Pricing European Options Under the 4/2 Stochastic Volatility Model

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1 The 4/2 stochastic volatility model

The diffusion term of 4/2 volatility model is obtained simply by combining that of the Heston model and the 3/2 model :

$$\frac{dS_t}{S_t} = rdt + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})dZ_t \tag{1}$$

where the stochastic variance factor V follows a CIR process as in the Heston model :

$$dV_t = \kappa(\theta - V_t)dt + \sigma V_t^{1/2} dW_t \tag{2}$$

with parameters $r, \kappa, \theta, \sigma \in \mathbb{R}_+$; $a, b \in \mathbb{R}$; $V_0 = v \in \mathbb{R}_+$.
Z and W are two Wiener processes with correlation ρ :

$$d\langle W, Z\rangle_t = \rho dt$$

The CIR process remains positive if Feller's condition $2\kappa\theta > \sigma^2$ is satisfied.

By taking $X_t = V_t^{-1}$, we have the dynamic X_t including a power 3/2 term :

$$dX_t = \tilde{\kappa} X_t(\tilde{\theta} - X_t)dt + \tilde{\sigma} X_t^{3/2} dW_t$$

where $\tilde{\kappa} = \kappa\theta - \sigma^2$, $\tilde{\theta} = \kappa/(\kappa\theta - \sigma^2)$ and $\tilde{\sigma} = -\sigma$.

By taking resp. $(a = 1;\ b = 0)$ and $(a = 0;\ b = 1)$, we are able to recover Heston model and 3/2 model.

In order to make the Heston term and the 3/2 term comparable, we need to have $b \approx aV_0$. We can for example take a, b such that $a\sqrt{V_0} + b/\sqrt{V_0} = \sqrt{V_0}$ to satisfy the initial condition on the volatility.

We note $\Psi_T(u) = \mathbb{E}[e^{uY_T}]$ the characteristic function of the log price $Y_T = \log(S_T)$.

2 Direct calculation of the European option prices

Knowing the cumulative distribution function $F_{Y_T}(x) = \mathbb{P}(Y_T \leqslant x)$, we are able to valuate a European put option with the formula thanks to a numerical integration :

$$P = e^{-rT}\mathbb{E}[(K - S_T)^+] = e^{-rT}\int_{-\infty}^{k} e^x F_{Y_T}(x)dx$$

The cumulative distribution function is obtained from the characteristic function $\Psi_T(u)$ using another integration given by the inverse theorem of Gil-Pelaez (1951) :

$$F_{Y_T}(x) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-\mathbf{i}ux}\Psi_T(\mathbf{i}u)}{\mathbf{i}u}du$$

Thus, we have a double inproper integral to evaluate, it involves the control of both truncation error and discretization error.

3 Adaptation of the FFT method to the 4/2 model

In the paper of Carr and Madan (1999) we are able to calculate the option price using only one integral, moreover we are able to apply the Fast Fourier Transform (FFT) alogrithm for the discretized sum.

$$C_T(k) \approx \frac{\exp(-łphak)}{\pi}\sum_{j=0}^{N-1} e^{-\mathbf{i}v_j k}\widehat{c_T}(v_j)\eta$$

where α is chosen such that $\Psi_T(\alpha+1) = \mathbb{E}[S_T^{\alpha+1}] < \infty$ and $\widehat{c_T}(v)$ the Fourier tranform of $c_T(k) = \exp(\alpha k)C_T(k)$ with $\alpha > 0$, it is defined by :

$$\widehat{c_T}(v) = \int_{-\infty}^{+\infty} e^{\mathbf{i}vk}c_T(k)dk = \frac{e^{-rT}\Psi_T[(\alpha+1)+\mathbf{i}v]}{\alpha^2+\alpha-v^2+\mathbf{i}(2\alpha+1)v}$$

As output, we have European call option prices corresponding to N values of log strike with a step λ. In order to apply the FFT algorithm of Cooley-Tukey (1965), the relation $\lambda\eta = 2\pi/N$ must be satisfied, where η is the grid for the discretized integral. By Simpson's rule, we finally have :

$$C_T(k_n) \approx \frac{\exp(-\alpha k_n)}{\pi}\sum_{j=0}^{N-1} e^{-\mathbf{i}\frac{2\pi}{N}jn}e^{\mathbf{i}(b-Y_0)v_j}\widehat{c_T}(v_j)\frac{\eta}{3}[3+(-1)^j-\delta_{j-1}]$$

4 Adaptation of Broadie and Kaya's method to the 4/2 model

4.1 Exact simulation of the CIR process

Given V_0, the initial value of the CIR process, the distribution of V_t is, up to a scale factor, a noncentral chi-square. Its transition law is given by :

$$V_t = \frac{\sigma^2(1-e^{-\kappa t})}{4\kappa}\chi'^2_d\left(\frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}V_0\right)$$

with $d = 4\theta\kappa/\sigma^2$ degrees of freedom and $\lambda = \frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}V_0$ the noncentrality parameter.

