Pricing of Variance and Volatility Swaps

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March 3, 2020

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1 Model specification

The method proposed by Lian, Chiarella and Kalev in [3] gives an optimal solution to compute the prices of variance and volatility swaps analytical solution based on the Fourier cosine series expansion method. We implement this method for the Heston model and compute prices relative to specifiable strikes for swaps (as opposed to a fair strike assumption).

The assumption of the Heston model is that the underlying follows a stochastic process with drift and diffusion characteristics. Under the risk neutral measure, the asset S_t is given by the following diffusion process where V_t denotes the instantaneous variance.

$$\begin{aligned} dS_u &= (r-\delta)S_u du + \sqrt{V_u} S_u dW_u^S \\ dV_u &= \kappa(\theta - V_u)du + \sigma\sqrt{V_t} dW_u^V, \\ \langle dW_t^S, dW_t^V \rangle &= \rho dt, \text{ with } \rho \in [-1,1]. \end{aligned}$$

Here, θ is the long term average variance, κ is the mean reverting parameter of variance, σ is the volatility applicable on the diffusion process of volatility. The characteristic function for continuous realised variance under the Heston model has been solved by Broadie and Jain in [1] as

$$f_H(\phi, \infty) = \mathbb{E}\left(\phi RV(\infty)\right) = \exp(A(T,\phi) + B(T,\phi)V_0), \text{ where}$$

$$\begin{cases} A(T,\phi) = \dfrac{2\kappa\theta}{\sigma^2} \ln\left(\dfrac{2\gamma(\phi)e^{\frac{(\kappa+\gamma(\phi))T}{2}}}{(\gamma(s)+\kappa)(e^{\gamma(s)T}-1)+2\gamma(s)}\right) \\ B(T,\phi) = \dfrac{2\phi(e^{\gamma(\phi)T}-1)}{T\left[(\gamma(\phi)+\kappa)(e^{\gamma(\phi)T}-1)+2\gamma(\phi)\right]} \\ \gamma(\phi) = \sqrt{\kappa^2 - 2\phi\sigma^2/T}. \end{cases}$$

2 Implementation and Numerical results

When characteristic functions of payoffs are available, multiple methods have been proposed in literature to price derivatives, often using numerical techniques. For volatility derivatives, to compute the expectation of the payoff as shown above, Lian,Chiarella and Kalev propose in [3] to use the Fourier cosine series expansion method which was initially suggested by Fang and Osterlee in [2]. In this method, the probability density function of realised variance $p(RV)$ is reconstructed by a Fourier cosine series expansion with the coefficients expressed in terms of the characteristic function $f(\phi)$. The density is given by

$$p(RV) = \sum_{j=0} \infty A_j \cos\left(\frac{j\pi(RV-a)}{b-a}\right), \text{ where}$$

$$A_0 = \frac{1}{b-a}, \quad A_j = \frac{2}{b-a}\mathrm{Re}\left[f\left(\frac{j\pi i}{b-a}\right)e^{\frac{ja\pi i}{b-a}}\right] = \frac{2}{b-a}\int_a^b p(RV)\cos\left(\frac{j\pi(RV-a)}{b-a}\right)dRV, \text{ for } j>0;$$

Here $f(\phi)$ would be the characteristic function that was derived in the previous section. We choose $a = 0$ and $b = 100 \times \mathbb{E}[RV]$ as the two limits of the integral, since realised variance is positive and typically very small. Thus, the price, denoted by P in general, of a volatility derivative with a payoff $h(x)$ can be computed approximately as (before performing interest rate discounting)

$$p \simeq \sum_{j=0}^{\infty} A_j \int_a^b h(x) \cos\left(\frac{j\pi(x-a)}{b-a}\right) dx \simeq \sum_{j=0}^{300} A_j \int_a^b \mathrm{Re}\left[h(x) \exp\left(j\pi i \frac{x-a}{b-a}\right) dx\right]$$

The formula highlighted above is the central equation used to compute the results in this paper. Since no numerical intergration is required, the computation is extremely easy and quick. The expressions for the integral within it, are given below: let us denote

$$I := \int_a^b \mathrm{Re}\left[h(x) \exp\left(j\pi i \frac{x-a}{b-a}\right) dx\right]$$

then in the case of

Variance Swaps $h(x) = x$, we have

$$I = \frac{(b-a)^2 \cos(j\pi)}{j\pi}, \text{ when } j > 0, \text{ and } I = \frac{b^2 - a^2}{2} \text{ when } j = 0.$$

In the case of

Volatility Swaps $h(x) = \sqrt{x}$

$$I = \mathrm{Re}\left[\frac{(a-b)^{3/2}\sqrt{i}}{2\pi j^{3/2}}\left(\mathrm{erf}\left(\sqrt{\frac{ia\pi j}{a-b}}\right) - \mathrm{erf}\left(\sqrt{\frac{ib\pi j}{a-b}}\right)\right)\exp\left(\frac{ia\pi j}{a-b}\right) - \frac{ie^{i\pi j}(\sqrt{a}-\sqrt{b})(b-a)}{j\pi}\right]$$

and

$$I = \frac{2}{3}(b^{3/2} - a^{3/2}), \text{ when } j = 0.$$

Finally, theses two prices are computed by the Premia code source `vol_Swaps`.

References

[1] Broadie, M., and Jain, A. The effect of jumps and discrete sampling on volatility and variance swaps, International Journal of Theoretical and Applied Finance (2008) 1

[2] Fang, F. and Oosterlee, C. A novel pricing method for european options based on fourier-cosine series expansions, SIAM Journal on Scientific Computing (2008). 1

[3] Lian, G. Chiarella, C. and Kalev, Petko S. Volatility swaps and volatility options on discretely sampled realized variance. J. Econom. Dynam. Control (2014) 1