

# Large/Small time asymptotics for Implied volatility under the Heston model

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## Premia 22

Most of what is presented here is taken from [1] and [2]

### 1 Small-time asymptotics for Implied Volatility

Consider the Heston stochastic volatility model for a forward price process  $S_t$ , defined by the following stochastic differential equations

$$\begin{cases} \frac{dS_t}{S_t} = \sqrt{V_t}dW_t^1, \\ dV_t = (a - bV_t)dt + \sigma\sqrt{V_t}dW_t^2, \quad d\langle W^1, W^2 \rangle_t = \rho dt, \end{cases} \quad (1.1)$$

According to Forde and Jacquier (cf [1]), if we consider  $(S, V)$  the unique solution of (1.1) starting with  $(s, v)$ , we get

$$\lim_{t \rightarrow \infty} t \log \mathbb{E}(K - S_t)_+ = -\Lambda^*(\log(\frac{K}{s})), \quad \text{pour } s > K$$

and

$$\lim_{t \rightarrow \infty} t \log \mathbb{E}(S_t - K)_+ = -\Lambda^*(\log(\frac{K}{s})), \quad \text{pour } s < K$$

where  $\Lambda^*$  is the Legendre transform of the function  $\Lambda$  defined as

$$\begin{cases} \Lambda(p) = \frac{vp}{\sigma(\sqrt{1-\rho^2} \cot(\frac{1}{2}\sigma p \sqrt{1-\rho^2}) - \rho)}, & \text{pour } p \in ]p_-, p_+[ , \\ \Lambda(p) = \infty, & \text{pour } p \in \mathbb{R} \setminus ]p_-, p_+[ , \end{cases}$$

where  $p_-$  and  $p_+$  are given by

$$p_- = \frac{\arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}} \mathbb{1}_{\rho < 0} - \frac{\pi}{\sigma} \mathbb{1}_{\rho = 0} + \frac{-\pi + \arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}} \mathbb{1}_{\rho > 0}, \quad (1.2)$$

$$p_+ = \frac{\pi + \arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}} \mathbb{1}_{\rho < 0} + \frac{\pi}{\sigma} \mathbb{1}_{\rho = 0} + \frac{\arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}} \mathbb{1}_{\rho > 0}, \quad (1.3)$$

The function  $\Lambda^*$  is given by

$$\Lambda^*(x) = xp^*(x) - \Lambda(p^*(x)), \quad (1.4)$$

where  $p^*(x)$  is unique solution of

$$x = \Lambda'(p^*(x)),$$

and the function  $\Lambda'$  is given by

$$\Lambda'(p) = \frac{v}{\sigma(\sqrt{1-\rho^2} \cot(\frac{1}{2}\sigma p \sqrt{1-\rho^2}) - \rho)} + \frac{\sigma vp(1-\rho^2) \csc^2(\frac{1}{2}\sigma p \sqrt{1-\rho^2})}{2\sigma(\sqrt{1-\rho^2} \cot(\frac{1}{2}\sigma p \sqrt{1-\rho^2}) - \rho)^2}$$

On the other hand, if we denote by  $\Sigma_t(x)$  the implied volatility defined as the unique solution of  $P(t, K e^{-x}, v) = P_{BS}(t, K e^{-x}, K; \Sigma_t(x))$ , où  $P_{BS}(t, s, k, \Sigma) = K N(\frac{-\log(s/k) + t\Sigma/2}{\sqrt{t\Sigma}}) - sN(\frac{-\log(s/k) - t\Sigma/2}{\sqrt{t\Sigma}})$ . Then we have (cf. [1], Theorem 2.4)

$$\lim_{t \rightarrow 0} \Sigma_t(x) = \frac{|x|}{\sqrt{2\Lambda^*(x)}} \quad (1.5)$$

## 2 Large-time asymptotics for Implied Volatility

We keep the notations of Forde-Jacquier-Mijatovic (cf. [2]). Under the assumption  $b - \rho\sigma > 0$ , we have for any  $p \in ]p_-, p_+[$ ,

$$V(p) = \lim_{t \rightarrow \infty} t^{-1} \log \mathbb{E}[\exp(p(X_t - x_0))] = \frac{a}{\sigma^2} (b - \rho\sigma p - d(-ip)),$$

with

$$d(-ip) = \sqrt{(b - \rho\sigma p)^2 + \sigma^2(p - p^2)},$$

where

$$p_{\pm} := \left( -2b\rho + \sigma \pm \sqrt{\sigma^2 + 4b^2 - 4b\rho\sigma} \right)$$

We consider the function  $p^* : \mathbb{R} \rightarrow ]p_-, p_+[$  defined as

$$p^*(x) := \frac{\sigma - 2b\rho + (a\rho + x\sigma) \left( \frac{\sigma^2 + 4b^2 - 4b\rho\sigma}{x^2\sigma^2 + 2xa\rho\sigma + a^2} \right)^{1/2}}{2\sigma(1 - \rho^2)}, \quad \text{pour tout } x \in \mathbb{R}.$$

For  $t$  large enough, we have for any  $x \in \mathbb{R}$ , (cf. [2])

$$\frac{1}{S_0} \mathbb{E}(S_t - S_0 e^{-x})_+ = 1 + \frac{A(0)}{\sqrt{2\pi t}} \exp(-(1 - p^*(0))x - V^*(0)t) (1 + O(1/t)),$$

where  $V^*$  is the Legendre transform of  $V$ , ie  $V^*(x) := \sup \{px - V(p), p \in ]p_-, p_+[ \}$  and  $A$  is the function defined in a neighborhood of 0 by

$$A(x) = \frac{-1}{\sqrt{V''(p^*(x))}} \frac{U(p^*(x))}{p^*(x)(1 - p^*(x))},$$

with

$$U(p) := \left( \frac{2d(-ip)}{b - \rho\sigma p + d(-ip)} \right)^{\frac{2a}{\sigma^2}} \exp\left(\frac{v}{a} V(p)\right)$$

Similarly, the implied volatility can be written as (cf. [2], Theorem 3.2)

$$\Sigma_t^2(x) = 8V^*(0) + a_1(x)/t + o(t), \tag{2.1}$$

where

$$a_1(x) = -8 \log \left( -A(0) \sqrt{2V^*(0)} \right) + 4(2p^*(0) - 1)x$$

In particular, for any  $x \in \mathbb{R}$ , we have

$$\lim_{t \rightarrow +\infty} \Sigma_t^2(x) = 8V^*(0).$$

## References

- [1] Forde, M. and Jacquier, A. Small-time asymptotics for implied volatility under the Heston model, with A. Jacquier, *International Journal of Theoretical and Applied Finance*, Volume 12, issue 6, pp. 861-876, (2009)
- [2] Forde, F, Jacquier, A, and Mijatovic, A. Asymptotic formulae for implied volatility under the heston model. Forthcoming in Proceedings of the Royal Society A, 2009. [1](#), [2](#)
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