On the application of spectral filters in a Fourier option pricing technique: Implementation in PREMIA

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March 3, 2020

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Abstract

Applying the pricing method based on Fourier-Cosine series expansion applying spectral filters proposed in [2], we implement the algorithms for pricing European put option under the Variance Gamma model with the spectral filters.

1 Introduction

The Fourier-Cosine pricing method (COS, in short) first proposed in [1], is based on the risk-neutral option valuation formula (discounted expected payoff approach). For an European option, the option value at time s given the state variable of the underlying asset taking value x is

$$v(x,s) = \mathrm{e}^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}[v(X_t,t)|X_s = x] = \mathrm{e}^{-r\Delta t}\int_{-\infty}^{\infty} v(y,t) f_{X_t|X_s}(y|x)\mathrm{d}y, \quad (1)$$

where r is the interest rate, t and s are the expiration date and the initial date respectively and $\Delta t := t - s$, X_t for $t \leq 0$ is the state variable which can be any monotone functions of the underlying asset at time t. Function $v(y,t)$, which equals the payoff of the European option, is known. However the conditional density function $f_{X_t|X_s}(y|x)$ usually maintains unknown.

The $v(x,s)$ can be approximated in a chosen sufficient wide domain $[a,b]$ instead of $(-\infty,\infty)$,

$$v(x,s) \approx \mathrm{e}^{-r\Delta t}\int_{a}^{b} v(y,t) f_{X_t|X_s}(y|x)\mathrm{d}y. \quad (2)$$

and $[a,b]$ is given by

$$[a,b] := \left[c_1 - L\sqrt{c_2+\sqrt{c_4}}, c_1 + L\sqrt{c_2+\sqrt{c_4}}\right],$$

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where $c_1, ..., c_4$ are the cumulants of the underlying stochastic process $X(t)$, given in [1],Table 11. $L = 10$ is consistent with [2].

$f_{X_t|X_s}(y|x)$ can be approximated in $[a, b]$ as follows:

$$f_{X_t|X_s}(y|x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ \phi\left(\frac{k\pi}{b-a}; x\right) \exp\left(-i\frac{ak\pi}{b-a}\right)\right\} \cos\left(k\pi\frac{y-a}{b-a}\right), \tag{3}$$

where $\phi(u; x)$ represents the characteristic function of $f_{X_t|X_s}(y|x)$, and $\sum'$ means that for $k = 0$ the expansion should be multiplied by $\frac{1}{2}$.

Substitute (3) in (2), $v(x, s)$ can be rewritten as

$$v(x, s) \approx \mathrm{e}^{-r\Delta t} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ \phi\left(\frac{k\pi}{b-a}; x\right) \exp\left(-i\frac{ak\pi}{b-a}\right)\right\} V_k(t) \tag{4}$$

where

$$V_k(t) = \frac{2}{b-a} \int_a^b v(y, t) \cos\left(k\pi\frac{y-a}{b-a}\right) \mathrm{d}y, \quad k = 0, \cdots, N-1 \tag{5}$$

And the details of COS method for European options can be found in [1].

When Fourier techniques are employed to specific cases with non-smooth functions, the Gibbs phenomenon may become apparent, which seriously impacts the efficiency and accuracy of the valuation. The Gibbs phenomenon means there is always an overshoot at a jump discontinuity in a finite Fourier series, which makes it more difficult to approximate a discontinuous function. Meanwhile, since the decay of the Fourier coefficients is decided by the smoothness of the involved function, the Gibbs phenomenon also lead to a slowly converging Fourier series. Spectral filters are applied to the COS method (called filter-COS, in short) to deal with the Gibbs phenomenon under Variance Gamma model, which has a non-smooth density, in [2]. And in our implementation, *Exponential filter* is used.

