Option pricing and implied volatility in a 2-hypergeometric stochastic volatility model

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Abstract

We derive closed form analytical approximations in terms of series expansions for option prices and implied volatilities in a 2-hypergeometric stochastic volatility model with correlated Brownian motions. Our computation of implied volatilities exhibits the well known skew and smile phenomena on implied volatility surfaces, depending on the values of the correlation parameter.

Keywords: Stochastic volatility; 2-hypergeometric model; implied volatility; series expansions.
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1 Introduction

Stochastic volatility models have been introduced as realistic models for the motion of asset prices in financial markets. The most well-known of such models is the Heston [5] model, which however has one major drawback as its stochastic volatility may reach zero in finite time unless one imposes the Feller condition, and this poses potential problems in model calibration, cf. e.g. § 6.5.2 of Henry-Labordère [4]. In view of this, the α-hypergeometric stochastic volatility model has been introduced by Da Fonseca and Martini [1] to ensure strict positivity of volatility. In the α-hypergeometric model the dynamics of the asset price S_t at time t and the volatility V_t are governed by

$$dS_t = S_t e^{V_t} dW_t^1, \quad dV_t = \left(a - \frac{c}{2} e^{\alpha V_t}\right) dt + \eta dW_t^2, \tag{1}$$

$c > 0$, $\eta > 0$, $a \in \mathbb{R}$, $\alpha > 0$, and W_t^1 and W_t^2 are correlated Brownian motions satisfying $\langle W^1, W^2 \rangle_t = \rho t$. In this model the risk free rate r is taken to be equal to 0 and the value of c can be used to set the price of volatility risk.

Stochastic volatility models generally do not admit explicit solutions, and this has motivated the development of approximate expansions. In Fouque et al. [2] a method to obtain series expansions for European option prices has been proposed in the Heston model. This expression does not depend on the value of stochastic volatility which is a key quantity in the Heston model. A more accurate approximation has been proposed in Han et al. [3] for European option prices in the Heston model, see also Kim [6] under stochastic interest rates. In this paper we extend the method of [3], see also [7], in order to derive series expansions in the 2-hypergeometric model of [1]. In particular, our analytical approximate solution depend on the underlying stochastic volatility. We derive implied volatility estimates which display the well known phenomena of skew and smile. We also derive the delta estimates.

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2 Stochastic volatility

We start with a general class of stochastic volatility models in which the dynamics of the asset price and volatility processes are given by

$$dS_t^\varepsilon = S_t^\varepsilon p(t, V_t^\varepsilon) dW_t^1, \quad dV_t^\varepsilon = u(t, V_t^\varepsilon) dt + \varepsilon h(t, V_t^\varepsilon) dW_t^2,$$

where $\varepsilon > 0$. Recall that under absence of arbitrage, the vanilla option price of an option with payoff $g(S_T^\varepsilon)$ takes the form

$$f(t, S_t^\varepsilon, V_t^\varepsilon) := \mathbb{E}[g(S_T^\varepsilon) \mid \mathcal{F}_t]$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by $(W_t^1, W_t^2)_{t\in[0,T]}$, and the function $f(t,x,v)$ solves the PDE

$$\frac{\partial f}{\partial t} + u(t,v)\frac{\partial f}{\partial v} + \frac{x^2}{2}p^2(t,v)\frac{\partial f}{\partial x^2} + \varepsilon\rho x p(t,v)h(t,v)\frac{\partial^2 f}{\partial x \partial v} + \frac{\varepsilon^2}{2}h^2(t,v)\frac{\partial^2 f}{\partial v^2} = 0, \tag{2}$$

cf. e.g. (2.17) in [2], with the terminal condition $f(T,x,v) = g(x)$.

