

CDO Pricing : Copula

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Premia 22

1 Link between CDOs and Copulas

We will thereafter consider a synthetic CDO with some given maturity T . This is based upon n CDS with nominals $N_j, j = 1, \dots, n$ and maturity also equal to T . We denote by δ_j the *recovery rate* for credit j and by $M_j = (1 - \delta_j)N_j$ the corresponding *loss given default*.

For the n names in the collateral pool, we consider the associated default times τ_1, \dots, τ_n defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$. In the following, we will consider only reduced-form models of default times defined by

$$\tau_i = \inf \left\{ u \in \mathbb{R}^+, \int_0^u h_i(v) dv \geq -\log(U_i) \right\}, \quad (H_\tau)$$

where the h_i are deterministic and continuous positive functions, the U_i are some uniform random variables.

In order to compute (by a semi-analytic approach) the price of one CDO tranche, all we need is the portfolio loss distribution *i.e.* the portfolio aggregate loss on the credit portfolio at time t :

$$L(t) = \sum_{j=1}^n M_j \mathbf{1}_{\{\tau_j \leq t\}}$$

which is a pure jump process. This distribution depend on the joint distribution of the default times τ_1, \dots, τ_n that we modelling using a classical factor approach and Copula functions.

We denote by F and S respectively the *joint distribution* and survival functions such that for all $(t_1, \dots, t_n) \in [0, T]^n$, $F(t_1, \dots, t_n) = \mathbb{P}(\tau_1 \leq t_1, \dots, \tau_n \leq t_n)$ and $S(t_1, \dots, t_n) = \mathbb{P}(\tau_1 > t_1, \dots, \tau_n > t_n)$. F_1, \dots, F_n represent the *marginal* distribution functions and S_1, \dots, S_n the corresponding survival functions. By the assumption (H_τ) , we have

$$S_i(t) = \mathbb{P}(\tau_i > t) = \exp\left(-\int_0^t h_i(v) dv\right). \quad (1)$$

We refer to Appendix for the proof of (1).

We will consider now a latent factor V such that conditionally on V , the default times are independent. We will denote by $p_t^{i|V} = \mathbb{P}(\tau_i \leq t|V)$ and $q_t^{i|V} = 1 - p_t^{i|V}$ the conditional default and survival probabilities. It is easy to check that

$$S(t_1, \dots, t_n) = \int \prod_{i=1}^n q_t^{i|v} f(v) dv,$$
$$F(t_1, \dots, t_n) = \int \prod_{i=1}^n p_t^{i|v} f(v) dv.$$

So, if we can easily compute the conditional default probabilities and integrate along the density of the factor V , we are able to compute the joint distribution of the default times.

Définition 1. A copula C is a multivariate joint distribution on the m -dimensional unit cube $[0, 1]^m$ such that every marginal distribution is uniform on the interval $[0, 1]$.

$$C : (u_1, \dots, u_m) \in [0, 1]^m \mapsto \mathbb{P}(U_1 \leq u_1, \dots, U_m \leq u_m)$$

U_1, \dots, U_m is a random vector whose marginals are uniform on $[0, 1]$.

2 Gaussian Copula

We consider a standard Gaussian random variable V , and we define the Gaussian vector (X_1, \dots, X_n) by

$$X_i = \rho V + \sqrt{1 - \rho^2} V_i$$

where V_i are *independent* ($\forall i, j, V_i \perp V_j$ and $\forall i, V_i \perp V$) standard Gaussian random variables. We define the uniform random variable $U_i = 1 - \mathcal{N}(X_i)$ where \mathcal{N} is the cumulative distribution function of a standard Gaussian variable. The joint distribution of (U_1, \dots, U_n) is known as the Gaussian copula. Then, we get

$$p_t^{i|V} = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(F_i(t)) - \rho V}{\sqrt{1 - \rho^2}}\right). \quad (2)$$

with $F_i(t) = 1 - \exp(-\int_0^t h_i(v) dv)$.

We refer to Appendix for the proof of (2).

