BACKWARD INDUCTION METHOD WITH SPLITTING AND MATRIX EXPONENTIALS FOR OPTION PRICING UNDER LÉVY PROCESSES

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Abstract. We describe "The method of pseudodifferential operators" (MPsDO-method) for pricing options for a wide class of Lévy processes. The method solves backward jump-diffusion PIDEs with splitting and matrix exponentials. The key idea behind the approach involves representing a jump operator as a pseudodifferential operator with subsequent transforming into operator exponential.The method implemented into Premia 19 for European options is based on the one developed in Itkin (Algorithmic Finance 3:233-250, 2014; J. Comput. Finance 19:29-70, 2016).

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1. Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

By now, there exist several large groups of relatively universal numerical methods for pricing of American and barrier options under exponential Lévy processes. The number of publications is huge, and, therefore, an exhaustive list is virtually impossible. We concentrate on the one-dimensional case.

Existing numerical methods in literature can be categorized into three groups: Monte Carlo simulation, partial-(integro) differential equation (PIDE) methods, and backward induction methods. We will consider the last group.

The backward induction methods are based on the fact that the risk-neutral valuation formula for the European option can be seen as a convolution of the payoff function with the transition density. The key idea is to set up a time lattice and view the option as of European type between two adjacent dates. Hence, the backward induction method requires the transition density to be known in closed-form, which is the case in e.g. the Black-Scholes model and Merton's jump-diffusion model. The approximation proposed by Geske and Johnson (1984) uses the discretization of the time parameter and the backward induction for pricing American options in the GBM model. The method was extended in Boyarchenko and Levendorskiǐ (2002) for some Lévy models, and its applications can be founded e.g. in Kudryavtsev and Levendorskiǐ (2006) and Levendorskiǐ et al. (2006). If there is no an explicit formula for the probability density, it can be recovered by inverting the characteristic function, so the method can be used for a wide range of Lévy models.

Since convolutions can be handled very efficiently by means of the Fast Fourier Transform (FFT), an overall complexity of the method is $O(mn \ln n)$, where m and n are the numbers of points on the grid in time and space, respectively. The FFT-based backward induction method was applied in Jackson et al. (2008), see also Lord et al. (2008). In terms of the theory of pseudodifferential operators (PDOs), at each time step, the FFT-based backward induction method implements action of the PDO which symbol is the characteristic function.

The method suggested in Itkin (2014,2016) solves backward jump-diffusion PIDEs for option prices by splitting the related operator into differential and jump parts. The key idea behind the approach involves representing a jump operator as a PDO with subsequent transforming into operator exponential.

2. Lévy processes: basic facts

2.1. **General definitions.** A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato [20]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\le 1})\nu(dy), \tag{2.1}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2.2}$$

If the jump component is a process of finite variation, equivalently, if

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, |y|\}F(dy) < +\infty, \tag{2.3}$$

then the last term in the integrand in (2.1) can be integrated out and added to the drift term. Then we obtain

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y})F(dy), \tag{2.4}$$

with a different μ, and the new μ is the drift of the Gaussian component.

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\mathrm{Im}\,\xi \in (-1, 0)$ and continuous continuation into the closed strip $\mathrm{Im}\,\xi \in [-1, 0]$.

The infinitesimal generator of X, denote it L, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x) - y\mathbf{1}_{|y|\le 1}\frac{\partial u}{\partial x}(x))\nu(dy). \tag{2.5}$$

The infinitesimal generator L also can be represented as a pseudo-differential operator (PDO) with the symbol $-\psi(\xi)$: $L = -\psi(-i\partial_x)$. Recall that a PDO $A = a(-i\partial_x)$ acts as follows:

$$Au(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi, \tag{2.6}$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx.$$

Note that the inverse Fourier transform in (2.6) is defined in the classical sense only if the symbol $a(\xi)$ and function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform by duality.