This document is organized as follows: The Variance Gamma will be shown in Section 2; The spectral filter will be introduced and the pricing formula of filter-COS method will be derived in Section 3; The price without filter and with filter in different orders ($p = 2, 4, 6, 8, 10, 12$) are compared with those in Premia in Section 4; Section 5 presents the manual of using the implementation in Premia.

2 Variance Gamma

The Variance Gamma (VG) process is obtained by evaluating a Brownian motion with drift θ and volatility σ at a random time given by a gamma process $\gamma(t)$ with mean rate one and variance rate ν,

$$X(t) = \theta\gamma(t) + \sigma W(\gamma(t)). \tag{6}$$

The asset price is then defined as $S(t) = S_0 \mathrm{e}^{X(t)}$. And the characteristic function $\phi(u|x) = \mathbb{E}[\mathrm{e}^{iuX(T)}|X(t_0) = x] = \mathrm{e}^{iux}\varphi_{VG}(u)$, is given by:

$$\varphi_{VG}(u) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-\Delta t/\nu} \sim \mathcal{O}(u^{-2\Delta t/\nu}).$$

3 Spectral filter and the filter-COS method

3.1 Definition of the Spectral filter

Definition 2.1. *Any $C^\infty([0,1])$ even function $\hat{s}$, whose support is $[-1,1]$ with $\hat{s}(0)=1$ is called a filter.*

The *filtered* partial sum of a Fourier series is defined by

$$f_N^{\hat{s}}(y) = \sum_{|n|\le N} \hat{s}(n/N)\hat{f}_n \mathrm{e}^{iny} \tag{7}$$

which can be rewritten as a convolution in physical space:

$$\begin{aligned} f_N^{\hat{s}}(y) &= \sum_{|n|\le N} \hat{s}(n/N)\hat{f}_n \mathrm{e}^{iny} = \frac{1}{2\pi}\int_0^{2\pi} \sum_{|n|\le N} \hat{s}(n/N)\mathrm{e}^{iny}\mathrm{e}^{-int} f(t)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} s(y-t)f(t)\mathrm{d}t, \end{aligned} \tag{8}$$

with filter function

$$s(x) = \sum_{|n|\le N} \hat{s}(n/N)\mathrm{e}^{inx} = \sum_{|n|\le\infty} \hat{s}(n/N)\mathrm{e}^{inx}, \quad x\in[0,2\pi] \tag{9}$$

Note that $\hat{s}(n/N)=0$ for $|n|>N$. The filter function is the representation of the spectral filter $\hat{s}(n/N)$ in physical space. Filtering does not affect the total mass of the resulting approximation (which should be one for a probability density), since the first coefficient is never altered. To be precise,

$$\int_0^{2\pi} f_N^{\hat{s}}(y)\mathrm{d}y = \sum_{|n|\le N} \hat{s}(n/N)\hat{f}_n \int_0^{2\pi} \mathrm{e}^{iny}\mathrm{d}y = \hat{s}(0/N)\hat{f}_0 2\pi = \int_0^{2\pi} f(y)\mathrm{d}y. \tag{10}$$

Definition 2.2. *(Fourier space filter of order p) A real and even function $\hat{s}(\eta)$ is called a filter of order p, if*

1. $\hat{s}(0)=1$ and $\hat{s}^l(0)=0$, $1\le l\le p-1$,

2. $\hat{s}(\eta)=0$ for $|\eta|\ge 1$,

3. $\hat{s}(\eta)\in C^{p-1}$, $\eta\in(-\infty,\infty)$.

Conditions 2 and 3 imply $\hat{s}^l(1)=0$, $0\le l\le p-1$.

3.2 Exponential filters

Exponential filter is one of the general p-th order spectral filters:

$$\hat{s}(\eta) = \exp(-\alpha\eta^p),$$

where p must be even. $\hat{s}(1)=\mathrm{e}^{-\alpha}$, so the formal requirements of a p-th order filter do not hold. However, we will choose $\alpha=-\log\epsilon_m$, where ϵ_m represents the machine epsilon, so that $\hat{s}(1)=\epsilon_m\approx 0$ within machine precision.