3 Clayton Copula

We consider a positive random variable V following a standard Gamma distribution $\Gamma(\lambda, \alpha)$ with parameters $\lambda = 1, \alpha = 1/\theta$ where $\theta > 0$. Its probability density is given by $f(x) = \frac{1}{\Gamma(\theta-1)} \exp(-x)x^{(1-\theta)/\theta}$ for $x > 0$. We define the uniform random variables U_1, \dots, U_n

$$U_i = 1 - \Psi\left(-\frac{\log(\bar{U}_i)}{V}\right),$$

where $\bar{U}_1, \dots, \bar{U}_n$ are independent uniform random variables also independent from V , and Ψ is the Laplace transform of f . The joint distribution of (U_1, \dots, U_n) is known as the Clayton copula.

The conditional default probabilities can be expressed as

$$p_t^{i|V} = \exp\left(V(1 - F_i(t))^{-\theta}\right),$$

with $F_i(t) = 1 - \exp(-\int_0^t h_i(v) dv)$.

4 NIG Copula

For more details on the NIG copula, we refer to [1]. The Normal Inverse Gaussian distribution (NIG) is a mixture of normal and inverse Gaussian distributions. A non-negative r.v.

Y has inverse Gaussian distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function is of the form:

$$f_{IG}(y; \alpha, \beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}} y^{-3/2} \exp\left(-\frac{(\alpha-\beta y)^2}{2\beta y}\right) & \text{if } y > 0 \\ 0 & y \leq 0. \end{cases}$$

A r.v. X follows a Normal Inverse Gaussian (NIG) distribution with parameters α, β, μ and δ if:

$$\begin{aligned} X|Y = y &\sim \mathcal{N}(\mu + \beta y, y) \\ Y &\sim IG(\delta\gamma, \gamma^2) \text{ with } \gamma := \sqrt{\alpha^2 - \beta^2}, \end{aligned}$$

with parameters satisfying the following conditions: $0 \leq |b| < a$, $\delta > 0$. We write $X \sim \mathcal{NIG}(\alpha, \beta, \mu, \delta)$ and the density function is given by:

$$f_{NIG}(x; \alpha, \beta, \mu, \delta) = \frac{\delta\alpha \exp(\delta\gamma + \beta(x - \mu))}{\pi\sqrt{\delta^2 + (x - \mu)^2}} K_1(\alpha\sqrt{\delta^2 + (x - \mu)^2}),$$

where $K_1(w) = \frac{1}{2} \int_0^\infty \exp(-\frac{1}{2}w(t + t^{-1}))dt$ is the modified Bessel function of the third kind. The probability function is given by

$$\begin{aligned} F_{NIG}(x) &:= \int_{-\infty}^x f_{NIG}(t)dt = \int_0^\infty \mathcal{N}\left(\frac{x - (\mu + \beta y)}{\sqrt{y}}\right) f_{IG}(y; \delta\gamma, \gamma^2) dy \\ &= \int_0^1 \mathcal{N}\left(\frac{x - (\mu - \beta \log(t))}{\sqrt{-\log(t)}}\right) f_{IG}(-\log(t); \delta\gamma, \gamma^2) \frac{1}{t} dt \end{aligned}$$

The first equality is due to the fact that the NIG distribution stems from a convolution of the normal and the inverse Gaussian distribution. The second one follows from the change of variable $t = \exp(-y)$.

The main properties of the NIG distribution class are the scaling property

$$X \sim \mathcal{NIG}(\alpha, \beta, \mu, \delta) \implies cX \sim \mathcal{NIG}\left(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta\right)$$

and the closure under convolution for independent r.v. X and Y

$$\begin{aligned} X &\sim \mathcal{NIG}(\alpha, \beta, \mu_1, \delta_1), Y \sim \mathcal{NIG}(\alpha, \beta, \mu_2, \delta_2) \\ \implies X + Y &\sim \mathcal{NIG}(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2) \end{aligned}$$

In our implementation, we consider a random variable V following a NIG distribution with parameters

$$V \sim \mathcal{NIG}\left(\alpha, \beta, -\frac{\alpha\beta}{\gamma}, \alpha\right)$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$, and we define the vector (X_1, \dots, X_n)

$$X_i = \rho V + \sqrt{1 - \rho^2} V_i,$$

where V_i are independent (and independent from V) NIG random variables with parameters

$$V_i \sim \text{NIG}\left(\frac{\sqrt{1-\rho^2}}{\rho}\alpha, \frac{\sqrt{1-\rho^2}}{\rho}\beta, -\frac{\sqrt{1-\rho^2}}{\rho}\frac{\alpha\beta}{\gamma}, \frac{\sqrt{1-\rho^2}}{\rho}\alpha\right).$$

