FRAME PROJECTION METHOD FOR PRICING BERMUDAN AND BARRIER OPTIONS UNDER LÉVY PROCESSES

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Abstract. We describe "The frame projection method" (PROJ-method) for pricing discretely monitored barrier options for a wide class of Lévy processes, which is implemented into Premia 21. The method uses a backward induction approach and the Fast Fourier Transform algorithm for the efficient computation of the convolution with the probability density. The key idea behind the approach involves approximating the log-return density by its orthogonal projection onto a space of compactly supported basis elements. Discrete convolution of the projected density with a set of value coefficients at each time step is represented as a Toeplitz matrix-vector multiplication, which can be efficiently implemented by means of the fast Fourier transform (FFT). The method is implemented into Premia 21 for knock-out barrier options based on Kirkby (Applied Mathematical Finance, 24(4), 337–386, 2017), for Bermudan options based on Kirkby (Journal of Computational Finance, 22(3), 89–148, 2018), and for European options based on Kirkby (SIAM J. Financial Math., 6(1), 713-747, 2015).

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1. Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

By now, there exist several large groups of relatively universal numerical methods for pricing of American and barrier options under exponential Lévy processes. The number of publications is huge, and, therefore, an exhaustive list is virtually impossible. We concentrate on the one-dimensional case.

Existing numerical methods in literature can be categorized into three groups: Monte Carlo simulation, partial-(integro) differential equation (PIDE) methods, and backward induction methods. We will consider the last group.

The backward induction methods are based on the fact that the risk-neutral valuation formula for the European option can be seen as a convolution of the payoff function with the transition density. The key idea is to set up a time lattice and view the option as of European type between two adjacent dates. Hence, the backward induction method requires the transition density to be known in closed-form, which is the case in e.g. the Black-Scholes model and Merton's jump-diffusion model. The approximation proposed by Geske and Johnson (1984) uses the discretization of the time parameter and the backward induction for pricing American options in the GBM model. The method was extended in Boyarchenko and Levendorskiǐ (2002) for some Lévy models, and its applications can be founded e.g. in Kudryavtsev and Levendorskiǐ (2006) and Levendorskiǐ et al. (2006). If there is no an explicit formula for the probability density, it can be recovered by inverting the characteristic function, so the method can be used for a wide range of Lévy models.

Since convolutions can be handled very efficiently by means of the Fast Fourier Transform (FFT), an overall complexity of the method is $O(mn\ln n)$, where m and n are the numbers of points on the grid in time and space, respectively. The FFT-based backward induction method was applied in Jackson et al. (2008), see also Lord et al. (2008). In terms of the theory of pseudodifferential operators (PDOs), at each time step, the FFT-based backward induction method implements action of the PDO which symbol is the characteristic function.

The method suggested in Itkin (2014,2016) solves backward jump-diffusion PIDEs for option prices by splitting the related operator into differential and jump parts. The key idea behind the approach involves representing a jump operator as a PDO with subsequent transforming into operator exponential.

In series of papers Kirkby (2015, 2017a, 2017b), a backward induction method based on the frame projection approach (PROJ) was developed. In particular, in Kirkby (2017a) the approach was applied for robust pricing discretely monitored barrier derivatives under exponential Lévy models. Coefficient functionals of the orthogonally projected transition density are given by its convolution with a dual B-spline scaling function of the first order, using the characteristic function of the underlying asset.

The method's efficiency is derived in part from the use of frame projected transition densities, which transform the problem into the Fourier domain, and accelerate the convergence of intermediate expectations. These expectations are approximated by Toeplitz matrix-vector multiplications, resulting in a fast implementation by means of the Fast Fourier Transform. Additionally, the method includes proper truncating support of the transition density. In Kirkby (2017b), the frame projection approach is generalized for a case of B-spline scaling functions of an arbitrary order.

