Finite difference methods for pricing of Swing options in Lévy-driven models

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Premia 22

1 Lévy processes

1.1 General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\leq 1})\nu(dy), \tag{1.1}$$

where $\sigma^2 \geq 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{1.2}$$

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ

must admit the analytic continuation into a strip $\Im\xi \in (-1, 0)$ and continuous continuation into the closed strip $\Im\xi \in [-1, 0]$.

The infinitesimal generator of X, denote it L, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x) - y\mathbf{1}_{|y|\leq 1}\frac{\partial u}{\partial x}(x))\nu(dy). \tag{1.3}$$

2 The multiple optimal stopping problem for Swing options

We consider a price process which evolves according to the formula :

$$S_t \quad = \quad e^{X_t},$$

where $\{X\}_{t\geq 0}$, the driving process, is an adapted Lévy process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.

Let T be the option's maturity time and let $\mathcal{T}_{t,T}$ be the set of $\mathbb{F}$-stopping times with values in $[t, T]$. Consider a Swing option that gives the right to multiple exercise with $\delta > 0$ refracting period which separates two successive exercises. We consider the possibility of n put exercises. We shall denote $\mathcal{T}^n$ the collection of all vectors of stopping times $(\tau_1, \tau_2, ..., \tau_n)$, such that

- $\tau_1 \leq T$ a.s.
- $\tau_i - \tau_{i-1} \geq \delta$ on $\{\tau_{i-1} \leq T\}$ a.s., for all $i = 2, .., n$

Denote by $v^{(i)}(t, x)$ the Swing option value with the possibility of i exercises at spot level $S = e^x$ and time $t \leq T$. Following [3], the multiple exercise problem can be solved computing

$$v^{(n)}(0, x) = \sup_{(\tau_1,...,\tau_n)\in\mathcal{T}^n} \sum_{i=1}^{n} E[e^{-r\tau_i}\phi(X_{\tau_i})] \tag{2.1}$$

where

$$\phi(x) = (K - e^x)_+$$

is the payoff function.

For solving the multiple optimal stopping problem Carmona and Touzi (2008) introduce the idea of a inductive hierarchy. In fact, they reduce the multiple stopping problem to a cascade of n optimal single stopping problems. Define the value function for $i = 1, ..., n$

$$v^{(i)}(t, x) = \sup_{\tau\in\mathcal{T}_{t,T}} E[e^{-r\tau}\phi^{(i)}(\tau, X_\tau^{t,x})] \tag{2.2}$$

where the reward function $\phi^{(i)}$ is now defined as

$$\phi^{(i)}(t, x) = \phi(x) + E[e^{-r\delta}v^{(i-1)}(t+\delta, X_{t+\delta}^{t,x})], \quad t \leq T - \delta, \tag{2.3}$$

$$\phi^{(i)}(t, x) = \phi(x), \quad t > T - \delta. \tag{2.4}$$

The problem could be solved using Monte Carlo algorithm. Let be $t_0 = 0 < t_1 < t_2 < ... < t_N = T$ a time discretization grid. The price of a Swing option can be actually computed by the backward induction procedure:

$$\begin{cases} v^{(i)}(t_N, x) = \phi(x) \\ v^{(i)}(t_{k-1}, x) = \max\left\{\phi^{(i)}(t_{k-1}, x); e^{-r(t_k - t_{k-1})} E[v^{(i)}(t_k, X_{t_k}^{t_{k-1},x})]\right\}, k = N, ..., 1. \end{cases}$$

Carmona-Touzi (2008) and Mnif-Zeghal (2006) considered Monte Carlo Malliavin-based algorithm to compute the price, respectively, in the Black-Scholes and jump models frameworks. Barrera-Esteve *et al.* (2006) used a regression based method in order to approximate conditional expectations. In the next sections, we propose two PIDE-based approaches.

3 The finite difference scheme for pricing Swing options

We can compute the Swing option price using the formulation given in (2.2) with an analytic approach. In fact, we propose to solve the following system of variational inequalities associated to the Swing options formulation

$$\begin{cases} \max\left(\phi^{(i)}(t,x) - v^{(i)}(t,x), \frac{\partial v^{(i)}}{\partial t} + Lv^{(i)} - rv^{(i)}\right) = 0, \ (t,x) \ \text{ in } [0,T[\times\mathbb{R}, \\ v^{(i)}(T,x) = \phi^{(i)}(T, e^x). \end{cases} \tag{3.1}$$

with $i = 1, ..., n$, where the integro-differential operator L is defined in (**??**).