We start by expanding $f(t,x,v)$ as

$$f(t,x,v) = f_0(t,x,v) + \varepsilon f_1(t,x,v) + o(\varepsilon). \tag{3}$$

By plugging in the expansion (3) into the pricing PDE (2) we get the system of equations

$$\frac{\partial f_n}{\partial t} + \mathcal{L}_0 f_n + \mathcal{L}_1 f_{n-1} + \mathcal{L}_2 f_{n-2} = 0, \quad n \in \mathbb{N},$$

with $f_n = 0$, $n \leq -1$, $f_0(T,x,v) = g(x)$ and $f_n(T,x,v) = 0$, $n \geq 1$. In particular the operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\mathcal{L}_0 = u(t,v)\frac{\partial}{\partial v} + \frac{x^2}{2}p^2(t,v)\frac{\partial^2}{\partial x^2}, \quad \mathcal{L}_1 = \rho x p(t,v)h(t,v)\frac{\partial^2}{\partial x \partial v}, \quad \mathcal{L}_2 = \frac{1}{2}h^2(t,v)\frac{\partial^2}{\partial v^2}. \tag{4}$$

3 Deterministic volatility

When $n = 0$ we have $\dfrac{\partial f_0}{\partial t} + \mathcal{L}_0 f_0 = 0$, $(S_t^0)_{t\in[0,T]}$ and $(V_t^0)_{t\in[0,T]}$ are given by

$$dS_t^0 = S_t^0 p\left(t, V_t^0\right) dW_t^1, \quad dV_t^0 = u\left(t, V_t^0\right) dt$$

and the vanilla option price

$$f_0\left(t, S_t^0, V_t^0\right) = \mathbb{E}\left[g\left(S_T^0\right) \,\middle|\, \mathcal{F}_t\right]$$

can be computed by the Black-Scholes formula as

$$f_0\left(t, S_t^0, V_t^0\right) = \mathbb{E}\left[\left(S_T^0 - K\right)^+ \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\left(S_t^0 \exp\left(Z\sigma\left(t, V_t^0\right) - \frac{1}{2}\sigma^2\left(t, V_t^0\right)\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right],$$

where $Z \simeq \mathcal{N}(0,1)$ is independent of $\mathcal{F}_t$ and

$$\gamma^2\left(t, V_t^0\right) := \int_t^T p^2\left(u, V_u^0\right) du, \qquad t \in [0,T].$$

We note that in the α-hypergeometric model (1) with $\eta = 0$ the integral $\int_t^T e^{\alpha V_u^0} du$ can be computed in closed form as

$$\int_t^T e^{\alpha V_u^0} du = \frac{2}{\alpha c}\log\left(1 + \frac{\alpha c}{2}e^{\alpha V_t^0}\int_0^{T-t} e^{\alpha a s} ds\right) = \frac{2}{\alpha c}\log\left(1 + \frac{\alpha c}{2}e^{\alpha V_t^0}\frac{e^{\alpha a(T-t)} - 1}{\alpha a}\right),$$

cf. § 2.1.1 of [1], and this yields the following proposition.

Proposition 1. *In the 2-hypergeometric model (1) with $\eta = 0$ the European call price*

$$f_0\left(t, S_t^0, V_t^0\right) = \mathbb{E}\left[\left(S_T^0 - K\right)^+ \middle| \mathcal{F}_t\right]$$

under the terminal condition $f_0(T, x, v) = (x - K)^+$ is given by

$$f_0(t, x, v) = x\Phi\left(d_+(t, x, v)\right) - K\Phi\left(d_-(t, x, v)\right),$$

where Φ is the standard Gaussian cumulative distribution function,

$$d_\pm(t, x, v) = \frac{1}{\gamma(t, v)}\left(\log\left(\frac{x}{K}\right) \pm \frac{\gamma^2(t, v)}{2}\right), \text{ and } \quad \gamma^2(t, v) = \frac{1}{c}\log\left(1 + ce^{2v}\frac{e^{2a(T-t)} - 1}{2a}\right). \tag{5}$$

In the case of a put option the function $f_0(t, x, v)$ can be obtained as

$$f_0(t, x, v) = -x\Phi\left(-d_+(t, x, v)\right) + K\Phi\left(-d_-(t, x, v)\right), \quad t \in [0, T],$$

by a standard call-put parity argument. In the remainder of this paper we work in the 2-hypergeometric model with $\alpha = 2$.

