LRS model

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Premia 22

1 LRS

1.1 HJM

Heath, Jarrow and Morton (1992) assumed that, for a fixed maturity T, the instantaneous forward rate $f(t,T)$ evolves, under a given measure, according to the following diffusion process :

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

But this dynamics is not necessarily arbitrage-free. As in the Black & Scholes framework, the main hypothesis to avoid arbitrage situations is : *there exists a probability measure, equivalent to the given measure, under which the price of the zero coupon bond is a martingale. It is called the risk-neutral measure*. To verify this hypothesis, the drift and the volatility cannot be independant.

So, for a one factor model, we must have :

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)ds$$

so that the f dynamics is :

$$df(t,T) = \sigma(t,T) \left(\int_t^T \sigma(t,s)ds \right) dt + \sigma(t,T)dW(t)$$

and r_t can be expressed as :

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(u,t) \int_u^t \sigma(u,s)dsdu + \int_0^t \sigma(s,t)dW(s) \quad (1)$$

The dynamics of the zero coupon bond price is also the following :

$$dP(t,T) = P(t,T) \left[r(t)dt - \left(\int_t^T \sigma(t,s)ds \right) dW(t) \right]$$

1.2 The Lee, Ritchken and Sankarasubramanian Framework

The problem is now the choice of the volatility process. An arbitrary specification of the forward-rate volatility will likely lead to a non-Markovian instantaneous short-rate process. In such a case, we would soon encounter major computational problems when discretizing the dynamics (1) for the pricing of a general derivative.

These pricing problems are solved by Ritchken and Sankarasubramanian. They thought that, even though the short-rate process is not Markovian, there may exist a higher-dimensional Markov process having the short-rate as one of its component. Precisely, they proved the following.

Proposition 1 (Ritchken and Sankarasubramanian)**.** *Consider a one-factor HJM model. If the volatility function $\sigma(t,T)$ is differentiable with respect to T, a necessary and sufficient condition for the price of any (interest-rate) derivative to be completely determined by a two-state Markov process $\chi(.) = (r(.), \phi(.))$ is that the following condition holds :*

$$\sigma(t,T) = \eta(t) \exp\left(- \int_t^T \kappa(x)dx \right)$$

where η is an adapted process and κ is a deterministic (integrable) function. In such a case, the second component of the process χ is defined by

$$\phi(t) = \int_0^t \sigma(s,t)^2 ds$$

Accordingly, zero-coupon-bond prices are explicitly given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\frac{1}{2}\Lambda^2(t,T)\phi(t) + \Lambda(t,T)[f(0,t) - r(t)]\right)$$

where

$$\Lambda(t,T) = \int_t^T \exp\left(-\int_t^u \kappa(x)dx\right) du$$

Differentiation of equation (1) shows that, under the Ritchken and Sankarasubramanian class of volatilities, the process χ, and hence the instantaneous short-rate r, evolve according to

$$d\chi(t) = \begin{pmatrix} dr(t) \\ d\phi(t) \end{pmatrix} = \begin{pmatrix} \mu(r,t)dt + \eta(t)dW(t) \\ \left[\eta^2(t) - 2\kappa(t)\phi(t)\right] dt \end{pmatrix} \tag{2}$$

with

$$\mu(r,t) = \kappa(t)[f(0,t) - r(t)] + \phi(t) + \frac{\partial}{\partial t} f(0,t)$$

The yield curve dynamics described by (2) can be, therefore, discretized. This was suggested by Li, Ritchken and Sankarasubramanian. We use their method here in our algorithm under the particular case where

$$\sigma(t,T) = s\,[r(t)]^b\, e^{-\kappa(T-t)}$$

which is the so called LRS model.

2 Tree method

See here