Now recall that for $t \le T - \delta$

$$\phi^{(i)}(t,x) = (K - e^x)_+ + E[e^{-r\delta} \ v^{(i-1)}(t+\delta, X_{t+\delta})].$$

Let us define for $t \le T - \delta$

$$u^{(i)}(t,x) = E[e^{-r\delta} \ v^{(i)}(t+\delta, X_{t+\delta}^{t,x})].$$

By the Feyman-Kac theorem, $u^{(i)}(t,x) = z(0,x)$, where $z(t,x)$ is the solution of the following partial integro-differential equation (PIDE)

$$\begin{cases} \frac{\partial z}{\partial t} + Lz - rz = 0, \ (t,x) \in [0,\delta[\times\mathbb{R}, \\ z(\delta, x) = v^{(i)}(t+\delta, x), \end{cases} \tag{3.2}$$

which can be numerical computed using a finite difference approach. In order to price a Swing option we can therefore solve the system of variational inequalities (3.1) computing the reward payoff function $\phi^{(i)}(t,x)$ in the following way.

$$\phi^{(i)}(t,x) = \phi(x)$$

for $T - \delta < t \le T$, and

$$\phi^{(i)}(t,x) = (K - e^x)_+ + u^{(i-1)}(t,x)$$

for $t \le T - \delta$.

As said before, the reward payoff function can be computed numerically using a finite difference scheme. The numerical solution of each variational inequalities (3.1) requires to solve numerically each PIDE problem (3.2). In order to solve (3.1) and (3.2), we perform the following steps:

- *Localization.* It means that we choose a spatial bounded computational domain Ω_l. This implies that we have to choose some artificial boundary conditions.

- *Truncation of large jumps.* This corresponds to truncate the integration domain in the integral part.

- *Discretization.* The derivatives of the solution are replaced by usual finite differences and the integral terms are approximated using the trapezoidal rule. The problem is then solved using an implicit-explicit scheme (see Briani *et al* (2004), Cont and Voltchkova (2005) and its program implementation PREMIA). In particular, we introduce a time grid $t = s\Delta t$, $s = 0, .., M$, where $\Delta t = \frac{T}{N}$ is the time step. This produces to solve at each time step a linear system for the linear problem (3.2) and a linear complementarity problem for the non linear problem (3.1).

- *Treatment of the variational inequalities.* We solve each variational inequalities (3.1) using the splitting method of Barles *et al* (1995). The splitting methods can be viewed as an analytic version of dynamic programming. The idea contained in such scheme is to split the American problem in two steps: we construct recursively the approximate solution $v^{(i)}(s\Delta t, x)$ at each time step $s\Delta t$ starting from $v^{(i)}(N\Delta t, x) = \phi(x)$ and computing at each time step $v^{(i)}(s\Delta t, x)$ for $s = N-1, .., 0$ as follows:

  - Compute the solution of the following linear Cauchy problem on $[s\Delta t, (s+1)\Delta t[\times\Omega_l$ using an implicit-explicit scheme:

$$\begin{cases} \frac{\partial w^{(i)}(s\Delta t,x)}{\partial t} + Lw^{(i)}(s\Delta t, x) - rw^{(i)}(s\Delta t, x) = 0, & \text{in } [s\Delta t, (s+1)\Delta t[\times\Omega_l \\ w^{(i)}((s+1)\Delta t, x) = v^{(i)}((s+1)\Delta t, x) \end{cases}$$

  - Apply the early exercise $v^{(i)}(s\Delta t, x) = \max(w^{(i)}(s\Delta t, x), \phi^{(i)}(s\Delta t, x))$, where the reward function $\phi^{(i)}(s\Delta t, x)$ is obtained by solving the linear problem (3.2) with an implicit-explicit finite difference method.

References

[1] Barles, G., Daher, C. and Romano, M. Convergence of numerical Schemes for problems arising in Finance theory. *Mathematical models and methods in applied Sciences*, 1995, 5(1), 125–143.

[2] Briani, M., La Chioma, C. and Natalini, R. Convergence of numerical schemes for viscosity solutions to integro-differential degenerate parabolic problems arising in financial theory. *Numer. Math.*, 2004, 98(4), 607–646.

[3] Carmona, C. and Touzi, N. Optimal Multiple Stopping and Valuation of Swing Options. *Mathematical Finance*, 2008, 18(2), 239–268. 2

[4] Carr, P. and Faguet, D. Fast accurate valuation of American options, 1994. Working paper, Cornell University, Ithaca.

[5] Cont, R. and Voltchkova, E. A finite difference scheme for option pricing in jump-diffusion and exponential Lévy models. *SIAM Journal on Numerical Analysis*. 2005, 43(4), 1596–1626.

[6] Mnif, M. and Zhegal, A.B. Optimal Multiple stopping and Valuation of Swing Options in Lévy Models. *International Journal of Theoretical and Applied Finance*, 2006, 9(8), 1267–1298.

[7] PREMIA: An Option Pricer Project CERMICS - INRIA.
http://www.premia.fr