Further, if the riskless rate, r, is constant, and the stock pays dividends q, then the discounted price process must be a martingale. Equivalently, the following condition must hold

$$r - q + \psi(-i) = 0, \tag{2.7}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - q - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty}(1 - e^y + y\mathbf{1}_{|y|\le 1})F(dy). \tag{2.8}$$

Example 1. [**Tempered stable Lévy processes**] The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{2.9}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (2.9) is derived in Boyarchenko and Levendorskiǐ (2000, 2002) from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$F_\mp(dy) = ce^{\lambda_\pm y}|y|^{-\nu-1}dy; \tag{2.10}$$

in the first two papers, the name extended Koponen family was used. Later, the same class of processes was used in Carr et al. (2002) under the name CGMY-model. More general version with $c_\pm$ instead of c, and the different exponents $\nu_\pm$ is known as a Tempered Stable Lévy model. In this case, we have for $\nu_+, \nu_- \in (0,2), \nu_+, \nu_- \neq 1$

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}], \tag{2.11}$$

where $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2. [Normal Inverse Gaussian processes] A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see Barndorff-Nielsen (1998))

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{2.12}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example 3. [Variance Gamma processes] The Lévy density of a Variance Gamma process is of the form (2.10) with $\nu = 0$, and the characteristic exponent is given by (see Madan et al. (1998))

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{2.13}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 4. [Kou model] If $F_\mp(dy)$ are given by exponential functions on negative and positive axis, respectively:

$$F_\mp(dy) = c_\pm(\pm\lambda_\pm)e^{\lambda_\pm y},$$

where $c_\pm \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain Kou model. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{2.14}$$

The version with one-sided jumps is due to Das and Foresi (1996), the two-sided version was introduced in Duffie, Pan and Singleton (2000), see also S.G. Kou (2002).

3. Solution to pure jump equation

We shortly describe a numerical framework of Itkin (2014,2016) to value options in exponential Lévy models. As a basic example to illustrate the method we consider pricing European put options under Tempered stable Lévy processes. Let T, K be the maturity, strike, and the stock price $S_t = e^{X_t}$ is an exponential Lévy process under a chosen risk-neutral measure. The riskless rate is assumed constant $r > 0$.

Then the payoff at maturity is $G(x) = (K - e^x)_+$, and the no-arbitrage price of the European option at time $t < T$ and $X_t = x$ is given by

$$V(t,x) = V(T,G;t,x) = E^{t,x}\left[e^{-r(T-t)}G(X_T)\right]. \tag{3.1}$$

It is well-known that $V(t,x)$ is the solution to the following problem

$$(\partial_t + L - r)V(t,x) = 0, t < T; \tag{3.2}$$

$$V(T,x) = G(x). \tag{3.3}$$

It follows from (2.5), that the infinitesimal generator of a Lévy process is the sum of the infinitesimal generator of the diffusion component (with drift) and pure jump component, which we denote by L_G and L_J, respectively. Then we can rewrite (2.5) as

$$Lu = L_G u + L_J u. \tag{3.4}$$

Consider equally spaced dates t_k, $k = 0, 1, \ldots, m$, where $t_0 = 0$, $t_m = T$. Set $\Delta\tau := T/m$. Using splitting technique (for further reading see Marchuk (1978)) as described in Itkin (2014,2016), we approximate $V(x,t)$ in the correspondent discrete time model (3.1) as follows. We have

$$V(x, t_m) = (K - e^x)_+. \tag{3.5}$$

For $k = m-1, m-2, \ldots,$ the numerical scheme includes three steps.

$$V_1(x, t_{k+1}) = \exp\left(\frac{\Delta\tau}{2}(L_G - r)\right) V(x, t_{k+1}); \tag{3.6}$$

$$V_2(x, t_{k+1}) = \exp\left(\Delta\tau L_J\right) V_1(x, t_{k+1}); \tag{3.7}$$

$$V(x, t_k) = \exp\left(\frac{\Delta\tau}{2}(L_G - r)\right) V_2(x, t_{k+1}). \tag{3.8}$$

Thus, instead of an unsteady PIDE, we obtain one PIDE with no drift and diffusion (the second equation in (3.7) and two unsteady PDEs ((3.6) and (3.8)). Recall that $\exp(\tau L)$ is the operator exponential, which acts exactly like the Taylor series expansion of $\exp(\tau L)$ around $\tau = 0$. Let ψ_G and ψ_J be the Gaussian and jump parts of the characteristic exponent ψ in (2.1). Hence, we have

$$L_G u(x) = -\psi_G(-i\partial_x)u(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x), \tag{3.9}$$

$$L_J u(x) = -\psi_J(-i\partial_x)u(x). \tag{3.10}$$

The steps (3.6)–(3.8) can be numerically implemented by using finite difference method. Let ∇_x denote the discrete analogue of ∂_x obtained by finite difference discretization of ∂_x on the space grid $\mathbf{x} = \{x_l\}$. Accordingly, let us define the matrix $A_G = -\psi_G(-i\nabla_x)$ and the $A_J = -\psi_J(-i\nabla_x)$ to be the discrete analogues of the operators L_G and L_J, respectively.