To simplify notations we denote $F_{\text{NIG}(s)}(x)$ the cumulative distribution of a NIG random variable with parameters $\text{NIG}\left(s\alpha, s\beta, -s\frac{\alpha\beta}{\gamma}, s\alpha\right)$. Using the scaling property and stability under convolution of NIG distribution we get $X_i \sim \text{NIG}(1/\rho)$. We define the uniform random variable $U_i = 1 - F_{\text{NIG}(1/\rho)}(X_i)$. The joint distribution of (U_1, \dots, U_n) is known as the NIG copula. We get

$$p_t^{i|V} = F_{\text{NIG}(\sqrt{1-\rho^2}/\rho)}\left(\frac{F_{\text{NIG}(1/\rho)}^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\right).$$

5 Student Copula

Définition 2. Let G and Y be two independent r.v. s.t. $G \sim \mathcal{N}(0, 1)$ and $Y \sim \chi^2(\nu)$. Then, $X := \sqrt{\frac{\nu}{Y}}G$ is a Student r.v. with parameter ν . The density of the Student law is given by

$$t_\nu(t) := \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}} \frac{1}{\left(1 + \frac{t^2}{\nu}\right)^{(\nu+1)/2}}, t \in \mathbb{R}.$$

In our implementation, we define for $i = 1, \dots, n$

$$X_i = \sqrt{\frac{\nu}{Y}} \left(\rho V + \sqrt{1-\rho^2} V_i \right),$$

where V , Y and V_1, \dots, V_n are independent r.v. s.t. $V \sim \mathcal{N}(0, 1)$, $V_i \sim \mathcal{N}(0, 1)$ and $Y \sim \chi^2(\nu)$. X_i is a Student r.v. with parameter ν . Let us define $U_i := 1 - T_\nu(X_i)$, where T_ν is the cumulative distribution function of a Student r.v. with parameter ν . The joint distribution of (U_1, \dots, U_n) is known as the Student copula. We get

$$p_t^{i|V} = \mathcal{N}\left(\frac{T_\nu^{-1}(F_i(t))\sqrt{\frac{\nu}{Y}} - \rho V}{\sqrt{1-\rho^2}}\right). \quad (3)$$

The proof of (3) is postponed to the Appendix.

6 Double T Copula

Let M be a Student r.v. s.t. $M \sim S(\nu)$. We define the vector (X_1, \dots, X_n) by

$$X_i := \rho \sqrt{\frac{\nu-2}{\nu}} M + \sqrt{1-\rho^2} \sqrt{\frac{\bar{\nu}-2}{\bar{\nu}}} Z_i,$$

where $(Z_i)_{i=1\dots n}$ are independent Student variables with parameter $\bar{\nu}$. Let $T_{\nu, \bar{\nu}}$ denote the cumulative distribution of X_i . The definition of $T_{\nu, \bar{\nu}}$ can be found by using the law of M and Z_i :

$$\mathbb{P}(X_i \leq z) = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left(\rho \sqrt{\frac{\nu-2}{\nu}} x + \sqrt{1-\rho^2} \sqrt{\frac{\bar{\nu}-2}{\bar{\nu}}} y \right) \mathbf{1}_{x+y \leq z} t_\nu(x) t_{\bar{\nu}}(y).$$

Then, we define $U_i := 1 - T_{\nu, \bar{\nu}}(X_i)$. The joint distribution of (U_1, \dots, U_n) is known as the Double-t copula. We get

$$p_t^{i|V} = T_{\bar{\nu}}\left(\frac{T_{\nu, \bar{\nu}}^{-1}(F_i(t)) - \rho\sqrt{\frac{\nu-2}{\bar{\nu}}}M}{\sqrt{1 - \rho^2}\sqrt{\frac{\bar{\nu}-2}{\bar{\nu}}}}\right). \quad (4)$$

The proof of (4) is similar to the proof of (3).