4 Second order expansion

In this section we consider small values of the volatility of volatility by replacing η in (1) with $\varepsilon\eta e^{V_t^\varepsilon}\gamma^4(t, V_t^\varepsilon)$, $\varepsilon > 0$, i.e. we have

$$dS_t^\varepsilon = S_t^\varepsilon e^{V_t^\varepsilon} dW_t^1, \quad dV_t^\varepsilon = \left(a - \frac{c}{2}e^{2V_t^\varepsilon}\right)dt + \varepsilon\eta e^{V_t^\varepsilon}\gamma^4(t, V_t^\varepsilon)dW_t^2,$$

and from (4) the operators $\mathcal{L}_0$, $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ are given by

$$\mathcal{L}_0 = \left(a - \frac{c}{2}e^{2v}\right)\frac{\partial}{\partial v} + \frac{x^2}{2}e^{2v}\frac{\partial^2}{\partial x^2}, \quad \tilde{\mathcal{L}}_1 = \eta\rho x e^{2v}\gamma^4(t, v)\frac{\partial^2}{\partial x \partial v}, \quad \tilde{\mathcal{L}}_2 = \frac{\eta^2}{2}e^{2v}\gamma^8(t, v)\frac{\partial^2}{\partial v^2}.$$

In particular, we look for an expansion of the form

$$f(t, x, v) = f_0(t, x, v) + \varepsilon\tilde{f}_1(t, x, v) + \varepsilon^2\tilde{f}_2(t, x, v) + o(\varepsilon^2), \tag{6}$$

where

$$\frac{\partial f_0}{\partial t} + \mathcal{L}_0 f_0 = 0, \quad \frac{\partial \tilde{f}_1}{\partial t} + \mathcal{L}_0\tilde{f}_1 + \tilde{\mathcal{L}}_1 f_0 = 0, \quad \frac{\partial \tilde{f}_2}{\partial t} + \mathcal{L}_0\tilde{f}_2 + \tilde{\mathcal{L}}_1\tilde{f}_1 + \tilde{\mathcal{L}}_2 f_0 = 0,$$
$$f_0(T, x, v) = (x - K)^+, \quad \tilde{f}_1(T, x, v) = 0, \quad \tilde{f}_2(T, x, v) = 0.$$

Note that our approximation $(S_t^\varepsilon, V_t^\varepsilon)_{t\in[0,T]}$ does not lie within the class of 2-hypergeometric models.

Proposition 2. *The first and second order coefficients appearing in the expansion (6) are given by*

$$\tilde{f}_1(t, x, v) = -\eta\rho K\frac{d_-(t, x, v)}{c^3\gamma^2(t, v)}\phi\left(d_-(t, x, v)\right)\left(e^{-c\gamma^2(t,v)}\left(c^2\gamma^4(t, v) + c^2\gamma^2(t, v) + 2\right) - 2 + \frac{c^3}{3}\gamma^6(t, v)\right),$$

$$\tilde{f}_2(t, x, v) = \frac{\eta^2}{c}K\phi\left(d_-(t, x, v)\right)\left(\frac{A_3(t, v)}{\gamma(t, v)} + d_-(t, x, v)B_3(t, v) + \frac{\left(d_-(t, x, v)\right)^2}{\gamma(t, v)}B_3(t, v)\right) + \eta^2\rho^2 K\phi\left(d_-(t, x, v)\right)\left(C_3(t, v)\right.$$
$$\left. + \frac{2D(t, v)}{3c^7\gamma^4(t, v)}\left(\frac{\left(d_-(t, x, v)\right)^4}{3\gamma(t, v)} - d_-(t, x, v) + \frac{\left(d_-(t, x, v)\right)^3}{3}\right) + \frac{\left(d_-(t, x, v)\right)^2}{\gamma^5(t, v)}E_3(t, v)\right), \quad t \in [0, T],$$

where $\phi(x)$ is the standard Gaussian probability density function, $\gamma(t, v)$ is defined in (5) and the functions A_i, B_i, C_i, D, E_i are given below for $i = 1, 2, 3$.

