Larsson and Trolle-Schwartz

Pu Jiang

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1 Swaptions

In this last section, we will present the commodity swap options, or swaptions. We will present briefly the idea of the approximation given by Larsson (2011) in [2] in the general HJM framework. Then we will give an approximation formula for the Gaussian 2-factor model that we implemented.

1.1 Commodity Swap and Swaption

We consider a sequence of dates $t < T_1 < \cdots < T_n$, where t denotes the current time. At each date i, the holder of a commodity swap pays the strike price K and receive the underlying commodity's spot price S_{T_i}. The discounted payoff of the swap at t is

$$\sum_{i=1}^{N} e^{-\int_t^{T_i} r_s ds}(S_{T_i} - K) \tag{1}$$

where r_t is the spot interest rate.
By taking the expression above under the risk-neutral probability, we obtain the value of the swap as:

$$V(t) = \sum_{i=1}^{N} B(t, T_i)(F(t, T_i) - K) \tag{2}$$

where

- $B(t,T) = e^{-\int_t^T f(t,s)ds}$ is the bound price, and $f(t,s)$ represents the continuously compounded forward rate
- $F(t,T) = \dfrac{\mathbb{E}\left[e^{-\int_t^T r_s ds} S_T\right]}{B(t,T)} = S_t e^{-\int_t^T (f(t,s) - q(t,s))ds}$ is the forward price and $q(t,s)$ represents the forward convenience yield.

  We are going to give more details in the following subsection, which presents the HJM framework.

We can also express the swap value as

$$V(t) = A(t)(y(t) - K) \tag{3}$$

where

- $A(t) := \sum_{i=1}^{N} B(t, T_i)$ is often called par swap price
- $y(t) := \frac{\sum_{i=1}^{N} B(t,T_i)F(t,T_i)}{\sum_{i=1}^{N} B(t,T_i)}$

A commodity swaption is an option on a commodity swap. The swaption gives the holder the right to enter at a future date T, a swap maturing at $T_n > T$. The swaption's price is therefore:

$$P = \mathbb{E}[e^{-\int_t^T r_s ds} V(T)^+] \tag{4}$$

1.2 HJM framework

We present here the HJM framework, introduced by Heath, Jarrow and Morton (1992) to model the evolution interest rate.
In this model, S_t, $f(t,s)$ and $q(t,s)$ have the following dynamics:

$$\begin{array}{rl} dS_t & = S_t[\mu_S(t)dt + \sigma_S(t)dW_t] \\ df(t,s) & = \mu_f(t,s)dt + \sigma_f(t,s)dW_t \\ dq(t,s) & = \mu_q(t,s)dt + \sigma_q(t,s)dW_t \end{array} \tag{5}$$

We will use the following notation later:

$$\begin{array}{l} \Sigma_f(t,T) := \int_t^T \sigma_f(t,s)ds \\ \Sigma_q(t,T) := \int_t^T \sigma_q(t,s)ds \end{array} \tag{6}$$

where W_t is a multi-dimensional standard Brownian motion under the risk-neutral probability.

Heath, Jarrow and Morton showed that the following relations must be satisfied in absence of arbitrage:

$$\begin{array}{rl} \mu_S(t) & = r_t - q_t \\ \mu_f(t,s) & = \sigma_f(t,T)\Sigma_f(t,T) \\ \mu_q(t,s) & = \sigma_q(t,T)\Sigma_q(t,T) \end{array} \tag{7}$$

1.3 Approximation of the swaption

The approximation consists in approximating the following values by their initial values:

$$\begin{aligned} \omega_f(t,T_i) &= \frac{B(t,T_i)}{\sum_{k=1}^n B(t,T_k)} \\ \omega_q(t,T_i) &= \frac{B(t,T_i)F(t,T_i)}{\sum_{k=1}^n B(t,T_k)F(t,T_k)} \end{aligned} \tag{8}$$

They are stochastic weights, positive and whose sum is one, so their variations are not big in magnitude.
This approximation leads to the following pricing formula, whose proof is not given here, see [2].

