

Backward Convolution Algorithm for Discretely Sampled Asian Options

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Premia 22

Abstract

This note give a summary of the backward price convolution algorithm used in [1] to price discretely sampled Asian options. For more details see [1].

1 Introduction

Let $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a probability space. We consider an underlying price of a risky asset given by

$$S_j = S_0 \exp \left(\sum_{k=1}^j Z_k \right) \quad \text{for } j = \{1, \dots, n\}, \quad (1.1)$$

and a process of partial sums defined by

$$I_j = \sum_{k=0}^j \lambda_k S_k \quad \text{for } j = \{1, \dots, n\}, \quad (1.2)$$

where $(Z_k)_{1 \leq k \leq n}$ is a collection of independent random variables and the deterministic process λ depends of the type of the Asian option (see Table 1).

As proved by [1], the price of an Asian option amounts to calculating the quantity

$$\mathbb{E} \left(I_n^+ \right). \quad (1.3)$$

Introduce a new filtration $\mathcal{G} := (G_i)_{1 \leq i \leq n}$

$$\mathcal{G}_i = \sigma(Z_n, Z_{n-1}, \dots, Z_{n-i+1}),$$

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Option type	λ_0	$\lambda_1, \dots, \lambda_{n-1}$	λ_n
Call, fixed strike	$\frac{\gamma}{n+\gamma} - \frac{K}{S_0}$	$\frac{1}{n+\gamma}$	$\frac{1}{n+\gamma}$
Call, floating strike	$-\frac{\gamma\alpha}{n+\gamma}$	$-\frac{\alpha}{n+\gamma}$	$1 - \frac{\alpha}{n+\gamma}$
Put, fixed strike	$\frac{K}{S_0} - \frac{\gamma}{n+\gamma}$	$-\frac{1}{n+\gamma}$	$-\frac{1}{n+\gamma}$
Put, floating strike	$\frac{\gamma\alpha}{n+\gamma}$	$\frac{\alpha}{n+\gamma}$	$\frac{\alpha}{n+\gamma} - 1$

Table 1: Choice of λ corresponding to different types of Asian options. $\alpha > 0$ is the coefficient of partiality for floating strike options. Coefficient γ takes value 1 (0) when S_0 is (is not) included in the average.

and a process X defined by

$$\begin{aligned} X_k &:= \lambda_{n-k} + X_{k-1} \exp(Z_{n+1-k}) \\ X_0 &:= \lambda_n : \end{aligned}$$

Proposition 2.1 of [1] shows that X is a \mathcal{G} -Markov process under \mathbb{P} .

2 Backward Price Convolution Algorithm

To price Asian options [1] have used a backward algorithm described in the following theorem (Theorem 3.1 in [1]). Note that we should have $\lambda_k > 0$ for any $k \in \{1, \dots, n\}$. This is the case for the fixed strike Asian call.

Theorem 2.1. *Assume that for all k the CDF of Z_{n+1-k} has a probability density function f_k with respect to the Lebesgue measure on \mathbb{R} , satisfying*

$$\mu_k := \int_{\mathbb{R}} e^z f_k(z) dz < \infty.$$

Consider constants $\lambda_k > 0$, $0 < k \leq n$ and $\lambda_0 \in \mathbb{R}$. Define functions $p_k: \mathbb{R} \rightarrow \mathbb{R}$ for $0 < k \leq n$ and $q_k, h_k: \mathbb{R} \rightarrow \mathbb{R}$ for $0 \leq k < n$ as follows

$$\begin{aligned} p_n(y) &:= (e^y + \lambda_0)^+; \\ h_k(y) &:= \log(e^y + \lambda_{n-k}), \quad 0 < k < n, \\ q_{k-1}(x) &:= \int_{\mathbb{R}} p_k(x+z) f_k(z) dz, \quad 0 < k \leq n, \\ p_{k-1}(y) &:= q_{k-1}(h_{k-1}(y)), \quad 1 < k \leq n. \end{aligned}$$

The following statements hold:

1. *The forward price of an Asian call contract with parameters $(\lambda_j)_{0 \leq j \leq n}$ is given by*

$$\mathbb{E}(I_n^+) = S_0 \mathbb{E}(X_n^+) = S_0 q_0(\log(\lambda_n)).$$

2. There are positive constants a_k, b_k such that for all $x, y \in \mathbb{R}$

$$\begin{aligned} 0 &\leq p_k(y) \leq a_k e^y + b_k, \\ 0 &\leq q_k(x) \leq a_k e^x + b_k + 1. \end{aligned}$$

These constants are given recursively by

$$\begin{aligned} a_n &= 1, \quad b_n = \lambda_0^+ \\ a_{k-1} &= a_k \mu_k \\ b_{k-1} &= b_k + a_{k-1} \lambda_{n-k+1}. \end{aligned}$$

The range of integration in the above theorem must be curtailed. So functions p_k and q_k are approximated by functions \bar{p}_k and \bar{q}_k defined by

$$\begin{aligned} \bar{q}_{k-1}(x) &:= \int_{\mathbb{R}} \bar{p}_k(x+z) f_k(z) dz \mathbb{1}_{[\bar{L}_{k-1}, \bar{U}_{k-1}]}(x), \quad 0 < k \leq n \\ \bar{p}_{k-1}(y) &:= \bar{q}_{k-1}(h_{k-1}(y)) \mathbb{1}_{[L_{k-1}, U_{k-1}]}(y), \quad 1 < k \leq n \\ \bar{p}_n(y) &:= p_n(y) \mathbb{1}_{[L_n, U_n]}(y). \end{aligned}$$

The curtailed ranges $[\bar{L}_k, \bar{U}_k]$ for $k \in \{0, \dots, n-1\}$ and $[L_k, U_k]$ for $k \in \{1, \dots, n\}$ are defined in Theorem 3.2 in [1]. The pricing error caused by the curtailment is also estimated in the latter. The idea in [1] is to evaluate \bar{p}_k and \bar{q}_k in the Fourier space. Then, for $0 \leq k < n$ and $x \in (\bar{L}_k, \bar{U}_k)$

$$\bar{q}_{k-1}(x) = \mathcal{F}^{-1} \left(\mathcal{F}(\bar{p}_k) \bar{\phi}_k \right) (x),$$

where \mathcal{F} denotes the Fourier transform, ϕ_k is the characteristic function of Z_{n-k} and $\bar{\phi}_k$ denotes its complex conjugate. The Fourier transform is approximated by the general discrete Fourier transform (see [1], Definition 4.3). The backward algorithm for numerical implementations is summarized by Proposition 4.3 of [1].

References

- [1] CERNY, A., KYRIAKOU I. An Improved Convolution Algorithm for Discretely Sampled Asian Options. To appear in Quantitative Finance (2010). [1](#), [2](#), [3](#)

References