ULTRA FAST PRICING FIRST TOUCH DIGITAL OPTIONS UNDER SPECTRALLY NEGATIVE LÉVY PROCESSES

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Abstract. We describe an approximate method for efficient pricing first touch digital options developed by S. Z. Levendorskiĭ, International Journal of Theoretical and Applied Finance (2017). The method was implemented into program platform Premia for the underlying log-price process being a Brownian motion with the embedded spectrally negative KoBoL.

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Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

In the case of pricing European options in one-factor exponential Levy models, the most popular approach is the Fourier transform method which was applied in [4, 2] and many others. In all these papers, as in most others, the inverse Fourier integral representation is used, and the option price is represented as the integral over an appropriate line in the complex plane parallel to the real axis. A numerical realization of the inverse Fourier transform (iFT) can be handled very efficiently by means of the Fast Fourier Transform (FFT), if we need a set of option prices at different spot/strike levels.

In [5], it was given fairly simple and efficient recommendations for choosing the parameters of the numerical scheme and suggest families of the conformal contour deformations, which greatly increases the rate of convergence of the integral. The resulting pricing formula was called "parabolic iFT" because it can be described as a change of variables in the standard Fourier inversion formula, resembling the analytical expression for a fractional parabola. In cases in which the standard inverse Fourier transform realization may require thousands or even millions of terms, parabolic iFT may sufficiently reduce the number of terms in the integral sum. Notice that parabolic iFT cannot be applied in combination with the FFT technique introduced to finance in [4]. If prices of European options at less than one or two hundred points are needed, then parabolic iFT remains faster than the standard iFT with FFT.

In [11], the fractional-parabolic deformation technique [5, 10] and Wiener-Hopf method used to approximate the price of first touch digital options under spectrally negative Lévy models. Different numerical methods for pricing first touch digitals under Lévy processes that apply Wiener-Hopf approach can be found e.g. in [2, 3, 9, 8].

The method developed in [11] has been implemented into the Premia platform for the one-sided KoBoL model with the non-negligible Brownian motion component.

1. Lévy processes: a short reminder

A Lévy process is a process with stationary independent increments (for details, see e.g. [14]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. We denote it by $F(dy)$. A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1})F(dy), \tag{1.1}$$

where σ^2 is the variance of the Gaussian component, and $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty. \tag{1.2}$$

One says that X is spectrally negative if the measure $F(dy)$ is carried by $(-\infty, 0)$, that is $F((0, +\infty)) = 0$.

Assume that under a risk-neutral measure chosen by the market, the stock has the dynamics $S_t = e^{X_t}$. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\operatorname{Im}\xi \in (-1, 0)$ and continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$. Further, if the riskless rate, r, is constant, and the stock pays dividends, then the following condition must hold

$$r - q + \psi(-i) = 0, \tag{1.3}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - q - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty}(1 - e^{y} + y\mathbf{1}_{|y|\le 1})F(dy). \tag{1.4}$$

Example 1. **[One-sided KoBoL processes]** The characteristic exponent of a spectrally negative KoBoL process (Tempered stable Lévy process) of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^{\nu} - (\lambda_+ + i\xi)^{\nu}], \tag{1.5}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_+ > 0$. The formulas for the characteristic exponents of the general KoBoL model were derived in [1, 2].

Example 2. [One-sided KoBoL processes with the non-negligible Brownian motion component] A Brownian motion with the embedded spectrally negative KoBoL process (Tempered stable Lévy process) of order $\nu \in (0, 2), \nu \neq 1$ can be described by the characteristic exponent of the form

$$\psi(\xi) = \frac{\sigma^2}{2} - i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu], \tag{1.6}$$

where $\sigma > 0$, $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_+ > 0$. We will refer to the model as the BMSNTSL model (Brownian motion and spectrally negative Tempered stable Lévy).

