Computing CVA with ADI and Monte-Carlo

Ludovic Goudenège

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1 Problem formulation

We consider the following model for the stock price:

$$\begin{cases} dS_t &= (r-q)S_t dt + \sqrt{V_t} S_t dW_t^S \\ dV_t &= \alpha(\beta - V_t)dt + \omega\sqrt{V_t} dW_t^V, \end{cases}$$

with α, β and $\omega \in \mathbb{R}_+$, where W_t^S and W_t^V are Gaussian processes, where r is the interest rate, q is a foreign interest or dividend, and with the correlation between the two implied Gaussian processes given by

$$\langle dW_t^S, dW_t^V \rangle = \rho dt$$

Using a martingale approach for an european or an american option (call or put), we can prove that the price is given by the solution of the following partial differential equation Under these assumptions, we solve the following PDE

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{1}{2}s^2 v \frac{\partial^2 U}{\partial s^2} + \frac{1}{2}\omega^2 v \frac{\partial^2 U}{\partial v^2} + \rho\omega s v \frac{\partial^2 U}{\partial s \partial v} \\ &\quad + rs\frac{\partial U}{\partial s} + \alpha(\beta - v)\frac{\partial U}{\partial v} - rU, \end{aligned}$$

with the following boundary conditions for the call option

$$\begin{aligned} U(s,v,t) &= 0 && \text{whenever } s = 0, \\ \frac{\partial U}{\partial s}(s,v,t) &= \exp(-q\ t) && \text{whenever } s = S_{\max}, \\ \frac{\partial U}{\partial v}(s,v,t) &= 0 && \text{whenever } v = V_{\max}, \end{aligned}$$

and the following boundary conditions for the put option

$$\begin{aligned} U(s,v,t) &= K\exp(-rt) && \text{whenever } s = 0, \\ \frac{\partial u}{\partial s}(s,v,t) &= 0 && \text{whenever } s = S_{\max}, \\ \frac{\partial U}{\partial v}(s,v,t) &= 0 && \text{whenever } v = V_{\max}. \end{aligned}$$

By default, the initial values are $S = S_0 = 100$ and $V = V_0 = 0.01$, the maturity T is one year and the strike value is 100, such that $C(x, v, 0) = (b(S_0 \exp(x) - K))^+$ where $b = 1$ for the call and $b = -1$ for the put. In the case of the american options, we should add the possibility to exercice the option before the maturity which is easily implemented in the partial differential equation by taking the maximum compared to the pay-off at any time.

Following [2], we define the exposure of an option at a futur time $t < T$ by

$$E(t) := \max(U(S_t, V_t, t), 0)$$

where $U(S_t, V_t, t)$ is the (mark-to-market) value of financial derivatives contract at time t. The present Expected Exposure at a future time $t < T$ is defined by

$$EE(t) := \mathbb{E}\left[E(t)|\mathcal{F}_0\right]$$

where $\mathcal{F}_0$ is the filtration at time $t = 0$ and the expectation is computed under the risk-neutral measure $\mathbb{Q}$.

Finally the credit valuation adjustment (CVA) is given by

$$CVA(0, T) := (1 - R) \int_0^T$$

where R is the recovery rate, $D(0, t)$ is the risk-free discount factor and $PD(t)$ denotes the default probability of the counter-party at time t.

2 ADI finite difference scheme

We refer to [1] where a similar method is described to solve the partial differential equation. We have used the same grids whose sizes are given respectively for time, S-space, V-space by N_t, N_s and N_v. The default values are 40, 100 and 20. This choice ensures very good estimations for the prices of call or put options in a large variety of parameters in less than 1 second.

The Douglas scheme described in [1] has been implemented, but the methods for all the others schemes are potentially already in the code, since all the necessary functions are already implemented. See also the documentation for the Heston model.

3 Computing CVA by Monte-Carlo procedure

The method developed in [2] is based on a Monte-Carlo procedure using all the prices already computed at all times. It suffices to describe how to compute the expectation and the integral to obtain a complete methodology.

First the recovery rate R is a fixed value (default is $0.4 = 40\%$). Then we have to define the default probability of the counter-party at time t, which is actually given by an exponential survival probability with hazard rate λ defined by

$$PD(t)dt = \exp(-\lambda t) - \exp(-\lambda(t + dt)),$$

and given a time-step Δ_t we can define the factor $q(t, \Delta_t) := \exp(-\lambda t) - \exp(-\lambda(t + \Delta_t))$ and $D(0, t) = \exp(-rt)$.

Now the CVA can be approximated by the following formula

$$CVA(0,T) \simeq (1 - R) \sum_{k=1}^{N_t} D(0, k\Delta_t) q(k\Delta_t, \Delta_t) EE(k\Delta_t).$$

Finally we need to compute the expected exposure which is given by a Monte-Carlo procedure of M scenari on asset S and variance V.

$$CVA(0,T) \simeq \frac{(1 - R)}{M} \sum_{m=1}^{M} \sum_{k=1}^{N_t} D(0, k\Delta_t) q(k\Delta_t, \Delta_t) \max(U(S^m_{k\Delta_t}, V^m_{k\Delta_t}, t), 0).$$

4 Implementation

The main program fixes the variables and compute the grid in space and variance variables. It calls the function `compute_CVA` which first call the function `compute_all_prices`. This function computes all the prices for all time t in the time grid, for all asset value S in the asset grid, and for all variance value V in the variance grid. It uses the ADI technics described in [1] returning a three-dimensional array $U[Nt \times Ns \times Nv]$.

Next the function `compute_CVA` simulates the asset and variance scenario, see also the Alfonsi algorithm for the Heston model. For all values of $S^m_{k\Delta_t}$ and $V^m_{k\Delta_t}$ at time $k\Delta_t$ we do an interpolation in the array U using neighborhood points. We make the sum of all this values over M scenari (default value is 15000).

References

[1] Tinne Haentjens and Karel J. in't Hout. ADI finite difference schemes for the Heston-Hull-White PDE. J. Comp. Finan. 16, 83-110 (2012). 2, 3

[2] Cornelis S. L. de Graaf, Drona Kandhai and Peter M.A. Sloot. Efficient Estimation of Sensitivities for Counterparty Credit Risk with the Finite Difference Monte-Carlo Method. 2