m+\Delta\tau_m)e^{-i(z-x)\xi}\exp[-\Delta\tau_m\psi(\xi)]d\xi dz \\ &= (2\pi)^{-1}\int_{-\infty}^{+\infty}\hat{f}(\xi,t_{m+1})\exp[ix\xi - \Delta\tau_m(r+\psi(\xi))]d\xi, \end{aligned} \tag{14}$$

where $\hat{f}(\xi,t_{m+1})$ is the Fourier transform of a function $f(z,t_{m+1})$ in the first variable. The integral operator in the RHS of the formula (14) can be represented as a pseudo-differential operator (PDO) with the symbol

$$\Psi_{\Delta\tau}(\xi) = \exp[-\Delta\tau(r+\psi(\xi))]. \tag{15}$$

Recall that a PDO $A = a(D)$ with the symbol $a(\xi)$ acts as follows:

$$Au(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi, \tag{16}$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx.$$

Thus, in terms of PDO, we can rewrite the algorithm in the following form.

$$f(x,t_m) = \Psi_{\Delta\tau_m}(D)f(x,t_{m+1}). \tag{17}$$

To improve the convergence, one can introduce new function:

$$f_\omega(x,t_m) = e^{\omega x}f(x,t_m), m = 0,1,\dots,M, \tag{18}$$

where $\omega \in \mathbf{R}$ is chosen in a such way that $f_\omega(x,t_m)$ is absolutely integrable. Then taking into account that $e^{\omega x}\Psi_{\Delta\tau}(D)e^{-\omega x} = \Psi_{\Delta\tau}(D+i\omega)$, we obtain the following backward recursion for $f_\omega(x,t_m)$.

$$f_\omega(x,t_m) = \Psi_{\Delta\tau}(D+i\omega)f_\omega(x,t_{m+1}). \tag{19}$$

Formally, the action of a PDO A with the constant symbol $a(\xi)$ can be described as the composition

$$Au(x) = F^{-1}_{\xi\to x}a(\xi)F_{x\to\xi}u(x) \tag{20}$$

If the functions u and a are represented as arrays suitable for application of the Fast Fourier Transform and inverse Fast Fourier Transfrom algorithms (FFT and iFFT), then (20) can be programmed as $Au = iFFT(a.*(FFT(u)))$.

Let d be the step in x-space, $d\xi$–the step in ξ-space, and $N = 2^n$ the number of the points on the grid; decreasing d and increasing (even faster) N, we obtain a sequence of approximations to the option price. An approximation for the $\Psi(D)$ operator action can be efficiently computed by using the Fast Fourier Transform

(FFT) implemented into the PNL used in Premia. The discrete Fourier transform (DFT) in the PNL is defined by

$$G_l = DFT[g](l) = \sum_{k=1}^{N} g_k e^{2\pi i(k-1)(l-1)/N}, \quad l = 1, ..., N. \tag{21}$$

The DFT maps N complex numbers (the g_k's) into N complex numbers (the G_l's). The formula for the inverse DFT which recovers the set of g_k's exactly from G_l's is:

$$g_k = iDFT[G](k) = \frac{1}{N}\sum_{l=1}^{N} G_l e^{-2\pi i(k-1)(l-1)/N}, \quad k = 1, ..., N. \tag{22}$$

5. The algoritm implemented into Premia

The algorithm implemented into the program platform Premia for Tempered Stable Lévy models consist of the following steps.

- Define the Tempered Stable Lévy model parameters
- Define the option parameters: Call/Put, Strike in volatility points, Time to Expiry, Spot
- Define the method parameters:
  n – the power of 2 to define the number of space points $N = 2^n$
  L – the scale parameter to define the maximal size of increments in the log-domain $[-L; L]$
  Kz – the scale parameter to define the size of Z-grid;
- Set $x_{min} = -L$, $d = 2 \cdot L/N$, $x_k = x_{min} + (k-1) \cdot dx, p_k = x_k, k = 1, \dots N$, $\mathbf{x} = \{x_k\}$;
- Set $\xi_{min} = 2\pi/d$, $d\xi = 2\pi/(N \cdot d)$, $\xi_k = \xi_{min} + (k-1) \cdot d\xi, k = 1, \dots N$, $\xi = \{\xi_k\}$;
- Set the number of time steps $M = 252 \cdot T$, $\Delta t = \frac{T}{M}$, $t_m = m\Delta t$, $m = 0, 1, \dots M$;
- Define a uniform Z-grid with a separate last Z point at 1. Set the number of Z-points $Nz = 80$, $z_{min} = 0$, $dz = Kz \cdot K^2/(252 \cdot Nz)$, $z_k = z_{min} + (k-1) \cdot dz, k = 1, \dots Nz$, $z_{Nz-1} = \max(1., Kz \cdot K^2/252)$, $\mathbf{z} = \{\xi_k\}$;
- For $m = M$ to 1
  - For each $(p_j; z_k)$, determine $U(\mathbf{x}; t_m; p_j; z_k)$ using the updating rule (7).
    - If $m = M$
      Apply the terminal payoff function in (9) directly
    - Else
      Use an interpolation method
  - Compute $U(\mathbf{x}; t_m^+; p_j; z_k)$ in (8) via FFT using (19) and (20)
  - If $m = 1$
    Output $U(\mathbf{x}; t_0; P_0; Z_0)$; return
  - Else
    Store $U(p_j; t_m; p_j; z_k)$

- Next j; k

- Next m.

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