7 Numerical values for the Copula parameters

Type	Parameters (example value)
Gaussian	Correlation ρ (0.03)
Clayton	θ (0.2)
NIG	Correlation ρ (0.06), α (1.2), β (-0.2)
Student	Correlation ρ (0.02), Degree of freedom $t1$ (5)
Double t	Correlation ρ (0.03), Degree of freedom $t1$ (5), Degree of freedom $t2$ (7)

8 Appendix

8.1 Proof of (1)

We have

$$\begin{aligned} \tau_i &= \inf\{u \in \mathbb{R}^+ : -\int_0^u h_i(v)dv \geq -\log(U_i)\}, \\ &= \inf\{u \in \mathbb{R}^+ : \exp(-\int_0^u h_i(v)dv) \leq U_i\}. \end{aligned}$$

Since $u \mapsto \exp(-\int_0^u h_i(v)dv)$ is a decreasing function (denoted f) with values in $[0, 1]$, we get

$$\mathbb{P}(\tau_i > t) = \mathbb{P}(\inf\{u \in \mathbb{R}^+ : f(u) \leq U_i\} > t) = \mathbb{P}(f(t) > U_i) = f(t) = \exp(-\int_0^t h_i(v)dv).$$

8.2 Proof of (2)

U_i is a uniform r.v.: $1 - U_i = \Phi(X_i)$ and Φ is the cumulative density function of X_i , then $\Phi(X_i) \sim \mathcal{U}[0, 1]$. ($\mathbb{P}(U_i \leq k) = \mathbb{P}(\Phi(X_i) \leq k) = \mathbb{P}(X_i \leq \Phi^{-1}(k)) = \Phi(\Phi^{-1}(k)) = k$).

From the definition of τ_i , we get $\tau_i = \inf\{t : S_i(t) \leq U_i\} = \inf\{t : 1 - F_i(t) \leq U_i\} = \inf\{t : F_i(t) \geq 1 - U_i\} = \inf\{t : F_i(t) \geq \Phi(X_i)\} = \inf\{t : X_i \leq \Phi^{-1}(F_i(t))\}$. Then

$$\begin{aligned} \mathbb{P}(\tau_i \leq t|V) &= \mathbb{P}(\inf\{u \in \mathbb{R}^+ : X_i \leq \Phi^{-1}(F_i(u))\} \leq t|V) \\ &= \mathbb{P}(\Phi^{-1}(F_i(t)) > X_i|V) \text{ since } u \mapsto \Phi^{-1}(F_i(u)) \text{ is increasing} \\ &= \mathbb{P}(\rho V + \sqrt{1 - \rho^2}\bar{V}_i < \Phi^{-1}(F_i(t))|V) \\ &= \mathbb{P}\left(\bar{V}_i < \frac{\Phi^{-1}(F_i(t)) - \rho V}{\sqrt{1 - \rho^2}}|V\right) \\ &= \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho V}{\sqrt{1 - \rho^2}}\right). \end{aligned}$$

8.3 Clayton Copula

The Laplace transform of f is $\Psi(s) = \frac{1}{(1+s)^{\theta-1}}$. Indeed,

$$\begin{aligned}\Psi(s) &= \frac{1}{\Gamma(\theta-1)} \int_0^\infty e^{-sx} e^{-x} x^{\frac{1}{\theta}-1} dx \\ &= \frac{1}{\Gamma(\theta-1)} \int_0^\infty e^{-(s+1)x} x^{\frac{1}{\theta}-1} dx \\ &= \frac{1}{\Gamma(\theta-1)} \int_0^\infty e^{-t} t^{\frac{1}{\theta}-1} dt \frac{1}{(s+1)^{\theta-1}}.\end{aligned}$$