2. Lévy processes: basic facts

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato [27]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$

(we confine ourselves to the one-dimensional case). If X_t has probability density p_t, then we have

$$e^{-t\psi(\xi)} = \int_{-\infty}^{+\infty} e^{i\xi y} p_t(y)dy \tag{2.1}$$

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\le 1})\nu(dy), \tag{2.2}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2.3}$$

If the jump component is a process of finite variation, equivalently, if

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, |y|\}F(dy) < +\infty, \tag{2.4}$$

then the last term in the integrand in (2.2) can be integrated out and added to the drift term. Then we obtain

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y})F(dy), \tag{2.5}$$

with a different μ, and the new μ is the drift of the Gaussian component.

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\mathrm{Im}\,\xi \in (-1, 0)$ and continuous continuation into the closed strip $\mathrm{Im}\,\xi \in [-1, 0]$.

Further, if the riskless rate, r, is constant, and the stock pays dividends q, then the discounted price process must be a martingale. Equivalently, the following condition must hold

$$r - q + \psi(-i) = 0, \tag{2.6}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - q - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^{y} + y\mathbf{1}_{|y|\le 1})F(dy). \tag{2.7}$$

Example 1. [Tempered stable Lévy processes] The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{2.8}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (2.8) is derived in Boyarchenko and Levendorskiĭ (2000, 2002) from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$F_\mp(dy) = ce^{\lambda_\pm y}|y|^{-\nu-1}dy; \tag{2.9}$$

in the first two papers, the name extended Koponen family was used. Later, the same class of processes was used in Carr et al. (2002) under the name CGMY-model. The following relations between parameters of KoBoL model and C, G, M, Y parameters of CGMY-model is valid:

$$C = c,\ Y = \nu,\ G = \lambda_+,\ M = -\lambda_-.$$

More general version with $c_\pm$ instead of c, and the different exponents $\nu_\pm$ is known as a Tempered Stable Lévy model. In this case, we have for $\nu_+, \nu_- \in (0, 2), \nu_+, \nu_- \neq 1$

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}], \tag{2.10}$$

where $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2. [**Normal Inverse Gaussian processes**] A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see Barndorff-Nielsen (1998))

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{2.11}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example 3. [**Variance Gamma processes**] The Lévy density of a Variance Gamma process is of the form (2.9) with $\nu = 0$, and the characteristic exponent is given by (see Madan et al. (1998))

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{2.12}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 4. [**Kou model**] If $F_\mp(dy)$ are given by exponential functions on negative and positive axis, respectively:

$$F_\mp(dy) = c_\pm(\pm\lambda_\pm)e^{\lambda_\pm y},$$

where $c_\pm \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain Kou model. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{2.13}$$

The version with one-sided jumps is due to Das and Foresi (1996), the two-sided version was introduced in Duffie, Pan and Singleton (2000), see also S.G. Kou (2002).

3. The frame projection method

We briefly describe the numerical framework of Kirkby (2017a) to value exotic options in exponential Lévy models. Throughout, the riskless rate r and the dividend rate q are assumed to be constant. We consider here the special case of frame projection onto a linear spline basis, and more theoretical details and general B-spline basis results can be found in [13, 16]. After reviewing the implementation for European options in Section

3.1, we consider the case of knock-out barrier options in Section 3.2, and Bermudan options in Section 3.3.

3.1. **European Options.** Let T, K be the contract maturity and strike, and the stock price $S_t = S_0 e^{X_t}$ is an exponential Lévy process under a chosen risk-neutral measure (see (2.6)). We seek to value a on option with European style payoff of $G(x)$ at time T. Recall that for call option $G(x) = (S_0 e^x - K)_+$, and for put option $G(x) = (K - S_0 e^x)_+$.

Then the no-arbitrage price of the European option at time $t_0 = 0$ is given by

$$f(x, t_0) = E^x\left[e^{-rT} G(X_T)\right] \tag{3.1}$$

For consistency in notation with exotic options below, let $\Delta\tau$ denote the increment of time between *monitoring dates* of the contract. For a European option, $\Delta\tau = T - t_0$.

