MODEL INDEPENDENT BOUNDS FOR FORWARD-START OPTIONS

1. Introduction

The following method proposed in [1] deals with a method for obtaining model independent bounds for exotic options using linear programming methods.

2. Theoretical Framework

We consider a fixed exotic option depending only on the value of a single asset S at some discrete times $t_1 < \ldots < t_n$ whose payoff we denote with Φ.
The standard no arbitrage approach consist in fixing a model, that is a probability measure P on $\mathbb{R}^n$ under which the coordinate process $(S_i)_{i=1}^n$ is required to be a martingale, calibrated to the existing call options prices $\mathcal{C}(t_i, K)$ $K \in \mathbb{R}$, and then giving as a fair price $\mathrm{E}^{\mathrm{P}}[\Phi]$. This procedure is equivalent to require that each of the one dimensional marginals of P follows a specific law μ_i with Cumulative Distribution function given by

$$F_i(K) = 1 - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\mathcal{C}(t_i, K) - \mathcal{C}(t_i, K + \varepsilon)\right].$$

The primal optimal transport problem consists in considering the set $\mathcal{M}(\mu_1, \ldots, \mu_n)$ of all the martingale measures P on the space $\mathbb{R}^n$ having fixed marginals $\mathrm{P}_1 = \mu_1, \ldots, \mathrm{P}_n = \mu_n$ and finding

$$P = \inf\{\mathrm{E}^{\mathrm{P}}[\Phi],\, \mathrm{P} \in \mathcal{M}(\mu_1, \ldots, \mu_n)\}. \tag{2.1}$$

To this primal problem is associated a dual problem which consist in the construction of the semi-static sub-hedging strategy consisting in the sum of vanilla options and a delta strategy. The strategy we consider are of the form

$$\Psi_{(u_i),(\Delta_j)}(s_1, \ldots, s_n) = \sum_{i=1}^{n} u_i(s_i) + \sum_{j=1}^{n-1} \Delta_j(s_1, \ldots, s_j)(s_{j+1} - s_j) \tag{2.2}$$

for some μ_i-integrable function u_i and bounded function Δ_j, and they are sub-hedging in the sense that

$$\Psi_{(u_i),(\Delta_j)} \leq \Phi.$$

For each martingale measure $\mathrm{P} \in \mathcal{M}(\mu_1, \mu_n)$, we have

$$\mathrm{E}^{\mathrm{P}}[\Phi] \geq \mathrm{E}^{\mathrm{P}}[\Psi_{(u_i),(\Delta_j)}] = \mathrm{E}^{\mathrm{P}}[\sum_{i=1}^{n} u_i(S_i)] = \sum_{i=1}^{n} \mathrm{E}^{\mu_i}[u_i(S_i)], \tag{2.3}$$

which leads to consider the following dual problem:

$$D = \sup\left\{\sum_{i=1}^{n} \mathrm{E}^{\mu_i}[u_i(S_i)] \,\text{such that} \exists \Delta_1, \ldots, \Delta_{n-1} : \Psi_{(u_i),(\Delta_j)} \leq \Psi\right\}. \tag{2.4}$$

It has been proved in [1] that under the condition

$$\Phi(s_1, \ldots, s_n) \geq -K(1 + |s_1| + \ldots + |s_n|).$$

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for some $K > 0$ there is no duality gap, i.e. $P = D$. Moreover if $\Phi(s_1, \ldots, s_n) \leq K(1 + |s_1| + \ldots + |s_n|)$

$$\sup\{\mathrm{E}^{\mathrm{P}}[\Phi],\, \mathrm{P} \in \mathcal{M}(\mu_1, \ldots, \mu_n)\} = \inf \left\{ \sum_{i=1}^{n} \mathrm{E}^{\mu_i}[u_i(S_i)]\, \text{s.t. } \exists \Delta_1, \ldots, \Delta_{n-1} : \Psi_{(u_i),(\Delta_j)} \leq \Psi \right\}$$

In [1] it has also been showed that it is enough to consider sub-hedging (and super-hedging) strategies in which the functions u_i are linear combinations of call options, which restrict us to consider:

$$b + \sum_{i=1}^{n} \sum_{l=1}^{m_i} \mathcal{A}_{i,l}(s_i - K_{i,l})^+ + \sum_{j=1}^{n-1} \Delta_j(s_1, \ldots, s_j)(s_{j+1} - s_j) \tag{2.5}$$

which give price

$$b + \sum_{i=1}^{n} \sum_{l=1}^{m_i} \mathcal{A}_{i,l} \mathcal{C}(t_i, K_{i,l})\,.$$