When $d > 1$, a noncentral chi-squared distribution of d degrees of freedom is linked to a ordinary chi-squared distribution of $d-1$ degrees of freedom by the following relation :

$$\chi'^2_d(\lambda) = \chi'^2_1(\lambda) + \chi^2_{d-1}$$

where $\chi_1'^2(\lambda)$ is equal in law to $(G+\sqrt{\lambda})^2$ with G a standard normal variable.

More generally, for $d>0$, the simulations are also made possible possible for $0<d<1$, *i.e.* when $4\theta\kappa/<\sigma^2$. In fact, we have

$$X^2_{d+2N_\lambda} \stackrel{law}{=} X_d'^2(\lambda) \text{ with } N_\lambda \sim \text{Poisson}(\lambda/2).$$

4.2 Exact simulation of the 4/2 model

We note this time $\Phi_t(x)=\mathbb{E}[e^{\mathbf{i}xY_t}|V_t]$, the characteristic function conditional to the stochastic volatility, we are able to apply the algorithm of exact simulation in Broadie and Kaya (2006) to obtain a trajectory of $(S_{t_i}, V_{t_i})_{0\leqslant i\leqslant N}$.

We use the inversion formula of Gil-Pelaez once more for the conditional cumulative distribution function :

$$F_{Y_t|V_t}(u)=\mathbb{P}[Y_t\leqslant u|V_t]=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{\text{Im}[e^{-\mathbf{i}xu}\Phi_t(x)]}{x}dx$$

Then we can apply Newton's method, or preferably Brent-Dekker's method to inverse the function $F_{Y_t|V_t}(u)$.

5 Numerical schemes for the 4/2 model

5.1 Modified Euler's method

We first consider the classical Euler's scheme for the CIR process :

$$V_{t_{i+1}}=V_{t_i}+\kappa(\theta-V_{t_i})\Delta t+\sigma\sqrt{(V_{t_i})^+}\Delta W_{t_{i+1}}$$

For the dynamic of price under the 4/2 model, we have to introduce two positive functions f and g due to the square root and its inverse :

$$S_{t_{i+1}}=S_{t_i}[1+r\Delta t+(af(\sqrt{V_{t_i}})+\frac{b}{\sqrt{g(V_{t_i})}})(\rho\Delta W_{t_{i+1}}+\sqrt{1-\rho^2}\Delta W^\perp_{t_{i+1}})]$$

We can take for example $f(v)=(v)^+$ and $g(v)=\max(v,V_{\text{inf}})$ with for example $V_{\text{inf}}=V_0/16$.

5.2 Alfonsi's second order discretization scheme

As in Alfonsi (2010), we first provide the second order discretization scheme in terms of weak error. In our model where the Feller's condition is satisfied, we simply have $\hat{X}^x_t=\varphi(x,t,\sqrt{t}N)$: The value of the CIR process departing from x after a time t, where N is a standard normal random variable with :

$$\varphi(x,t,w)=e^{-\frac{kt}{2}}(\sqrt{(a-\sigma^2/4)\psi_k(t/2)+e^{-\frac{kt}{2}}x}+\frac{\sigma}{2}w))^2+(a-\sigma^2/4)\psi_k(t/2)$$

We also generalize the scheme for Heston model in the paper of Alfonsi :

We split the operator associated to our process into two operators, the first one gives :

$$\begin{cases} X_t^1 & = X_0^1 + \int_0^t \kappa(\theta - X_s^1)ds + \int_0^t \sigma\sqrt{X_s^1}dW_s \\ X_t^2 & = X_0^2 + \int_0^t rX_s^2 ds + \int_0^t (a\sqrt{X_s^1} + \frac{b}{\sqrt{X_s^1)}})X_s^2 dZ_s \\ X_t^3 & = \int_0^t a^2 X_1(s)ds \end{cases}$$

Using Itô's lemma, we can write :

$$\begin{aligned} X_t^2 = X_0^2 \ \exp[(r - ab - \frac{a\rho\kappa\theta}{\sigma} + \frac{b\rho\kappa}{\sigma})t + \frac{a\rho}{\sigma}(X_t^1 - X_t^0) \\ + \frac{b\rho}{\sigma}\log\frac{X_t^1}{X_t^0} + (\frac{\rho\kappa}{a\sigma} - \frac{1}{2})(X_t^3 - X_0^3) \\ + (\frac{\rho}{b\sigma}(\frac{\sigma^2}{2} - \kappa\theta) - \frac{1}{2})(X_t^4 - X_0^4)] \end{aligned}$$

Thus, the values can be calculated by trapezoidal rule.

For the second operator, we have $d\ln(X_t^2) = \sqrt{1-\rho^2}(a\sqrt{X_t^1} + \frac{b}{\sqrt{X_t^1}})dW_t^{\perp}$ and other operators zero.

Finally, as in Alfonsi's paper, we compose the transition probabilities associated to both operators, an appropriate second order discretization scheme can take the form :

$$\frac{1}{2}(\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}^1(t) \circ \hat{p}_x^2(t))$$

However, from a numerical point of view, we have to calculate twice the composition and need twice more costly. The idea is to take a Bernoulli variable with parameter 1/2. For example, when it is equal to 1 with probability 1/2 we apply $\hat{p}^2(t) \circ \hat{p}_x^1(t)$ and $\hat{p}^1(t) \circ \hat{p}_x^2(t)$ otherwise.

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