Pricing Formula. *Assume that the volatilities $\sigma_S(t)$, $\sigma_f(t,s)$ and $\sigma_q(t,s)$ are deterministic functions of t and s, then the price of the swaption is given by*

$$P \approx A(t)(y(t)N(d_1) - KN(d_2)) \tag{9}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(y(t)/K)}{\sqrt{\Psi(t,T)}} + \frac{\Psi(t,T)}{2} \\ d_2 &= \frac{\ln(y(t)/K)}{\sqrt{\Psi(t,T)}} - \frac{\Psi(t,T)}{2} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Psi(t,T) &= \sum_{1\le i,j\le N} \omega_f(t,T_i)\omega_f(t,T_j)\int_t^T \Sigma_f(u,T_i)\Sigma_f(u,T_j)du \\ &+ \sum_{1\le i,j\le N} \omega_q(t,T_i)\omega_q(t,T_j)\int_t^T \Sigma_q(u,T_i)\Sigma_q(u,T_j)du \\ &+ \int_t^T \|\sigma_S(u)\|^2 du - 2\sum_{1\le i,j\le N} \omega_f(t,T_i)\omega_q(t,T_j)\int_t^T \Sigma_f(u,T_i)\Sigma_q(u,T_j)du \\ &+ 2\sum_{i=1}^N \left[\left(\omega_f(t,T_i)\int_t^T \sigma_S(u)\Sigma_f(u,T_i)du\right) - \left(\omega_q(t,T_i)\int_t^T \sigma_S(u)\Sigma_q(u,T_i)du\right)\right] \end{aligned} \tag{11}$$

1.4 Gaussian 2-factor model

The approximation can be applied for the frequently used Gaussian 2-factor model:
The short rate r is assumed to be constant, so $B(t,T) = e^{-r(T-t)}$. W_t is a 2D standard Brownian motion under the risk-neutral measure and the volatilities are:

$$\begin{aligned} \sigma_S(t) = &\quad \sigma_S \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sigma_q(t,s) = &\quad \sigma_q e^{-\alpha(s-t)} \begin{pmatrix} \rho \\ \sqrt{1-\rho^2} \end{pmatrix} \end{aligned} \tag{12}$$

where the constants σ_S, σ_q, α are positive and $\rho \in [0,1]$.

In this model, we can explicit (11) a bit more, and we obtain the following explicit pricing formula that we implemented.

Pricing Formula. *The price of the commodity swaption at $t = 0$ is given by using the following explicit expression of $\Psi(0,T)$:*

$$\begin{aligned} \Psi(0,T) = &\quad \sum_{1\le i,j\le N} \omega_q(0,T_i)\omega_q(0,T_j)\mathcal{J}(T,T_i,T_j) + \sigma_S^2 T \\ &\quad -2\sigma_S\rho\sum_{i=1}^{n}\omega_q(0,T_i)\mathcal{I}(T,T_i) \end{aligned} \tag{13}$$

where

$$\begin{aligned} \mathcal{I}(T,T_i) = &\quad \frac{\sigma_q}{\alpha}\left(T - \frac{1}{\alpha}(e^{-\alpha(T_i-T)} - e^{-\alpha T_i})\right) \\ \mathcal{J}(T,T_i,T_j) = &\quad \frac{\sigma_q^2}{\alpha^2}\left(T - \frac{1}{\alpha}[(e^{-\alpha(T_i-T)} - e^{-\alpha T_i}) - (e^{-\alpha(T_j-T)} - e^{-\alpha T_j})]\right. \\ &\quad \left. + \frac{1}{2\alpha}[e^{-\alpha(T_i+T_j-T)} - e^{-\alpha(T_i+T_j)}]\right) \end{aligned} \tag{14}$$

The numerical results are the same as in [2], we do not repeat them here.

