

Cutting CVA's complexity

March 3, 2020

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The following method computes the credit value adjustment (CVA in the following) by using marked branching diffusions. It is based on the paper [1].

1 Introduction

This method computes the price of a Credit Value Adjustment in a Markovian model, in the presence of counterparty-risk or not. In each case, the mark-to-market value satisfies a non-linear PDE. The method developed to solve these PDEs is based on branching diffusions describing a marked Galton-Watson random tree.

2 Theoretical framework

We assume the issuer is allowed to dynamically trade one dimensional underlying asset X . Additionally, to hedge his credit risk on the counterparty name, he can trade a default risky bond, denoted P_t . The values of the underlyings are not altered by the counterparty default which is modeled by a Poisson jump process with constant intensity. We consider a long position in a single derivative denoted u , whose payoff is denoted Ψ . The processes X and P satisfy under the risk-neutral measure \mathbb{P}

$$\begin{aligned}dX_t &= r X_t dt + \sigma X_t dW_t, X_0 = x \\dP_t &= (r + \lambda) P_t dt - P_t dJ_t,\end{aligned}$$

where W is a Brownian motion and J is a jump Poisson process with intensity λ and r is the interest rate. The no arbitrage condition and the completeness of the market give that $e^{-rt}u(t, X_t)$ is a \mathbb{P} -martingale characterized by

$$\partial_t u + \mathcal{L}u + \lambda(\tilde{u} - u) - ru = 0,$$

where \mathcal{L} is the Itô generator of X and \tilde{u} is the derivative value after the counterparty has defaulted. At the default event, $\tilde{u} = RM^+ - M^-$ where M is the mark-to-market value of the derivative to be used in the unwinding of the position upon default and R is the recovery rate.

- When the mark-to-market value is calculated with provision for counterparty risk, $M = u$:

$$\partial_t u + \mathcal{L}u - (1 - R)\lambda u^+ - ru = 0, \quad u(T, x) = \Psi(x)$$

- When the mark-to-market value is calculated without provision for counterparty risk:

$$\begin{aligned}\partial_t u + \mathcal{L}u + \lambda(RM^+ - M^- - u) - ru &= 0, \quad u(T, x) = \Psi(x) \\ \partial_t M + \mathcal{L}M - rM &= 0, \quad M(T, x) = \Psi(x).\end{aligned}$$

By discounting and replacing u by $-u$, the two PDEs can be rewritten :

$$\partial_t u + \mathcal{L}u + \beta(u^+ - u) = 0, \quad u(T, x) = \Psi(x) \text{ PDE2}$$

$$\partial_t u + \mathcal{L}u + \frac{\beta}{1-R} \left((1-R)(\mathbb{E}_{t,x}(\Psi))^+ + R\mathbb{E}_{t,x}(\Psi) - u \right) = 0, \quad u(T, x) = \Psi(x) \text{ PDE1}$$

with $\beta = \lambda(1-R)$.

3 Market Branching diffusion

3.1 Case PDE2

In order to derive an algorithm solving PDE2, we first introduce the following PDE

$$\partial_t u + \mathcal{L}u + \beta(F(u) - u) = 0, \quad u(T, x) = \Psi(x), \quad (1)$$

where $F(u) = \sum_{k=0}^M a_k u^k = \sum_{k=0}^M \frac{a_k}{p_k} p_k u^k$, where $\sum_{k=0}^M p_k = 1$ and $\forall k, p_k \in [0, 1]$.

In order to approximate PDE2 by an equation of type (1), we have to approximate u^+ by a polynomial decomposition. Referring to [1, Equation (21)], we approximate u^+ by

$$F(u) := 0.0589 + 0.5u + 0.8164u^2 - 0.4043u^4. \quad (2)$$

Galton-Watson tree The probabilistic interpretation of equation (1) is the following. Let a single particle start at the origin, perform an Itô diffusion with generator \mathcal{L} , and after an exponential time with mean $\frac{1}{\beta}$ (independent of X) die and produce k descendants with probability p_k . Then, independently each descendant performs a Itô diffusion with generator \mathcal{L} , die and produce k descendants with probability p_k , and so on until T (If there are N_t particles at time t , the first time a particle dies follows an exponential law of parameter $N_t\beta$). We denote by $Z_t = (z_t^1, \dots, z_t^{N_t})$ the positions of the N_t particles alive at time t . From [1], we get that

