

fd_galerkin_discfem

Input parameters:

- SpaceStepNumber N
- TimeStepNumber M
- Theta $\frac{1}{2} \leq \theta \leq 1$
- Omega $1 \leq \omega \leq 2$
- Epsilon

Output parameters:

- Price
- Delta

We consider the European Black-Scholes Problem:

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f \text{ on }]0, T] \times \Omega_l \\ u(t, \cdot) = 0 \text{ on } \partial\Omega_l \\ u(0, \cdot) = \psi \text{ on } \Omega_l \end{cases} \quad (1)$$

where $\Omega_l = (-l, l)$, f is a smooth function and A is a second order linear operator assumed to be elliptic (i.e. $(Av, v) \geq 0$ for every function v in $H^1(\Omega_l)$). For instance, in the Black-Scholes model, the stock price process satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu(S_t)dt + \sigma(S_t)dB_t,$$

where $(B_t)_{0 \leq t \leq T}$ is a standard Brownian motion, μ is a continuous function and σ is a C^1 function uniformly bounded below. We have,

$$Au(x) = \frac{\sigma^2(x)}{2} \frac{\partial^2 u}{\partial x^2}(x) + \mu(x) \frac{\partial u}{\partial x}(x) - ru(x).$$

The discretisation of problem (1) is made by a discontinuous Galerkin finite element discretisation in space and time. We split the time interval $(0, T)$ into subintervals $I_n = (t_{n-1}, t_n]$, where $0 = t_0 < \dots < t_N = T$, with length $k_n = t_n - t_{n-1}$.

At each time step, we split the space interval Ω_l into M_n subintervals (x_i^n, x_{i+1}^n) , where $-l = x_0^n < \dots < x_{M_n}^n = l$, with length $h_i^n = x_1^n - x_{i-1}^n$. Let $(\phi_i^n)_{1 \leq i \leq M_n-1}$ the functions defined by

$$\phi_i^n(x) = \begin{cases} \frac{x - x_{i-1}^n}{h_i^n} & \text{if } x_{i-1}^n \leq x \leq x_i^n \\ \frac{x_{i+1}^n - x}{h_{i+1}^n} & \text{if } x_i^n \leq x \leq x_{i+1}^n \end{cases}$$

and let V_n the vector space generated by the function ϕ_i^n . Let us define the space-time finite element space

$$W(0, T; H_0^1(\Omega_a)) = \left\{ u \in L^2(0, T; H_0^1(\Omega_a)); u_t \in L^2(0, T; H^{-1}(\Omega_a)) \right\}.$$

For functions from this space, let us denote by

$$\begin{aligned} [v]_n &= v_n^+ - v_n^- \\ &= \lim_{s \rightarrow 0^+} v(t_n + s) - v(t_n) \end{aligned}$$

This discretization is based on a variational formulation which allows the use of discontinuous function in time. This method determines approximation U in $W_{N,q}$ by

$$B(U, V) = F(V) \text{ for } V \in W_{N,q}, \quad (2)$$

with the bilinear form

$$B(v, w) = \sum_{n=1}^N \int_{I_n} \left\{ \left(\frac{\partial v}{\partial t}, w \right) + a(v, w) \right\} + \sum_{n=2}^N ([v]_{n-1}, w_{n-1}^+) + (v_0^+, w_0^+),$$

and the linear functional

$$F(w) = (f, w) + (\psi, w_0^+).$$