Let A be a matrix which represents differential or jump operator. It follows from Itkin (2016) that the finite difference scheme

$$V(\mathbf{x}, t) = \exp(\Delta\tau A)\, V(\mathbf{x}, t + \Delta\tau)$$

is unconditionally stable in time τ and preserves nonnegativity of the vector $V(\mathbf{x}, t)$ if there exists an M-matrix B such that $\Delta\tau A = -B$, where τ is the time step of the scheme. Once the discretization is performed, we need to compute a matrix exponential $\exp(\Delta\tau A)$, and then a product of this exponential with $V(\mathbf{x}, t+\Delta\tau)$.

This statement gives us a recipe for the construction of the appropriate discretization of the operators L_G and L_J. Notice that (3.6) and (3.8) can be reduced to implicit finite difference schemes

$$\left(1+\frac{\Delta\tau}{2}(r-A_G)\right)V_1(x,t_{k+1}) = V(x,t_{k+1}); \tag{3.11}$$

$$\left(1+\frac{\Delta\tau}{2}(r-A_G)\right)V(x,t_k) = V_2(x,t_{k+1}); \tag{3.12}$$

Since a constant time step is used for computations, the matrix $\exp(\Delta\tau A_J)$ can be precomputed once the space grid.

In order to reach unconditional stability of the finite difference scheme in time τ in (3.11)-(3.12), we need to approximate ∂_x^2 using the central difference, and choose an approximation for the first spatial derivative depending on the drift sign. If $\mu > 0$, we use the forward differences, otherwise we use the backward ones.

Let $A_G = (d_{ij})$ and h be the uniform space step. In the case of Tempered Stable Lévy models (see Example 1), $\sigma = 0$, hence we need to approximate in L_G the first spatial derivative only. In particular, if $\mu > 0$, we set

$$\begin{aligned} d_{i,i} &= 1+\mu\frac{\Delta\tau}{2h}+r\frac{\Delta\tau}{2}; \\ d_{i,i+1} &= -\mu\frac{\Delta\tau}{2h}; \\ d_{i,i+j} &= 0, j\neq 0, j\neq 1; \end{aligned}$$

otherwise, we set

$$\begin{aligned} d_{i,i} &= 1-\mu\frac{\Delta\tau}{2h}+r\frac{\Delta\tau}{2}; \\ d_{i,i-1} &= \mu\frac{\Delta\tau}{2h}; \\ d_{i,i+j} &= 0, j\neq 0, j\neq -1; \end{aligned}$$

We implemented into Premia the case of $\nu_\pm \in (0,1)$. Consider a finite difference approximation for L_J. We represent the correspondent matrix A_J as follows

$$A_J = c_+\Gamma(-\nu_+)[(\lambda_+ I + A^B)^{\nu_+} - \lambda_+^{\nu_+} I] + c_-\Gamma(-\nu_-)[(-\lambda_- I - A^F)^{\nu_-} - (-\lambda_-)^{\nu_-} I], \tag{3.13}$$

where A^B and A^F are the backward and forward first order differences, respectively. It can be shown that A_J is the negative of an M-matrix and it gives $O(h)$ approximation of the operator L_J.

The computation of a real power of a matrix A^ν by definition uses the formula $A^\nu = \exp(\nu \ln A)$, which involves implementation of matrix exponential and matrix logarithm functions. The function of matrix exponential is available in the PNL-library. For the moment, the function of matrix logarithm $\ln A$ only works if A is diagonalizable. In our case, one need to use the Mercator series to compute the logarithms of matrix in A_J.

4. Implementation to the Premia 19

We implemented the MPsDO-method for call and put European options under the Tempered Stable Lévy model (see Example 1). One can use the routine for other types of Lévy processes by replacing the corresponding part with the computation of jump matrix A_J, according to the formulas in Itkin (2014,2016).

Note that in the program implemented to Premia 19 one can manage by three parameters of the algorithm: the space step h, the scale of logprice range L and the number of time steps N. Parameter L controls the size of the truncated region in x-space; it corresponds to the region $(-L; L)$. The typical values of the parameter are $L = 2$ and $L = 3$. To improve the results one should decrease h and/or increase N, when L is fixed.

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