Other examples of filters are listed in [2].

3.3 Filter-COS method

When applying exponential filter to COS method, the pricing formula (4) turns to be as follows

$$v(x,s) \approx \mathrm{e}^{-r\Delta t} \sum_{k=0}^{N-1'} \hat{s}\left(\frac{k}{N}\right) \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a};x\right) \exp\left(-i\frac{ak\pi}{b-a}\right)\right\} V_k(t), \qquad (11)$$

where $\hat{s}$ is shown in the above part.

4 The simulation result

The parameters used are the same as in the Premia, ie,

$$K = 100, S(0) = 100, r = \log(1.1), q = 0, \sigma = 0.12, \theta = -0.33, \nu = 0.16.$$

Table 1 shows the approximations of European put option with and without exponential filter (order $p = 2, 4, 6, 8, 10$ respectively) when $T = 0.025, 0.1, 1$ respectively, compared with the prices given in Premia in all three different methods.

Table 2 shows the differences between prices in filter-COS method and in the given methods in Premia.

Table 1: The results of filter-COS method compared with prices in Premia

T	0.025	0.1	1
no filter	0.662052	1.608412	3.286226
p=2	0.842128	1.720936	3.426798
p=4	0.667950	1.598055	3.286504
p=6	0.655917	1.605391	3.286210
p=8	0.654514	1.608860	3.286225
p=10	0.654817	1.610118	3.286226
p=12	0.655426	1.610522	3.286226
premia			
FD	0.652854	1.606742	3.288848
AP_Carr	0.678106	1.605805	3.286226
TR_Mss	0.669446	1.605760	3.293768

5 Program Manual

We implement the European put options pricing by Fourier Cosine expansion with Spectral filter. The program HAS TO work with the pnl libary.

Model Parameters:
theta: the drift in the Brownian motion, θ in (6).
sigma: the volatility of the Brownian motion, σ in (6).
vv: the variance rate of the gamma time change, ν in (6).
w: the drift correction term, which satisfies $\exp(-wt) = \phi(-i,t)$.

Table 2: The $\log_{10}$(error) based on three reference values

	T= 0.025			T=0.1		
ref. method	FD	AP_Carr	TR_Mss	FD	AP_Carr	TR_Mss
no filter	-2.036	-1.794	-2.131	-2.777	-2.584	-2.576
p=2	-0.723	-0.785	-0.763	-0.942	-0.939	-0.939
p=4	-1.821	-1.993	-2.825	-2.061	-2.111	-2.113
p=6	-2.514	-1.654	-1.869	-2.869	-3.383	-3.433
p=8	-2.780	-1.627	-1.826	-2.674	-2.515	-2.509
p=10	-2.707	-1.633	-1.835	-2.472	-2.365	-2.361
p=12	-2.590	-1.644	-1.853	-2.423	-2.326	-2.322

	T=1		
ref. method	FD	AP_Carr	TR_Mss
p=2	-0.860	-0.852	-0.876
p=4	-2.630	-3.556	-2.139
p=6	-2.579	-4.788	-2.122
p=8	-2.581	-5.937	-2.122
p=10	-2.581	-6.329	-2.122
p=12	-2.581	-6.325	-2.122

Parameters of the exponential filter:
p: the order of the exponential filter.

Parameters for Fourier-Cosine method:
N: number of Fourier-Cosine series, N in (3)

Parameters of the product:
S0: the initial value of stock price.
K: strike price of the European option.
T: the maturity of the European option.
r: the discount interest rate.
q: the dividend interest rate.

References

[1] Fang, F., Oosterlee, C.W., A novel pricing method for european options based on fourier-cosine series expansions, 2008,*SIAM Journal of Science Computation* 31: 826-848. 1, 2

[2] M. J. Ruijter, M. Versteegh, C. W. Oosterlee, On the application of spectral filters in a Fourier option pricing technique, 2013