To calculate the price in (3.1), note that in the general case, $p_{\Delta\tau}$ can be expressed in terms of the characteristic exponent $\psi(\xi)$, by using the Fourier transform

$$p_{\Delta\tau}(\nu) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi - \Delta\tau\psi(\xi)} d\xi. \tag{3.2}$$

For a fixed resolution $a > 0$, and a generator $\phi(\nu) = (1 - |\nu|)\mathbf{1}_{[-1,1]}$, we obtain the following analytical representation of the orthogonally projected density

$$p_{\Delta\tau}(\nu) \approx \sum_{k=1}^{N} \Big(\int_{-\infty}^{+\infty} p_{\Delta\tau}(y) \widetilde{\phi}_{a,k}(y) dy \Big) \phi_{a,k}(\nu) \tag{3.3}$$

onto a space of compactly supported basis elements $\phi_{a,k}(\nu) := a^{1/2}\phi(a(\nu - \nu_k))$, where ν_k are the points on a uniformly spaced grid of width $\Delta\nu = 1/a$. Using the Fourier transform technique and (3.2), one may rewrite (3.3) as follows

$$\begin{aligned} p_{\Delta\tau}(\nu) \;&\approx\; \sum_{k=0}^{N-1} \frac{a^{-1/2}}{\pi} \operatorname{Re}\Big(\int_0^{+\infty} \exp[i\nu_k\xi - \Delta\tau\psi(\xi)] h_a(\xi) d\xi \Big) \phi_{a,k}(\nu) \\ &\approx\; \frac{a^{5/2}}{N} \sum_{k=0}^{N-1} \beta_{a,k}\phi_{a,k}(\nu), \end{aligned} \tag{3.4}$$

where $\nu_k = \nu_1 + k\Delta\nu$, $k = 0, \ldots, N-1$,

$$h_a(\xi) = \frac{\sin^2(\xi/2a)}{\xi^2(2 + \cos(\xi/2a))}.$$

We choose ν_1 in order to cover the support of the transition density (see [15] for several viable approaches).

To evaluate European options, we have the valuation formula

$$f(x, t_0) \approx e^{-rT} \frac{a^{5/2}}{N} \sum_{k=0}^{N-1} \beta_{a,k} \int_{\nu_{k-1}}^{\nu_{k+1}} G(\nu)\phi_{a,k}(\nu) d\nu,$$

where the coefficients $\int_{\nu_{k-1}}^{\nu_{k+1}} G(\nu)\phi_{a,k}(\nu)$ are easy to evaluate numerically for arbitrary payoffs (see [15]), and are available in closed form for many standard payoffs (see [13]).

3.1.1. *Projection Coefficients by FFT.* Approximation for the coefficients $\beta_{a,k}$ can be efficiently computed by using the Fast Fourier Transform (FFT). Consider the algorithm (the discrete Fourier transform (DFT)) defined by

$$G_k = DFT[g](k) = \sum_{j=0}^{N-1} g_j e^{-2\pi ikj/N}, \quad k = 0, ..., N-1. \tag{3.5}$$

The DFT maps N complex numbers (the g_j's) into N complex numbers (the G_k's) (see Press, W. et al (1992) for technical details). The formula for the inverse DFT which recovers the set of g_j's exactly from G_k's is:

$$g_j = iDFT[G](j) = \frac{1}{N} \sum_{k=0}^{N-1} G_k e^{2\pi ikj/N}, \quad j = 0, ..., N-1. \tag{3.6}$$

In our case, the input data consist of the following complex-valued array $\{g_j\}_{j=0}^{M}$:

$$g_0 = 1/24a^2, g_j = \exp(-i\nu_1\xi_j)\exp[-\Delta\tau\psi(\xi_j)]h_a(\xi_j), j > 0. \tag{3.7}$$

Then we obtain

$$\beta_{a,k} = \mathrm{Re}(DFT[g](k)), k = 0, \dots, N-1. \tag{3.8}$$

3.2. **Barrier Options.** As a basic example to illustrate the method we consider pricing discretely monitored down-and-out call and put options under the CGMY (KoBoL) model. Let T, K, H be the maturity, strike and barrier, and the stock price $S_t = S_0 e^{X_t}$ is an exponential Lévy process under a chosen risk-neutral measure (see (2.6)). Denote by M the number of equally spaced monitoring dates t_k, $k = 0, 1, \dots, m$, where $t_0 = 0$ and $t_M = T$.

