Swaption approximation formula in the Libor Market Model

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March 4, 2020

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1 Libor Market Model

See the document 1 for the presentation of the LMM model.

We note $F_i(t)$ the libor rate set at T_i, payed at T_{i+1} and $\sigma_i(t)$ its volatility.

2 Swaption approximation formula

This part is taken from the book of Brigo&Mercurio, cf [1].

We consider a swaption striked at K, with a tenor $T_s, T_{s+1}, ..., T_e$ with T_s the maturity of the option, payement dates $T_{s+1}, ..., T_e$.

The rate of the underlying swap is a function of the libor rates, we can write it as follow :

$$S(t, T_e, T_s) = \sum_{i=s}^{e-1} \omega_i(t) F_i(t)$$

This is not a linear combination because the coeffients $\omega_i(t)$ depends on the rates $F_i(t)$ (cf [1] for an expression of the weights ω_i). A first approximation consist to freeze the weights $\omega_i(t)$ at time 0 :

$$S(t, T_e, T_s) \approx \sum_{i=s}^{e-1} \omega_i(0) F_i(t)$$

Then we calculate the percentage variance of the swap rate using the approximation above :

$$dS(t, T_e, T_s) \approx (...)dt + \sum_{i=s}^{e-1} \omega_i(0)\sigma_i(t) F_i(t) dW_i(t)$$

So the bracket of $S(t, T_e, T_s)$ is:

$$< \frac{dS(t, T_e, T_s)}{S(t, T_e, T_s)} > \approx \sum_{i,j=s}^{e-1} \frac{\omega_i(0)\omega_j(0) F_i(t) F_j(t) \sigma_i(t)\sigma_j(t)}{S(t, T_e, T_s)^2} dt$$

A second approximation is to freeze the $F_i(t)$ and $S(t, T_e, T_s)$ at time 0 :

$$< \frac{dS(t, T_e, T_s)}{S(t, T_e, T_s)} > \approx \sum_{i,j=s}^{e-1} \frac{\omega_i(0)\omega_j(0) F_i(0) F_j(0) \sigma_i(t)\sigma_j(t)}{S(0, T_e, T_s)^2} dt$$

Using this formula, we can compute the Black's volatility $v(T_e, T_s)$ of the swaption as the integral of the percentage variance of $S(t, T_e, T_s)$:

$$\begin{aligned} v(T_e, T_s)^2 &= \int_0^{T_e} < \frac{dS(t, T_e, T_s)}{S(t, T_e, T_s)} > \\ &\approx \int_0^{T_e} \sum_{i,j=s}^{e-1} \frac{\omega_i(0)\omega_j(0) F_i(0) F_j(0) \sigma_i(t)\sigma_j(t)}{S(0, T_e, T_s)^2} dt \\ &\approx \sum_{i,j=s}^{e-1} \frac{\omega_i(0)\omega_j(0) F_i(0) F_j(0)}{S(0, T_e, T_s)^2} \int_0^{T_e} \sigma_i(t)\sigma_j(t) dt \end{aligned}$$

We then put this quantity in the Black's formula to have the price of the swaption.

References

[1] D. Brigo, F. Mercurio , *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit (Springer Finance)*

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References