3. Numerical Algorithm

In our algorithm we consider the case of a Forward start option, that is

$$\Phi = (s_2 - K s_1)^+\,.$$

Due to the dependence only on two maturities, the dual problem associated to the upper bound becomes

$$\begin{cases} \inf b + \sum_{i=1}^{2} \sum_{l=1}^{m_i} \mathcal{A}_{i,l} \mathcal{C}(t_i, K_{i,l}) \\ s.t. \\ F(s_1, s_2) = b + \sum_{i=1}^{2} \sum_{l=1}^{m_i} \mathcal{A}_{i,l}(s_i - K_{i,l})^+ + \Delta_1(s_1)(s_2 - s_1) \geq (s_2 - K s_1)^+\,. \end{cases} \tag{3.1}$$

It is immediate to see that $s_2 \mapsto F(s_1, s_2) - (s_2 - K s_1)^+$ is piecewise linear in s_2 and obtains its extremal values for $s_2 = \{K_{2,l}\}_l, s_2 = 0, s_2 = \infty, s_2 = K s_1$. The above constraints thus reduce to a $m_2 + 3$ constraints parametrized by s_1. Due to the low dimension of the problem it can be easily solved with a simplex method discretizing the distribution of s_1. In the implementation we have used the library GLPK but we plan to soon substitute it with an In house one.

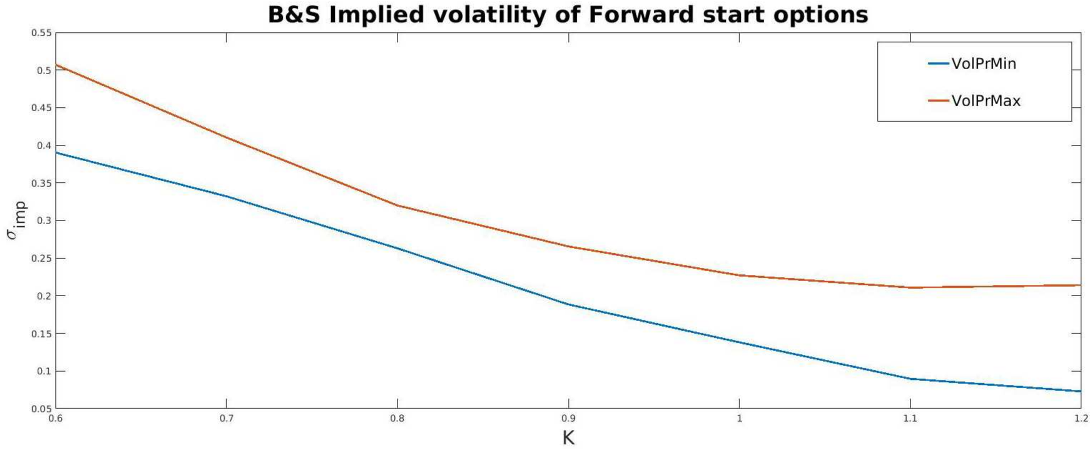
3.1. **Numerical experiments.** The measures μ_1 and μ_2 are deduced from the prices of call options written on the S&P500 with $t_1 = 0.25$ years and $t_2 = 0.78$ years with $m_1 = m_2 = 10$.

We discretized the values of s_1 on a spatial grid with $L = 201$ points and we obtained the results from Table 3.1.

Since in the Forward start option the constraints system has $(m_2 + 3) * L$ rows and $1 + m_1 + m_2 + L$ columns, in the case of study, the system has 2613 rows and 222 columns, for a total of 580086 elements of which only 30394 non zero. We have also tried to increase the number of discretization points up to 500, but we have noticed no significant variation in the optimal prices found.

We notice that the difference in implied volatility is quite significant, between 6 and 15 percentage points as can be also seen in Figure3.1.

K	0.6	0.7	0.8	0.9	1.0	1.1	1.2
Minimum Price	241.37	185.60	129.84	74.73	29.50	2.99	0.04
Maximum Price	249.95	193.44	137.38	88.33	48.47	23.47	11.74
impvol Minimum	0.39	0.33	0.26	0.19	0.14	0.09	0.07
impVol Maximum	0.51	0.41	0.32	0.27	0.23	0.21	0.22



4. Correlated problems: Asian options

A related problem which we studied is the case of Asian option treated in [2]. The main problem in this case is the fast growth of the dimension of the problem as the number of dates for which we study the dual increases.

In fact, even reducing the dimension of the constraints on the last variable as in the Forward start case, the system of constraints has $L^n * (m_2 + 3)$ rows and $1 + \sum_{1=1}^{n} m_i + L\frac{(L^{n-1}-1)}{L-1}$ columns.

References

[1] Beiglböck, M., Henry-Labordère, P. & Penkner, *Model-independent bounds for option prices—a mass transport approach*, F. Finance Stoch (2013) 17: 477. doi:10.1007/s00780-013-0205-8.

[2] Stebegg, F., *Model-independent pricing of Asian options via optimal martingale transport*, arXiv preprint arXiv:1412.1429 (2014).