Set $h = \ln H/S_0$ and $\Delta\tau := T/M$. Then the no-arbitrage price of the barrier option at time $t_0 = 0$ and $X_t = x > h$ is given by

$$f(x, t_0) = E^x \left[e^{-rT} \mathbf{1}_{m_1 > h} \mathbf{1}_{m_2 > h} \dots \mathbf{1}_{m_M > h} G(X_T) \right], \tag{3.9}$$

where $m_n = \min_{k=0,1,\dots,n} X_{t_k}$ is the processes of the minimum up to the nth monitoring date, $G(x)$ is the payoff at maturity. Recall that for call option $G(x) = (S_0 e^x - K)_+$, and for put option $G(x) = (K - S_0 e^x)_+$.

We have

$$f(x, t_M) = G(x),\ x > h, \tag{3.10}$$

and for all m,

$$f(x, t_m) = 0,\ x \le h. \tag{3.11}$$

For $m = M-1, M-2, \dots, 0$, and $x > 0$, the price $f(x, t_m)$ can be found as the price of the European option with the terminal payoff $f(X_{t_{m+1}}, t_{m+1})$ and the expiry date t_{m+1}:

$$f(x, t_m) = E[e^{-r\Delta\tau} f(X_{t_{m+1}}, t_{m+1}) \mid X_{t_m} = x],\ x > h. \tag{3.12}$$

If an explicit formula for the probability density $p_{\Delta\tau}$ of $X_{\Delta\tau}$ under EMM is known (e.g. GBM or NIG model), we can use it to write (3.12) in the form

$$f(x,t_m) = e^{-r\Delta\tau}\int_{-\infty}^{+\infty} p_{\Delta\tau}(y-x)f(y,t_{m+1})dy, x > h. \tag{3.13}$$

We will evaluate $f(x,t_m)$ along a grid of points in log asset space, $x_n = \ln H/S_0 + n\Delta x$, $n = 0,\ldots,N/2-1$, where $\Delta x = \Delta\nu$, using the frame projection approximation of $p_{\Delta\tau}$ defined in (3.4).

If the payoff G decays at $+\infty$ (see Section 3.2.1 for a treatment for unbounded payoffs), then truncating the integration domain in (3.13) from above by $u = x_{N/2-1}$ (see details in Kirkby (2017a)), we can rewrite (3.13) by using (3.4), and we obtain for x_n, $n = 0,\ldots,N/2-1$:

$$\begin{aligned} f(x_n,t_m) &\approx \frac{24a^2e^{-r\Delta t}}{N}\sum_{k=0}^{N/2-1}\beta_{a,N/2+k-n}a^{1/2}\int_h^u f(y,t_{m+1})a^{1/2}\phi(a(y-y_k))dy \\ &= C\sum_{k=0}^{N/2-1}\beta_{a,N/2+k-n}\theta_{m,k} \end{aligned} \tag{3.14}$$

where

$$\theta_{m,k} = a^{1/2}\int_h^u f(y,t_{m+1})a^{1/2}\phi(a(y-y_k))dy, \quad C = \frac{24a^2e^{-r\Delta t}}{N}. \tag{3.15}$$

The convolution (3.14) can be computed fast by using Fast Fourier Transform and the Toeplitz matrix theory. Set

$$a_j = \beta_{a,N+l-1}, j = -N/2+1\ldots,-1; \quad a_j = \beta_{a,N/2-j-1}, j = 0,\ldots N/2-1; \quad a_{-N/2} = 0.$$