$$\hat{u}(t, x) := \mathbb{E}_{t,x} \left[\prod_{k=0}^M \left(\frac{a_k}{p_k} \right)^{\omega_k} \prod_{i=1}^{N_T} \Psi(z_T^i) \right],$$

is the unique viscosity solution of (1). ω_k denotes, for each Galton-Watson tree, the number of particles that branch into k descendants with $k \in \{0, \dots, M\}$. $1 + \sum_{k=0}^M (k-1)\omega_k$ gives the total number of individuals produced by the branching $\omega = (\omega_0, \dots, \omega_M)$. These descendants are drawn with distribution p_k given by

$$p_k := \frac{|a_k| \|\Psi\|_\infty^k}{\sum_{i=0}^M |a_i| \|\Psi\|_\infty^i}.$$

This choice of probability is optimal for convergence (see [1, page 70]).

3.2 Case PDE1

We assume that the function $(1-R)x^+ + Rx$ can be well approximated by a polynomial function $F(x)$. We consider the following PDE, which approximates PDE1

$$\partial_t u(t, x) + \mathcal{L}u(t, x) + \frac{\beta}{1-R} (F(\mathbb{E}_{t,x}[\Psi(X_T)]) - u(t, x)) = 0, u(T, x) = \Psi(x).$$

From Feynman Kac formula, we have

$$u(t, x) = \mathbb{E}_{t,x}[\mathbb{1}_{\{\tau \geq T\}} \Psi(X_T)] + \mathbb{E}_{t,x}[\mathbb{1}_{\{\tau < T\}} F(\mathbb{E}_\tau(\Psi(X_T)))]$$

where τ is a Poisson default time with intensity $\frac{\beta}{1-R}$. The term $\mathbb{E}_{t,x}[\mathbb{1}_{\{\tau < T\}} F(\mathbb{E}_\tau(\Psi(X_T)))]$ can be calculated using the previous algorithm by imposing that the particle can default only once.

4 Algorithm

The algorithm for solving PDE1 and PDE2 is the following

1. Choose a polynomial approximation of $u^+ \sim \sum_{k=0}^M a_k u^k$ on the domain $[-1, 1]$
2. For each Monte Carlo simulation
 - (a) simulate the asset and the first Poisson default time with intensity β (resp. $\frac{\beta}{1-R}$) for PDE2 (resp. for PDE1)
 - (b) at each default time, produce k descendants with probability p_k . For PDE1, descendants produced after the first default become immortal
 - (c) evaluate the quantities

$$\prod_{k=0}^M \left(\frac{a_k}{p_k} \right)^{\omega_k} \prod_{i=1}^{N_T} \Psi(Z_T^i) \text{ for PDE2}$$

$$\left(\frac{a_1(1-R) + R}{p_1} \right)^{\omega_1} \prod_{k \neq 1}^M \left(\frac{a_k(1-R)}{p_k} \right)^{\omega_k} \prod_{i=1}^{N_T \in [0, M]} \Psi(Z_T^i) \text{ for PDE1}$$

In case of PDE1, $\sum_{k=0}^M \omega_k = 0$ or 1 .

5 Implementation and numerical experiments

We test the algorithm with the following parameters :

r	x	σ	R	$\Psi(x)$	K	β	MC
0.05	0.7	0.2	0.4	$1 - 2\mathbb{1}_{\{x > K\}}$	1	0.03	2^{22}

MC represents the number of Monte Carlo simulations.

We compute the value of the mark to market value at time $t = 0$, for different values of T

T	PDE1	PDE2
2	0.7040	0.7031
10	0.1060	0.1131

References

- [1] P. Henry-Labordère. Cutting CVA's complexity *Risk Magazine*, p 67-73, 2012. [1](#), [2](#)