1.5 Function usage

```
static int ap_schwartz([arguments])
```

The arguments are the following:

- `double r`
  The risk-free interest rate r
- `double divid`
  Convenience yield of the asset q
- `double sigma_q`
  Parameter σ_q in the Gaussian 2-factor model
- `double sigma_s`
  Parameter σ_S in the Gaussian 2-factor model
- `double alpha`
  Parameter α in the Gaussian 2-factor model
- `double rho`
  The correlation factor ρ in the Gaussian 2-factor model
- `double Nominal`
  Quantity of options bought
- `double tenor`
  The difference $T_{i+1} - T_i$
- `double opt_mat`
  Maturity T of the option
- `double swap_mat`
  Maturity T_n of the Swap
- `double K`
  Moneyness $K/y(0)$
- `double *ptprice`
  The pointer to the price computed using our function

2 Trolle-Schwartz formula

2.1 The Trolle-Schwartz "unspanned" model

We denote by $S(t)$ the time-t spot price of the commodity and by $y(t,T)$ the time-t instantaneous forward cost of carry at time T, the instantaneous spot cost of carry of $S(t)$ is given by $\delta(t) := y(t,t)$.
We assume that S and y depend on two volatility factors v_1 and v_2, their dynamics under the risk-neutral probability is given by (see [1]):

$$\frac{dS(t)}{S(t)} = \delta(t)dt + \sigma_{S1}\sqrt{v_1(t)}dW_1(t) + \sigma_{S2}\sqrt{v_2(t)}dW_2(t) \tag{15}$$

$$dy(t,T) = \mu_y(t,T)dt + \sigma_{y1}(t,T)\sqrt{v_1(t)}dW_3(t) + \sigma_{y2}(t,T)\sqrt{v_2(t)}dW_4(t) \tag{16}$$

$$dv_1(t) = (\eta_1 - \kappa_1 v_1(t) - \kappa_{12}v_2(t))dt + \sigma_{v1}\sqrt{v_1(t)}dW_5(t) \tag{17}$$
$$dv_2(t) = (\eta_2 - \kappa_{21} v_2(t) - \kappa_2 v_2(t))dt + \sigma_{v2}\sqrt{v_2(t)}dW_6(t) \tag{18}$$

where the W_i are standard Brownian motions under the risk-neutral probability measure. W_1, W_3 and W_5 can be correlated, and the same for W_2, W_4 and W_6, the corresponding correlation factors are denoted by ρ_{ij}.
The drift term μ_y must satisfy an analogous constraint to the HJM drift condition in absence of arbitrage:

$$\begin{aligned} \mu_y(t,T) = \quad & -v_1(t)\sigma_{y_1}(t,T)\left(\rho_{13}\sigma_{S1} + \int_t^T \sigma_{y1}(t,u)du\right) \\ & -v_2(t)\sigma_{y_2}(t,T)\left(\rho_{24}\sigma_{S2} + \int_t^T \sigma_{y2}(t,u)du\right) \end{aligned} \tag{19}$$

We will use the time-homogeneous specification for the volatilities of the Forward cost of carry:

$$\sigma_{y_i}(t,T) = \alpha_i e^{-\gamma_i(T-t)} \tag{20}$$

We want to compute the the price at time $t=0$ of a Put option maturing at T_0 on a futures contract maturing at $T_1 > T_0$.

2.2 Characteristic function

The characteristic function of the future contract's log-price is defined by :

$$\Phi(u,T_0,T_1) = \mathbb{E}[e^{u\log(F(T_0,T_1))}] \tag{21}$$

We have the following expression for Φ:

$$\Phi(u,T_0,T_1) = \exp[M(T_0) + N_1(T_0)v_1(0) + N_2(T_0)v_2(0) + u\log(F(0,T_1))] \tag{22}$$

where M, N_1 and N_2 solve the following system of ODEs with boundary conditions $M(0) = N_1(0) = N_2(0) = 0$:

$$\frac{dM}{d\tau} = N_1(\tau)\eta_1 + N_2(\tau)\eta_2 \tag{23}$$

$$\begin{aligned}\frac{dN_1}{d\tau} = & \quad -N_2(\tau)\kappa_{21} + N_1(\tau)(-\kappa_1 + u\sigma_{v1}(\rho_{15}\sigma_{S1} + \rho_{35}B_1(T_1))) \\ & \quad +\frac{1}{2}N_1(\tau)^2\sigma_{v_1}^2 + \frac{1}{2}(u^2-u)(\sigma_{S1}^2 + B_1(T_1)^2 + 2\rho_{13}\sigma_{S1}B_1(T_1))\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{dN_2}{d\tau} = & \quad -N_1(\tau)\kappa_{12} + N_2(\tau)(-\kappa_2 + u\sigma_{v2}(\rho_{26}\sigma_{S2} + \rho_{46}B_2(T_1))) \\ & \quad +\frac{1}{2}N_2(\tau)^2\sigma_{v2}^2 + \frac{1}{2}(u^2-u)(\sigma_{S2}^2 + B_2(T_1)^2 + 2\rho_{24}\sigma_{S2}B_2(T_1))\end{aligned} \tag{25}$$