The sequence $\{a_j\}_{j=-N/2+1}^{N/2-1}$ generates the truncated Toeplitz matrix $T(a)$:

$$T_{N/2}(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots & a_{-N/2+1} \\ a_1 & a_0 & a_{-1} & \ldots & a_{-N/2+2} \\ a_2 & a_1 & a_0 & \ldots & a_{-N/2+3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{N/2-1} & a_{N/2-2} & a_{N/2-3} & \ldots & a_0 \end{pmatrix}. \tag{3.16}$$

Then

$$\sum_{k=0}^{N/2-1}\beta_{a,N/2+k-n}\theta_{m,k} = T_{N/2}(a)\widetilde{\theta}_m,$$

where $\widetilde{\theta}_m = \{\theta_{m,0},\theta_{m,1},\ldots\theta_{m,N/2-1},\underbrace{0,0,\ldots,0,0}_{N/2 \text{ times}}\}$.

The symbol $a(\eta)=\sum_{j=-N/2+1}^{j=N/2-1} a_j e^{i\eta j}$ of the Toeplitz matrix $T(a)$ can be computed in the points $\eta_k=-2\pi k/N, k=0,...,N-1$ via the discrete Fourier transform (3.5):

$$a(\eta_k)=DFT[\widetilde{a}](k)=\sum_{j=0}^{N/2} a_j e^{-2\pi ikj/N}+\sum_{j=N/2+1}^{N/2-1} a_{j-N}e^{-2\pi ikj/N},\quad k=0,...,N-1. \tag{3.17}$$

where $\widetilde{a}=\{a_0,a_1,...a_{N/2},a_{-N/2+1},a_{-N/2+2},...,a_{-2},a_{-1}\}$.

It is easy to show that

$$T_{N/2}(a)\widetilde{\theta}_m=iDFT[DFT[\widetilde{a}]*DFT[\widetilde{\theta}_m]],$$

where $u*v$ is the element-wise product of vectors u with v.

Notice that $\theta_{M,k}$ could be computed explicitly, while the other coefficients $\theta_{m,k}$, $m<M$, are computed using polynomial interpolation (see details in Kirkby (2017a)).

3.2.1. *Treatment for Unbounded Payoffs.* If the payoff function $G(x)$ is unbounded at $+\infty$ (e.g. in the case of call option), then as in Kudryavtsev (2016) we choose real ω in a such way that $e^{\omega x}G(x)$ is absolutely integrable. In the case of the down-and-out call option and typical parameters of the Lévy model $\omega=-2$ is a good choice.

Then we can rewrite the algorithm in terms of new functions:

$$f_\omega(x,t_m)=e^{\omega x}f(x,t_m), m=0,1,\dots,M. \tag{3.18}$$

In this case, one should apply the frame projection method to the weighted transition density $e^{-\omega x}p_{\Delta\tau}(x)$ instead of the function $p_{\Delta\tau}(x)$.

Then taking into account that due (2.1)

$$\int_{-\infty}^{+\infty} e^{i\xi y}e^{-\omega x}p_t(y)dy=e^{-t\psi(\xi+i\omega)},$$

we may rewrite the formulas for (3.7) in (3.8) as follows:

$$g_0=1/24a^2\exp[-\Delta\tau\psi(i\omega)], g_j=\exp(-ix_1\xi_j)\exp[-\Delta\tau\psi(\xi_j+i\omega)]h_a(\xi_j), j>0.$$

If in the cross-barrier event the knock-out option provides the rebate $R>0$ to holders, one can represent $f(x,t_m)$ as $v(x,t_m)+R$ and adjust the algorithm accordingly.