Proof. The expression of $\tilde{f}_1$ and $\tilde{f}_2$ can be computed by similar arguments from the Feynman-Kac formula and the expected value. For simplicity of exposition we skip the corresponding computations, which are quite extensive. □

We have

$$A_1(t,v) = \frac{\sigma^8(t,v)}{2c} + \frac{5\gamma^6(t,v)}{4c^2} + \frac{2\gamma^4(t,v)}{c^3} + \frac{9\gamma^2(t,v)}{4c^4} + \frac{3}{2c^5} + \frac{3}{8c^6\gamma^2(t,v)}, \quad A_2(t,v) = -\frac{\gamma^8(t,v)}{c} - \frac{5\gamma^6(t,v)}{c^2} - \frac{16\gamma^4(t,v}{c^3}$$
$$- \frac{24}{c^4}\gamma^2(t,v) - \frac{48}{c^5} - \frac{24}{c^6\gamma^2(t,v)}, \quad A_3(t,v) = -\frac{\gamma^8(t,v)}{10c} + e^{-2c\gamma^2(t,v)}A_1(t,v) + A_2(t,v)e^{-c\gamma^2(t,v)} + \frac{93}{4c^5} + \frac{189}{8c^6\gamma^2(t,v)},$$
$$B_1(t,v) = -\frac{\gamma^6(t,v)}{4c^2} - \frac{\gamma^4(t,v)}{2c^3} - \frac{3\gamma^2(t,v)}{4c^4} - \frac{3}{4c^5} - \frac{3}{8c^6\gamma^2(t,v)}, \quad B_2(t,v) = \frac{\gamma^6(t,v)}{c^2} + \frac{4\gamma^4(t,v)}{c^3} + \frac{12\gamma^2(t,v)}{c^4} + \frac{24}{c^5}$$
$$+ \frac{24}{c^6\gamma^2(t,v)}, \quad B_3(t,v) = \frac{\gamma^8(t,v)}{10c} + e^{-2c\gamma^2(t,v)}B_1(t,v) + B_2(t,v)e^{-c\gamma^2(t,v)} - \frac{189}{8c^6\gamma^2(t,v)}, \quad C_1(t,v) = -\frac{\gamma^5(t,v)}{c^2} + \frac{\gamma^3(t,v}{2c^3}$$
$$+ \frac{3\gamma^2(t,v)}{c^4} + \frac{4\gamma(t,v)}{c^4} + \frac{9}{2c^5} + \frac{21}{2c^5\gamma(t,v)} + \frac{9}{2c^6\gamma^2(t,v)} + \frac{9}{c^6\gamma^3(t,v)} + \frac{9}{4c^7\gamma^4(t,v)} + \frac{15}{4c^7\gamma^5(t,v)}, \quad C_2(t,v) = -\frac{3\gamma^5(t,v)}{c^2}$$
$$- \frac{9\gamma^3(t,v)}{c^3} - \frac{6\gamma^2(t,v)}{c^4} - \frac{64\gamma(t,v)}{c^4} - \frac{36}{c^5} - \frac{120}{c^5\gamma(t,v)} - \frac{36}{c^6\gamma^2(t,v)} + \frac{24}{c^6\gamma^3(t,v)} - \frac{36}{c^7\gamma^4(t,v)} + \frac{24}{c^7\gamma^5(t,v)},$$
$$C_3(t,v) = -\frac{7\gamma^7(t,v)}{30c} - \frac{2\gamma(t,v)}{c^4} + C_1(t,v)e^{-2c\gamma^2(t,v)} + C_2(t,v)e^{-c\gamma^2(t,v)} + \frac{189}{2c^6\gamma^3(t,v)} + \frac{135}{4c^7\gamma^4(t,v)} - \frac{111}{4c^7\gamma^5(t,v)},$$
$$D(t,v) = e^{-2c\gamma^2(t,v)}\left(e^{c\gamma^2(t,v)}\left(c^3\gamma^6(t,v) - 3\right) + 3c\gamma^2(t,v)\left(\frac{c}{2}\gamma^2(t,v) + 1\right) + 3\right)^2, \quad E_1(t,v) = \frac{\gamma^{10}(t,v)}{c^2} + \frac{\gamma^8(t,v)}{c^3}$$
$$- \frac{15\gamma^6(t,v)}{c^4} - \frac{27\gamma^4(t,v)}{c^5} - \frac{51\gamma^2(t,v)}{2c^6} - \frac{12}{c^7}, \quad E_2(t,v) = \frac{2\gamma^{10}(t,v)}{c^2} + \frac{2\gamma^8(t,v)}{c^3} + \frac{17\gamma^6(t,v)}{c^4} + \frac{216\gamma^4(t,v)}{c^5} + \frac{24\gamma^2(t,v)}{c^6}$$
$$+ \frac{24}{c^7}, \quad E_3(t,v) = \frac{\gamma^{12}(t,v)}{15c} + \frac{4\gamma^6(t,v)}{c^4} - \frac{189\gamma^2(t,v)}{2c^6} + E_1(t,v)e^{-2c\gamma^2(t,v)} + e^{-c\gamma^2(t,v)}E_2(t,v) - \frac{492}{c^7}.$$

