

Pricing Asian CIR options by conditional moment matching

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Abstract

Asian yield options are priced in the CIR model using conditional moment matching for the gamma distribution. This method is fast and simple to implement, and it shows a high degree of accuracy without being subject to the numerical instabilities that can be encountered with more sophisticated approaches.

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Keywords: CIR model; Asian options; Asian caps, conditional moment matching; stratified approximation.

We consider Asian call options priced as

$$\text{AO}^c(K, T) := \mathbb{E} \left[e^{-\Lambda_T} \left(\frac{\Lambda_T}{T} - K \right)^+ \right]$$

where

$$\Lambda_T := \int_0^T S_t dt,$$

and $(S_t)_{t \in \mathbb{R}_+}$ is the Cox-Ingersoll-Ross (CIR) solution of the stochastic differential equation

$$dS_t = a(b - S_t)dt + \sigma \sqrt{S_t} dW_t. \quad (0.1)$$

Unconditional gamma approximation

Using moment matching, $\text{AO}^c(K, T)$ can be estimated as

$$\begin{aligned} \text{AO}^c(K, T) &:= \mathbb{E} \left[e^{-\Lambda_T} \left(\frac{\Lambda_T}{T} - K \right)^+ \right] \\ &\approx \frac{\theta_T}{(1 + \theta_T)^{\nu_T + 1}} \left(\nu_T Q \left(1 + \nu_T, KT + \frac{KT}{\theta_T} \right) - \left(KT + \frac{KT}{\theta_T} \right) Q \left(\nu_T, KT + \frac{KT}{\theta_T} \right) \right) \end{aligned} \quad (0.2)$$

under the unconditional gamma approximation, where

$$\theta_T := \frac{\text{Var}[\Lambda_T]}{\mathbb{E}[\Lambda_T]} \quad \text{and} \quad \nu_T := \frac{\mathbb{E}[\Lambda_T]}{\theta_T} = \frac{(\mathbb{E}[\Lambda_T])^2}{\text{Var}[\Lambda_T]},$$

and

$$\begin{aligned} \mathbb{E}[\Lambda_T] &= S_0 \frac{1 - e^{-aT}}{a} + ab \frac{e^{-aT} + aT - 1}{a^2}, \\ \text{Var}[\Lambda_T] &= \sigma^2 S_0 \frac{1 - 2aT e^{-aT} - e^{-2aT}}{a^3} + \sigma^2 ab \frac{5 - 2aT - e^{-2aT} - 4(aT + 1)e^{-aT}}{2(ab)^4}. \end{aligned}$$

Conditional gamma approximation

Under the conditional gamma approximation we find

$$\begin{aligned} \text{AO}^c(K, T) &= \mathbb{E} \left[e^{-\Lambda_T} \left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] \\ &\approx \frac{1}{T} \int_0^\infty \frac{\theta_T(y)}{(1 + \theta_T(y))^{\nu_T(y) + 1}} \left(\nu_T(y) Q \left(1 + \nu_T(y), KT + \frac{KT}{\theta_T(y)} \right) \right. \\ &\quad \left. - \left(KT + \frac{KT}{\theta_T(y)} \right) Q \left(\nu_T(y), KT + \frac{KT}{\theta_T(y)} \right) \right) f_{S_T}(y) dy, \end{aligned} \quad (0.3)$$

see § 4.4 of [4], where

$$\theta_T(y) := \frac{\text{Var}[\Lambda_T \mid S_T = y]}{\mathbb{E}[\Lambda_T \mid S_T = y]} \quad \text{and} \quad \nu_T(y) := \frac{\mathbb{E}[\Lambda_T \mid S_T = y]}{\theta_T(y)} = \frac{(\mathbb{E}[\Lambda_T \mid S_T = y])^2}{\text{Var}[\Lambda_T \mid S_T = y]},$$