and where B_i denotes

$$B_i(\tau) = \frac{\alpha_i}{\gamma_i}(1 - e^{-\gamma_i\tau}) \tag{26}$$

This system of ODE's are solved numerically using the 4th order Runge-Kutta algorithm.

2.3 The pricing formula

The price $P(T_0, T_1, K)$ of a Put option maturing at T_0 with strike K on a futures contract maturing at T_1 is given by:

$$\begin{aligned}P(T_0, T_1, K) \quad &= \mathbb{E}\left[e^{\int_t^{T_0} r(s)ds}(K - F(T_0, T_1))^+\right] \\ &= B(T_0)(KG_{0,1}(\log K) - G_{1,1}(\log K))\end{aligned} \tag{27}$$

where $B(T_0)$ is the price of a zero-coupon bond maturing at T_0 and

$$\begin{aligned}G_{a,b}(x) \quad &= \mathbb{E}\left[e^{a\log(F(T_0,T_1))}\mathbf{1}_{b\log(F(T_0,T_1))<y}\right] \\ &= \frac{\Phi(a, T_0, T_1)}{2} - \frac{1}{\pi}\int_0^\infty \Im(\Phi(a + iub, T_0, T_1)e^{-iux}\frac{du}{u}\end{aligned} \tag{28}$$

The price $C(T_0, T_1, K)$ of the Call counterpart is given by a similar formula:

$$\begin{aligned}C(T_0, T_1, K) \quad &= \mathbb{E}\left[e^{\int_t^{T_0} r(s)ds}(F(T_0, T_1) - K)^+\right] \\ &= B(T_0)(G_{1,-1}(-\log K) - KG_{0,-1}(-\log K))\end{aligned} \tag{29}$$

The integral is computed using the Gauss-Legendre quadrature formula, using $N/2$ integration points for $[0; 50]$ and $N/2$ for $[50; 400]$.

2.4 Function usage

```
static int ap_schwartztrolle([arguments]);
```

The arguments are the following:

- `NumFunc_1 *p`

  `p->Compute` can take the values: `\&Put` or `\&Call`
- `double f0T1`

  Price of the futures contract $F(0, T_1)$
- `double P0T0`

  Price of the zero-coupon bond $B(T_0)$
- `PnlVect *v0`

  The vector $(v_1(0), v_2(0))$
- `PnlVect *eta`

  The vector (η_1, η_2)
- `PnlVect *kappa`

  The vector $(\kappa_1, \kappa_2, \kappa_{21}, \kappa_{12})$
- `PnlVect *sigma`

  The vector $(\sigma_{S1}, \sigma_{S2}, \sigma_{v1}, \sigma_{v2})$
- `PnlVect *rho`

  The vector $(\rho_{13}, \rho_{15}, \rho_{35}, \rho_{24}, \rho_{26}, \rho_{46})$
- `PnlVect *alpha`

  The vector (α_1, α_2)
- `PnlVect *gammac`

  The vector (γ_1, γ_2)
- `double opt_mat`

  The time of maturity T_0 of the option
- `double fut_mat`

  The time of maturity T_1 of the futures contract
- `double *ptprice`

  The pointer to the price computed using our function

References

[1] A.B.Trolle E.Schwartz. Unspanned stochastic volatility and the pricing of commodity derivatives. *Review of Financial Studies*, 22-11:4423–4461, 2009. 8

[2] K.Larsson. Pricing commodity swaptions in multifactor models. *Journal of Derivatives*, 19-2:32–44, 2011. 3, 5, 6