Note that in the case of put options, only the function $f_0(t,x,v)$ is modified by the standard call-put parity argument, while higher order terms such as $\tilde{f}_1(t,x,v)$ and $\tilde{f}_2(t,x,v)$ remain unchanged.

5 Implied volatility

In this section we provide an estimation of the implied volatility. σ^{imp} which is determined by the equation

$$f^{\mathrm{BS}}\left(t,x,T,K,\sigma^{\mathrm{imp}}\right) = f(t,x,v),$$

where $f^{\mathrm{BS}}\left(t,x,T,K,\sigma^{\mathrm{imp}}\right)$ is the classical Black-Scholes function, cf. e.g. Da Fonseca and Grasselli [**?**] in multi-factor models.

Theorem 3. *The implied volatility σ^{imp} admits the series expansion*

$$\sigma^{\mathrm{imp}}(t,x,v) = \sigma_0(t,x,v) + \varepsilon\sigma_1(t,x,v) + \varepsilon^2\sigma_2(t,x,v) + o(\varepsilon^2),$$

where

$$\sigma_0(t,x,v) = \tfrac{\gamma(t,v)}{\sqrt{T-t}}, \quad \sigma_1(t,x,v) = \tfrac{\tilde{f}_1(t,x,v)}{K\sqrt{T-t}\phi(d_-(t,x,v))} \quad \textit{and}$$
$$\sigma_2(t,x,v) = \tfrac{\tilde{f}_2(t,x,v)}{K\sqrt{T-t}\phi(d_-(t,x,v))} - d_+(t,x,v)d_-(t,x,v)\tfrac{\sigma_1^2(t,x,v)}{2\sigma_0(t,x,v)}.$$

Proof. The implied volatility σ^{imp} is determined by equating

$$f^{\mathrm{BS}}\left(t,x,T,K,\sigma^{\mathrm{imp}}\right) = f(t,x,v) = f_0(t,x,v) + \varepsilon\tilde{f}_1(t,x,v) + \varepsilon^2\tilde{f}_2(t,x,v) + o(\varepsilon^2),$$

where f^{BS} is the classical Black-Scholes function with implied volatility σ^{imp}. Expressing the implied volatility as a power series

$$\sigma^{\mathrm{imp}}(t,x,v) = \sigma_0(t,x,v) + \varepsilon\sigma_1(t,x,v) + \varepsilon^2\sigma_2(t,x,v) + o(\varepsilon^2)$$

in ε, we expand $f^{\mathrm{BS}}\left(t,x,T,K,\sigma^{\mathrm{imp}}\right)$ and using a Taylor expansion in terms of ε to obtain

$$\begin{aligned}f^{\mathrm{BS}}\left(t,x,T,K,\sigma^{\mathrm{imp}}\right) &= f^{\mathrm{BS}}(t,x,T,K,\sigma_0(t,x,v)) + \left(\varepsilon\sigma_1(t,x,v)+\varepsilon^2\sigma_2(t,x,v)\right)\frac{\partial f^{\mathrm{BS}}}{\partial\sigma}(t,x,T,K,\sigma_0(t,x,v)) \\ &\quad +\frac{1}{2}\varepsilon^2\sigma_1^2(t,x,v)\frac{\partial^2 f^{\mathrm{BS}}}{\partial\sigma^2}(t,x,T,K,\sigma_0(t,x,v))+\cdots\end{aligned}$$

The first three terms of the implied volatility expansion are obtained by identification of coefficients in the above expressions. $\square$

6 Delta

In this section, we provide an estimation of the Delta, Δ which is approximated by Equation (6).