and

$$\begin{aligned} \mathbb{E}[\Lambda_T \mid S_T = y] &= -\frac{\sigma^2}{a^2} + \frac{1}{a(e^{aT} - 1)^2} \left(\frac{\sigma^2 T}{2} (e^{2aT} - 1) + (S_0 + y)(e^{2aT} - 2aT e^{aT} - 1) \right. \\ &\quad \left. + \sqrt{y S_0} e^{aT} (e^{aT} (aT - 2) + aT + 2) \frac{I_{2ab/\sigma^2} \left(\frac{2a\sqrt{y S_0}}{\sigma^2 \sinh(aT/2)} \right) + I_{2ab/\sigma^2 - 2} \left(\frac{2a\sqrt{y S_0}}{\sigma^2 \sinh(aT/2)} \right)}{I_{2ab/\sigma^2 - 1} \left(\frac{2a\sqrt{y S_0}}{\sigma^2 \sinh(aT/2)} \right)} \right). \end{aligned}$$

$$\begin{aligned}
\text{Var}[\Lambda_T \mid S_T = y] &= -2\frac{\sigma^4}{a^4} + \sigma^2 \frac{(e^{aT}(2aT+1) - 1)}{a^2(e^{aT} - 1)} \left(\frac{\sigma^2}{a^2} + \mathbb{E}[\Lambda_T \mid S_T = y] \right) \\
&+ \frac{1}{a(e^{aT} - 1)^2} \left(-\sigma^4 T^2 \frac{e^{2aT}}{a} - 2\frac{\sigma^2 T}{a} (S_0 + y) (e^{2aT} - e^{aT}(aT+1)) \right. \\
&- \frac{\sigma^2 T}{2a} \sqrt{yS_0 e^{aT}} (e^{aT}(3aT-4) + aT+4) \frac{I_{2ab/\sigma^2} \left(\frac{4a\sqrt{yS_0 e^{-aT}}}{\sigma^2(1-e^{-aT})} \right) + I_{2ab/\sigma^2-2} \left(\frac{4a\sqrt{yS_0 e^{-aT}}}{\sigma^2(1-e^{-aT})} \right)}{I_{2ab/\sigma^2-1} \left(\frac{2a\sqrt{yS_0}}{\sigma^2 \sinh(aT/2)} \right)} \\
&\left. + \frac{yS_0 e^{aT}}{a(1-e^{-aT})^2} (e^{aT}(aT-2) + aT+2)^2 \left(2 + \frac{I_{2ab/\sigma^2-3} + I_{2ab/\sigma^2+1}}{I_{2ab/\sigma^2-1}} - \frac{(I_{2ab/\sigma^2-2} + I_{2ab/\sigma^2})^2}{(I_{2ab/\sigma^2-1})^2} \right) \right),
\end{aligned}$$

with $I_z := I_z \left(\frac{2a\sqrt{yS_0}}{\sigma^2 \sinh(aT/2)} \right)$, and

$$I_\lambda(z) := \left(\frac{z}{2} \right)^\lambda \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k! \Gamma(\lambda + k + 1)}, \quad z, \lambda \in \mathbb{R},$$

is the modified Bessel function of the first kind, $z \in \mathbb{R}$. In (0.3) above, f_{S_T} is the non-central chi-square probability density function

$$f_{S_T}(y) := \frac{2a}{\sigma^2(1-e^{-aT})} \exp \left(-\frac{2a(S_0 + ye^{aT})}{\sigma^2(e^{aT} - 1)} \right) \left(\frac{ye^{aT}}{S_0} \right)^{ab/\sigma^2-1/2} I_{2ab/\sigma^2-1} \left(\frac{2a\sqrt{yS_0}}{\sigma^2 \sinh(aT/2)} \right), \quad (0.4)$$

$y > 0$, and

$$\Gamma(\lambda) := \int_0^\infty x^{\lambda-1} e^{-x} dx$$

denotes the gamma function.