Since $\Gamma(\theta-1) = \int_0^\infty e^{-t} t^{\frac{1}{\theta}-1} dt$, we get the result The inverse function is $\Psi^{-1}(s) = s^{-\theta} - 1$. The r.v. U_i are uniform:

$$\begin{aligned}\mathbb{P}(U_i \leq k) &= \mathbb{P}(1 - U_i \leq k) = \mathbb{P}\left(\Psi\left(-\frac{\log(\bar{U}_i)}{V}\right) \leq k\right) = \mathbb{P}\left(-\frac{\log(\bar{U}_i)}{V} \geq \Psi^{-1}(k)\right) \\ &= \mathbb{P}(-\log(\bar{U}_i) \geq V(k^{-\theta} - 1)) = \mathbb{P}(\bar{U}_i \leq \exp(V(1 - k^{-\theta}))) \\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\bar{U}_i \leq \exp(V(1 - k^{-\theta}))} | V]] = \mathbb{E}[\mathbb{P}(\bar{U}_i \leq \exp(V(1 - k^{-\theta})) | V)] \\ &= \mathbb{E}[\exp(V(1 - k^{-\theta}))] = \frac{1}{\Gamma(\theta-1)} \int_0^\infty \exp(v(1 - k^{-\theta})) e^{-v} v^{\frac{1}{\theta}-1} dv \\ &= \frac{1}{\Gamma(\theta-1)} \int_0^\infty e^{-vk^{-\theta}} v^{\frac{1}{\theta}-1} dv = \frac{1}{\Gamma(\theta-1)} \int_0^\infty e^{-x} x^{\frac{1}{\theta}-1} dx \frac{1}{k^{-\theta}} \left(\frac{1}{k^{-\theta}}\right)^{\frac{1}{\theta}-1} = k.\end{aligned}$$

Let us prove that the conditional default probability is $p_t^{i|V} = \exp(V(1 - F_i(t)^{-\theta}))$. To do so, we write $\tau_i = \inf\{t : S_i(t) \leq U_i\} = \inf\{t : 1 - F_i(t) \leq U_i\} = \inf\{t : F_i(t) \geq 1 - U_i\} = \inf\{t : F_i(t) \geq \Psi\left(-\frac{\log(\bar{U}_i)}{V}\right)\} = \inf\{t : \bar{U}_i \leq \exp(-V\Psi^{-1}(F_i(t)))\}$. Then, since $u \mapsto \exp(-V\Psi^{-1}(F_i(u)))$ is an increasing function, we get

$$\begin{aligned}\mathbb{P}(\tau_i \leq t | V) &= \mathbb{P}(\inf\{u \in \mathbb{R}^+ : \bar{U}_i \leq \exp(-V\Psi^{-1}(F_i(u)))\} \leq t | V) \\ &= \mathbb{P}(\bar{U}_i \leq \exp(-V\Psi^{-1}(F_i(t))) | V) = \exp(V(1 - F_i(t)^{-\theta})).\end{aligned}$$

8.4 Proof of (3)

From the definition of τ_i , we get $\tau_i = \inf\{t : S_i(t) \leq U_i\} = \inf\{t : 1 - F_i(t) \leq U_i\} = \inf\{t : F_i(t) \geq 1 - U_i\} = \inf\{t : F_i(t) \geq T_\nu^{-1}(X_i)\} = \inf\{t : X_i \leq T_\nu^{-1}(F_i(t))\}$. Then

$$\begin{aligned}\mathbb{P}(\tau_i \leq t | V) &= \mathbb{P}(\inf\{u \in \mathbb{R}^+ : X_i \leq T_\nu^{-1}(F_i(u))\} \leq t | V) \\ &= \mathbb{P}(T_\nu^{-1}(F_i(t)) > X_i | V) \text{ since } u \mapsto T_\nu^{-1}(F_i(u)) \text{ is increasing} \\ &= \mathbb{P}\left(\rho V + \sqrt{1 - \rho^2} V_i < \sqrt{\frac{Y}{\nu}} T_\nu^{-1}(F_i(t)) | V\right) \\ &= \mathbb{P}\left(V_i < \frac{\sqrt{\frac{Y}{\nu}} T_\nu^{-1}(F_i(t)) - \rho V}{\sqrt{1 - \rho^2}} | V\right) \\ &= \mathcal{N}\left(\frac{\sqrt{\frac{Y}{\nu}} T_\nu^{-1}(F_i(t)) - \rho V}{\sqrt{1 - \rho^2}}\right).\end{aligned}$$

References

- [1] Kalemánova, Schmid, Werner (2005) *The Normal Inverse Gaussian distribution for synthetic CDO pricing*. Preprint. [2](#)