A Multifactor Volatility Heston Model

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The following method computes the price of call and put options on a single risky asset whose volatility follows a multifactor Wishart process. It is based on the paper [1].

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1 Introduction

This method computes the price of call and put options on a single risky asset whose volatility follows a multifactor Wishart process by using the Fast Fourier Transform as in [2].

2 Theoretical framework

We consider an arbitrage-free frictionless financial market and a risky asset whose price follows

$$\frac{dS_t}{S_t} = r\,dt + Tr\left(\sqrt{\Sigma_t}\,dZ_t\right),$$

where r denotes the risk-free interest rate, Tr is the trace operator, $Z_t \in M_n$ (the set of square matrices of size n) is a matrix of Brownian motions (i.e. composed by n^2 independent Brownian motions) under the risk neutral measure and Σ_t belongs to the set of symmetric $n \times n$ positive definite matrices (as well as its square root $\sqrt{\Sigma_t}$). Σ_t is assumed to follow the dynamics

$$d\Sigma_t = (\beta Q^T Q + M\Sigma_t + \Sigma_t M^T)\,dt + \sqrt{\Sigma_t}dW_t Q + Q^T (dW_t)^T \sqrt{\Sigma_t}, \tag{1}$$

with M and Q in M_n, $W_t \in M_n$ is a matrix Brownian motion and $\beta \geq n-1$ (^{T} represents the transposition). M is a semi-negative matrix. (1) characterizes the Wishart process introduced by Bru (1991).

We also assume correlation between the noises driving the asset and the noises driving the volatility process :

$$\forall k, \mathrm{Cov}(Z_t^{i,k}, W_t^{k,j}) = R_{i,j}.$$

This leads to a constant matrix $R \in M_n$ which completely describes the correlation structure, i.e. $Z_t := W_t R^T + B_t\sqrt{\mathbb{I} - RR^T}$ ($\mathbb{I}$ represents the identity matrix).

3 The pricing problem solved by FFT

We consider a Call option with payoff $(S_T - K)_+$. In order to solve the pricing problem of such a vanilla option, it is enough to compute the conditional characteristic function of the underlying, or equivalently of the return process $Y_t = \ln(S_t)$, which satisfies

$$dY_t = (r - \frac{1}{2}Tr(\Sigma_t))dt + Tr\left(\sqrt{\Sigma_t}(dW w_t R^T + dB_t\sqrt{\mathbb{I} - RR^T})\right).$$

3.1 Characteristic function of the asset returns

From [1, Section 3.1], we get

$$\Phi_{\gamma,0}(T) := \mathbb{E}(\exp(\gamma Y_T)) = \exp(Tr(A(T)\Sigma_0) + b(T)\ln(S_0) + c(T)),$$

where for all t

$$\begin{aligned}
A(t) &= A_{22}(t)^{-1}A_{21}(t),\\
b(t) &= \gamma,\\
c(t) &= -\frac{\beta}{2}Tr\left(\ln(A_{22}(t) + (M^T + 2\gamma RQ)T\right) + \gamma rT
\end{aligned}$$

and

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = e^{T\begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ) \end{pmatrix}}$$

3.2 Fast Fourier Transform

The following method is issued from [2]. For fixed $\alpha > 0$, we consider the the scaled price at time 0 as

$$c_T(k) := e^{\alpha k}\mathbb{E}(e^{-rT}(S_T - K)_+) = e^{\alpha k}\mathbb{E}(e^{-rT}(e^{Y_T} - e^k)_+),$$

where $k = \ln(K)$. Its Fourier transform is given by

$$\begin{aligned}
\Psi_T(v) :&= \int_{-\infty}^{\infty} e^{ivk}c_T(k)dk,\\
&= e^{-rT}\frac{\Phi_{(\alpha+1)+iv,0}(T)}{(\alpha+iv)(\alpha+1+iv)}
\end{aligned}$$

The price of a Call option of strike K is given by

$$\begin{aligned}
Call(k) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\Phi_T(v)dv,\\
&= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-ivk}\Phi_T(v)dv \qquad (2)
\end{aligned}$$

4 Algorithm

In order to compute (2), we use a Simpson approximation to compute the integral on $[0, a]$

$$Call(k) \sim \frac{e^{-\alpha k}}{\pi} \left(\Psi_T(0) + \Psi_T(a) + \sum_{j=1}^{N-1} e^{-i\eta jk} \Psi_T(j\eta) \frac{\eta}{3} (3 + (-1)^{j+1}) \right)$$

where $\eta := \frac{a}{N}$. In the following, N and a are fixed ($N = 4096$ and $a = 200\pi$).
We can also use the FFT method, by writing

$$Call(k_u) \sim \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-\frac{2i\pi}{N}(j-1)(u-1)} x(j),$$

where $k_u := -\frac{N\pi}{a} + \frac{2\pi}{a}(u-1)$ and $x(j) = e^{i\pi(j-1)} \psi((j-1)\eta) \frac{\eta}{3} c_j$, where $c_1 = 1$, $c_N = 1$, $c_{2j} = 4$ and $c_{2j+1} = 2$.

5 Numerical experiments

We test the algorithm with the following parameters, for different maturities :

r	S_0	K	β	n
0	100	110	3	2

(β should be greater than $n - 1$) and

$$M = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, \ R = \begin{pmatrix} -0.7 & 0 \\ 0 & -0.7 \end{pmatrix}$$
$$Q = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}, \ \Sigma_0 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}$$

M should be a semi-definite negative matrix.

T	0.25	2
Price	10.554	19.034
Implied Volatility	0.16710	0.228

References

[1] J. Da Fonseca, M. Grasselli, C. Tebaldi. A Multifactor Volatility Heston Model *Quantitative Finance*, 2008, vol. 8, issue 6, pages 591-604. 1, 2

[2] P. Carr, D.B. Madan Option Valuation using the Fast Fourier Transform *Journal of Computational Finance*, 1992, vol. 2, issue 4. 1, 2