Numerical results

In Table 1 we compare our results to the joint density (JD) method of [2], [3], with the parameters of [1].

| Strike | Type | Maturity T = 0.1 | | | Maturity T = 0.5 | | |
|--------|-----------------|---------------------|------------------|-------------|---------------------|------------------|-------------|
| | | JD[3] | Stratified (0.3) | Gamma (0.2) | JD[3] | Stratified (0.3) | Gamma (0.2) |
| 0.08 | AO ^c | 0.0199 | 0.0199 | 0.0199 | 0.0201 | 0.0210 | 0.0201 |
| 0.12 | AO ^c | 0.0002 | 0.0002 | 0.0002 | 0.0018 | 0.0018 | 0.0018 |
| Strike | Type | T = 1 | | | T = 2 | | |
| | | JD[3] | Stratified (0.3) | Gamma (0.2) | JD[3] | Stratified (0.3) | Gamma (0.2) |
| 0.08 | AO ^c | 0.0193 | 0.0193 | 0.0194 | 0.0170 | 0.0170 | 0.0171 |
| 0.12 | AO ^c | 0.0023 | 0.0023 | 0.0023 | 0.0019 | 0.0019 | 0.0018 |
| Strike | Type | T = 5 | | | T = 10 | | |
| | | JD[3] | Stratified (0.3) | Gamma (0.2) | JD[3] | Stratified (0.3) | Gamma (0.2) |
| 0.08 | AO ^c | 0.0118 | 0.0118 | 0.0118 | 0.0069 | 0.0069 | 0.0069 |
| 0.12 | AO ^c | 0.0006 | 0.0006 | 0.0006 | 0.0001 | 0.0001 | 0.0001 |

Table 1: Asian prices, $S_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$.

Figure 1 presents the evolution of prices according to maturity times. All three methods show consistent numerical results.

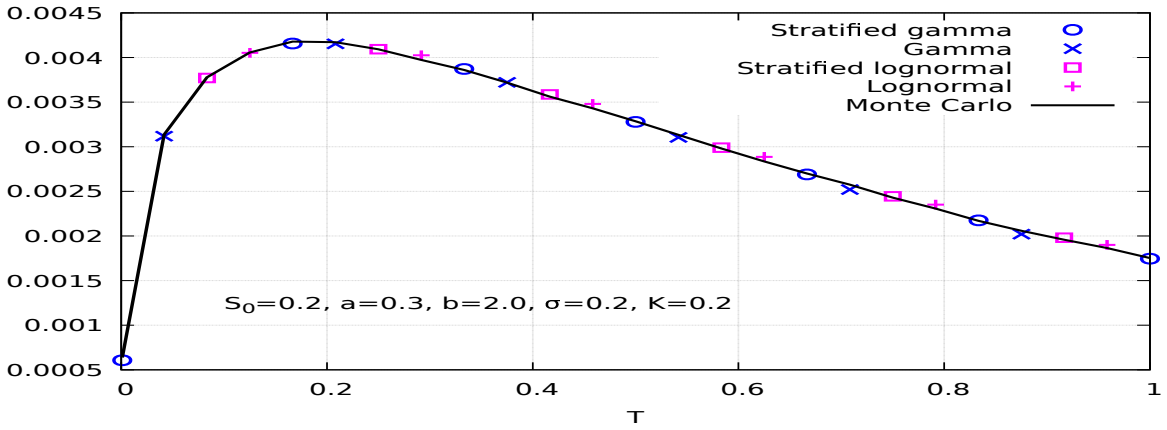


Figure 1: Regular Asian (A) cap (or call) prices for $T \in [0, 1]$.

Table 2 presents a sample of computation times for comparison of the different methods, cf. [4] for details.

| Parameters | | | | | | Time | | | |
|------------|-----|-------|----------|-----|-----|------------------|-------------|-------------|-------|
| S_0 | a | b | σ | T | K | Stratified (0.3) | Gamma (0.2) | Monte Carlo | JD[3] |
| 2.1 | 0.0 | -0.05 | 0.72 | 1.0 | 2.0 | 1.32e-02 | 2.60e-5 | 144.46 | 8.62 |