3.3. **American Options.** The frame projection method is extended to Bermudan/American options in [19] for Lévy processes, and [18] for stochastic volatility models. The approach is based on a value recursion, in terms of the frame projected approximation of the transition density $p_{\Delta\tau}$ as follows:

$$\begin{aligned}
&f(x,t_M)=G(x)\\
&\mathcal{C}(x,t_m)=e^{-r\Delta\tau}\int f(y,t_{m+1})p_{\Delta\tau}(y-x)dy,\quad m=M-1,...,0\\
&f(x,t_m)=\begin{cases}\max\{\mathcal{C}(x,t_m),G(x)\} & m=M-1,...,1\\ \mathcal{C}(x,t_m) & m=0\end{cases}
\end{aligned} \tag{3.19}$$

where $\mathcal{C}(x, t_m)$ is the *continuation value* at time t_m. We will again use the frame projection method with a discrete *log-asset grid* $\{x_k\}_{k=1}^N$ over $[l, u]$, which is chosen to ensure that $x_{\bar{k}} = \ln(K/S_0)$ for some fixed index $\bar{k}$, so that the payoff kink is aligned with the grid. The recursive valuation proceeds in a similar manner as for Barrier options described previously, and exploits the Toeplitz structure of the convolution with respect to the projected transition density.

To compute the recursive valuation, we use formula (3.14) where the value coefficients $\theta_{m,k}$ are computed in a similar manner as in (3.15), using

$$\begin{aligned}\theta_{m,k} &= a^{1/2}\int_l^u f(y, t_{m+1}) a^{1/2}\phi(a(y-y_k))dy \\ &= a\int_l^u \max\{\mathcal{C}(y, t_{m+1}), G(x)\}\phi(a(y-y_k))dy.\end{aligned} \tag{3.20}$$

To avoid a loss in convergence order due to the non-smoothness introduced by the $\max\{\cdot\}$ operation (and to determine the explicit optimal exercise policy), we estimate the *early-exercise* point directly before computing the integral. This point, x_*^m, should satisfy $G(x_*^m) = \mathcal{C}(x_*^m, t_m)$. For a Bermudan put option, we first note that the left bracketing index and grid point can be found easily (by binary search at a cost of $\mathcal{O}(\log(N))$) using

$$k_* = \max\{1 \le k \le \bar{k} : G(x_k) - \mathcal{C}(x_k, t_m) \ge 0\}, \qquad x_{k_*} = x_1 + (k_* - 1)\Delta, \tag{3.21}$$

so the early exercise point satisfies $x_{k_*} \le x_*^m < x_{k_*+1}$. Once k_* is found, we approximate x_*^m by

$$x_*^m \approx x_{k_*} + \Delta \frac{G(x_{k_*}) - \mathcal{C}(x_{k_*}, t_m)}{(G(x_{k_*}) - \mathcal{C}(x_{k_*}, t_m)) - (G(x_{k_*+1}) - \mathcal{C}(x_{k_*+1}, t_m))}. \tag{3.22}$$

We can then compute the integral in (3.20) with high accuracy by splitting the domain at the point x_*^m, and preserve the natural rate of convergence (see [19] for more details and closed-form algebraic expressions for $\theta_{m,k}$). Note that this procedure extends naturally to multi-early exercise contracts, as demonstrated in [17] for swing option pricing.

4. Implementation to the Premia 22

We implemented the PROJ-method for

- European options under the CGMY (KoBoL) model (see Example 1), NIG model (see Example 3) and the Kou model (see Example 4).
- American options under the CGMY (KoBoL) model (see Example 1), NIG model (see Example 3) and the Kou model (see Example 4).
- discretely monitored down-and-out and up-and-out call and put with a constant rebate under the CGMY (KoBoL) model (see Example 1). One can use the routine for other types of Lévy processes by replacing the corresponding part with the computation of the characteristic exponents (see the formulas in Examples 1-4).

Note that in the program implemented to Premia 22 for the barrier and American options one can manage by two parameters of the algorithm: the scale of log-price range L and the number of discrete monitoring points M. Parameter L controls the size of the truncated region in x-space. The typical values of the parameter for Lévy models are varying from $L = 8$ to $L = 15$. The number of the x-grid points is fixed at $N = 2^{14}$ inside the code. In the case of European options one can manage by the parameters L only.

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