2. Contour deformation method for pricing first touch digital options under spectrally negative Lévy processes, [11]

2.1. **Wiener-Hopf approach and fractional-parabolic deformation technique.** There are several forms of the Wiener-Hopf factorization. The Wiener-Hopf factorization formula used in probability reads:

$$E[e^{i\xi X_T}] = E[e^{i\xi \bar{X}_T}]E[e^{i\xi \underline{X}_T}], \quad \forall\ \xi \in \mathbf{R}, \tag{2.1}$$

where $T \sim \operatorname{Exp} q$, and $\bar{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$ are the supremum and infimum processes. Introducing the notation

$$\phi_q^+(\xi) \;=\; qE\left[\int_0^\infty e^{-qt} e^{i\xi\bar{X}_t}dt\right] = E\left[e^{i\xi\bar{X}_T}\right], \tag{2.2}$$

$$\phi_q^-(\xi) \;=\; qE\left[\int_0^\infty e^{-qt} e^{i\xi\underline{X}_t}dt\right] = E\left[e^{i\xi\underline{X}_T}\right] \tag{2.3}$$

we can write (2.1) as

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi). \tag{2.4}$$

Let T, H be the maturity and the barrier, and $S_t = e^{X_t}$ be the stock price under a chosen risk-neutral measure. The riskless rate r is assumed to be constant. We consider a first touch digital with a barrier from below H and and an expiration date T. The contract pays \$1, as a stock price S_t for first time the crosses the barrier H. If up to the date T the price does not cross the barrier H, the option becomes worthless.

Then the no-arbitrage price of the first touch digital option at time $t = 0$ and $X_0 = x > h(= \ln H)$ is given by

$$V_{f.t.d.}(h, T, x) = E\left[e^{-rT'}\mathbf{1}_{T'\le T} \mid X_0 = x\right], \tag{2.5}$$

where T' is the first entrance time of X_t into $(-\infty, h]$.

Now, we briefly describe the approximate method of computing (2.5) developed in [11] for the BMSNTSL model (see Example 2). We use the notation $x = \ln S$, $x' = x - h + \mu T$, where μ is the coefficient in the linear term $-i\mu\xi$ of the characteristic exponent $\psi(\xi)$ (see (1.6)).

As in [5], we set $\psi_0(\xi) = \psi(\xi)+i\mu\xi$, where ψ_0 is the sum of the characteristic exponents of the BM and jump components.

The Wiener-Hopf method leads to the following formula:

$$V_{f.t.d.}(h,T,x) = \frac{e^{-rT}}{2\pi i}\int_{\mathrm{Re}\,q=\sigma}\frac{e^{qT}}{q-r}\cdot\frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega}\frac{\exp[ix'\xi - T\psi_0(\xi)]}{\phi^+(\xi)(-i\xi)}\cdot\frac{q}{\psi(\xi)+q}d\xi, \tag{2.6}$$

where $\sigma > r$ and $\omega > 0$ under certain additional conditions (see details in [11]).

The method of [11] uses fractional-parabolic deformations as in [5, 10]. First, it is needed to deform the inner contour of integration, then one deforms the outer contour in (2.6) so that the repeated integral becomes absolutely convergent, and changes the order of integration using Fubini's theorem. If the deformation of the inner contour is done correctly, then, for each ξ on the new contour, it is possible to construct the parabolic deformation of the line of integration $\mathrm{Re}\,q = \sigma$ so that, for each ξ, a simple pole $q = ?\psi(\xi)$ of the integrand is crossed. Applying the Cauchy residue theorem as in [10], one finds the approximation to the price as the integral of the residues over the new contour. The resulting leading term can be calculated very fast provided the Wiener-Hopf factor $\phi^+_{-\psi(\xi)}$ can be calculated accurately and fast. The remainder is the repeated integral which is typically very small, especially if the log-price is not close to the barrier and time to maturity $T \geq 1$.

2.2. **Calculation of the leading term for the price of the first touch digital option with the barrier from below.** Let $x\prime \geq 0$. For $\alpha \in [1;2]$, introduce the conformal map χ_α defined on the half plane $\mathrm{Im}\,\eta < \lambda_+$ by

$$\chi_\alpha(\eta) = i\lambda_+ - i\lambda_+^{1-\alpha}(\lambda_+ + i\eta)^\alpha. \tag{2.7}$$