Theorem 4. *The Delta, Δ admits the series expansion*

$$\Delta(t,x,v)=\frac{\partial f_0(t,x,v)}{\partial x}+\varepsilon\frac{\partial\tilde{f}_1(t,x,v)}{\partial x}+\varepsilon^2\frac{\partial\tilde{f}_2(t,x,v)}{\partial x},$$

where

$$\begin{aligned}\frac{\partial f_0(t,x,v)}{\partial x} &= \begin{cases}\Phi\left(d_+(t,x,v)\right)+\frac{\phi(d_+(t,x,v))}{\gamma(t,v)}-\frac{K\phi(d_-(t,x,v))}{x\gamma(t,v)}, & \textit{for call}\\ -\Phi\left(-d_+(t,x,v)\right)+\frac{\phi(-d_+(t,x,v))}{\gamma(t,v)}-\frac{K\phi(-d_-(t,x,v))}{x\gamma(t,v)}, & \textit{for put}\end{cases}\\
\frac{\partial\tilde{f}_1(t,x,v)}{\partial x} &= -\eta\rho K\frac{1+\left(d_-(t,x,v)\right)^2}{xc^3\gamma^3(t,v)}\phi\left(d_-(t,x,v)\right)\left(e^{-c\gamma^2(t,v)}\left(c^2\gamma^4(t,v)+c^2\gamma^2(t,v)+2\right)-2+\frac{c^3\gamma^6(t,v)}{3}\right),\ \textit{for call}\\
\frac{\partial\tilde{f}_2(t,x,v)}{\partial x} &= \frac{\eta^2}{c}K\frac{d_-(t,x,v)}{x\gamma(t,v)}\phi\left(d_-(t,x,v)\right)\left(\frac{A_3(t,v)}{\gamma(t,v)}+d_-(t,x,v)B_3(t,v)+\frac{\left(d_-(t,x,v)\right)^2}{\gamma(t,v)}B_3(t,v)\right)\\
&\quad+\frac{\eta^2}{c}K\phi\left(d_-(t,x,v)\right)\left(\frac{A_3(t,v)}{\gamma(t,v)}+\frac{B_3(t,v)}{x\gamma(t,v)}+\frac{2d_-(t,x,v)}{x\gamma^2(t,v)}B_3(t,v)\right)\\
&\quad+\eta^2\rho^2K\frac{d_-(t,x,v)}{x\gamma(t,v)}\phi\left(d_-(t,x,v)\right)\left(C_3(t,v)+\frac{2D(t,v)}{3c^7\gamma^4(t,v)}\left(\frac{\left(d_-(t,x,v)\right)^4}{3\gamma(t,v)}-d_-(t,x,v)+\frac{\left(d_-(t,x,v)\right)^3}{3}\right)\right.\\
&\quad\left.+\frac{\left(d_-(t,x,v)\right)^2}{\gamma^5(t,v)}E_3(t,v)\right)\\
&\quad+\eta^2\rho^2K\phi\left(d_-(t,x,v)\right)\left(C_3(t,v)+\frac{2D(t,v)}{3c^7\gamma^4(t,v)}\left(\frac{4\left(d_-(t,x,v)\right)^3}{3x\gamma^2(t,v)}-\frac{1}{x\gamma(t,v)}+\frac{\left(d_-(t,x,v)\right)^2}{x\gamma(t,v)}\right)\right.\\
&\quad\left.+\frac{2d_-(t,x,v)}{x\gamma^6(t,v)}E_3(t,v)\right),\ \textit{for call and put.}\end{aligned}$$

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