Table 2: Computation times in seconds.*

Call-put parity

The relations

$$E \left[e^{\eta \Lambda_T} \right] = e^{-S_0 \psi(\eta) - ab \phi(\eta)}, \quad (0.5)$$

where $\bar{a} := \sqrt{a^2 - 2\eta\sigma^2}$ and

$$\psi(\eta) := \frac{2\eta(e^{-\bar{a}T} - 1)}{\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a)} \quad \phi(\eta) := \frac{1}{\sigma^2}(\bar{a} - a)T + \frac{2}{\sigma^2} \log \frac{\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a)}{2\bar{a}},$$

allow us to estimate the regular Asian floor (or put) price as

$$\begin{aligned} \text{AO}^f(K, T) &= \mathbb{E} \left[e^{-\Lambda_T} \left(K - \frac{\Lambda_T}{T} \right)^+ \right] \\ &= \mathbb{E} \left[e^{-\Lambda_T} \left(\frac{\Lambda_T}{T} - K \right)^+ \right] - \left(\frac{1}{T} \mathbb{E} [\Lambda_T e^{-\Lambda_T}] - K \mathbb{E} [e^{-\Lambda_T}] \right) \\ &= \text{AO}^c(K, T) + K \mathbb{E} [e^{-\Lambda_T}] - \frac{1}{T} \mathbb{E} [\Lambda_T e^{-\Lambda_T}], \end{aligned}$$

from

$$\begin{aligned} \mathbb{E} [\Lambda_T e^{-\Lambda_T}] &= -(S_0 \psi'(-1) + ab \phi'(-1)) e^{-S_0 \psi(-1) - ab \phi(-1)} \\ &= -S_0 \left(\frac{2\bar{a}(e^{-\bar{a}T} - 1) - 2\sigma^2 T e^{-\bar{a}T}}{\bar{a}(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))} - \frac{2\sigma^2(e^{-\bar{a}T} - 1)(1 - e^{-\bar{a}T}(\bar{a} - a)T + e^{-\bar{a}T})}{\bar{a}(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))^2} \right) \mathbb{E} [e^{-\Lambda_T}] \\ &\quad + \frac{ab}{\bar{a}} \left(T + 2 \frac{1 - e^{-\bar{a}T}(\bar{a} - a)T + e^{-\bar{a}T}}{\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a)} - \frac{2}{\bar{a}} \right) \mathbb{E} [e^{-\Lambda_T}], \end{aligned}$$

with $\bar{a}' = -\sigma^2/\bar{a}$ and $\eta = -1$, since

$$\begin{aligned} \psi'(\eta) &= \frac{2(e^{-\bar{a}T} - 1)}{\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a)} + \frac{2\eta T \sigma^2 e^{-\bar{a}T}}{\bar{a}(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))} \\ &\quad - \frac{2\eta(e^{-\bar{a}T} - 1)}{(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))^2} (-\sigma^2/\bar{a} + (\sigma^2 T/\bar{a})e^{-\bar{a}T}(\bar{a} - a) - e^{-\bar{a}T}(\sigma^2/\bar{a})) \\ &= \frac{2\bar{a}(e^{-\bar{a}T} - 1) + 2\eta T \sigma^2 e^{-\bar{a}T}}{\bar{a}(\bar{a} + b + e^{-\bar{a}T}(\bar{a} - a))} + \frac{2\eta \sigma^2(e^{-\bar{a}T} - 1)(1 - T e^{-\bar{a}T}(\bar{a} - a) + e^{-\bar{a}T})}{\bar{a}(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))^2}, \end{aligned}$$

and

$$\begin{aligned} \phi'(\eta) &= -\frac{T}{\bar{a}} + \frac{2(-\sigma^2/\bar{a} + (\sigma^2 T/\bar{a})e^{-\bar{a}T}(\bar{a} - a) + e^{-\bar{a}T}(-\sigma^2/\bar{a}))}{\sigma^2(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))} + \frac{2(2\sigma^2/\bar{a})}{\sigma^2(2\bar{a})} \\ &= -\frac{T}{\bar{a}} - 2 \frac{1 - e^{-\bar{a}T}(\bar{a} - a)T + e^{-\bar{a}T}}{\bar{a}(\bar{a} + a + e^{-\bar{a}T}(\bar{a} - a))} + \frac{2}{\bar{a}^2}. \end{aligned}$$

References

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