Fix $\omega \in (0, \lambda_+)$ and let L be the image of line $\mathrm{Im}\,\xi = \omega$ under the mapping χ_α. The new method developed in [11] leads to an approximation of the price of the first touch digital given by (2.6) with an integral over the contour L

$$V_{f.t.d,lead}(h,T,x) = \frac{e^{-rT}}{2\pi}\int_L \frac{\exp[ix'\xi - T\psi_0(\xi)]}{\phi^+_{-\psi(\xi)}(\xi)(-i\xi)}\cdot\frac{\psi(\xi)}{\psi(\xi)+r}d\xi. \tag{2.8}$$

Since $x\prime \geq 0$, the integral (2.8) has exactly the same form as the one for the put-like options considered in [5]. The case $x\prime < 0$ is treated similarly to the call-like options considered in [5].

It was shown in [11] that in the case of the BMSNTSL model

$$\phi^+_q(\xi) = \frac{\beta^+_q}{\beta^+_q - i\xi}, \quad \xi \in L$$

where β^+_q is analytic continuation of the unique positive root of the characteristic equation $q+\psi(-i\beta) = 0$ for $q > 0$. The root β^+_q can be calculated numerically for $q \in -\psi(L)$.

One can solve an appropriate ODE with the Newton's method or apply the Muller's algorithm [13]. We implemented the latter into the Premia platform.

In (2.8), change the variable $\xi = \chi_\alpha(\eta + i\omega)$, where $\eta \in \mathbf{R}$:

(2.9)
$$V_{f.t.d,lead}(h,T,x) = \frac{e^{-rT}}{\pi} \operatorname{Re} \int_0^{+\infty} e^{ix'\chi_\alpha(\eta+i\omega) - T\psi_0(\chi_\alpha(\eta+i\omega))} G_0(\chi_\alpha(\eta+i\omega))\chi'_\alpha(\eta+i\omega)d\eta,$$

where

$$G_0(\xi) = \frac{\psi(\xi)}{\phi^+_{-\psi(\xi)}(\xi)(-i\xi)(\psi(\xi)+r)}.$$

An efficient numerical realization of (2.9) starts with a discretization of the integral using the infinite trapezoid rule, denote the discretization step by $\Delta\xi$. Then we truncate the sum from the up and it leads to the final formula

$$V_{f.t.d,lead}(h,T,x) = \frac{e^{-rT}\Delta\xi}{\pi} \sum_{j=0}^{N} \operatorname{Re}(1-\delta_{j0}/2) e^{ix'\chi_\alpha(\eta_j+i\omega) - T\psi_0(\chi_\alpha(\eta+i\omega))} \tag{2.10}$$

$$\times \quad G_0(\chi_\alpha(\eta_j+i\omega))\chi'_\alpha(\eta_j+i\omega), \tag{2.11}$$

where $\eta_j = j\Delta\xi$ and N is the number of terms in the truncated sum.

According recommendations in [5, 11], we set $\alpha = 1.4$ in (2.7) for the case of the BMSNTSL model. For the typical values of the BMSNTSL parameters, the following choice ω, $\Delta\xi$ and N is typically optimal [11]. Assuming that the error tolerance $\epsilon > 0$ is small, we set

$$\omega = 0.4\lambda_+\left[1 - \left(\frac{\lambda_+ - \mu_+}{\lambda_+}\right)^\alpha\right], \tag{2.12}$$

$$\omega_\alpha = \lambda_+ - \lambda_+^{1-\alpha}(\lambda_+ - 1.8\omega)^\alpha, \tag{2.13}$$

$$\Delta\xi = \frac{1.8\pi\omega}{\ln(1/\epsilon_1) + \mu x'\omega_\alpha - T\psi_0(i\omega_\alpha)}. \tag{2.14}$$

where $\epsilon_1 = 2\pi\epsilon e^{rT}$ and $\mu_+ = \min\{\lambda_+, 0.2\ln(1/\epsilon_1)/(T\sigma^2)\}$.

More detailed recommendations about the most optimal choice of the algorithm parameters can be found in [11].

3. Implementation to the Premia

We implemented fractional-parabolic deformation method for first touch digital options with the barrier from below under the BMSNTSL model (see Example 2). One can use the routine for the other types of spectrally negative Lévy processes by replacing the corresponding part with the computation of the characteristic exponent provided that the Wiener-Hopf factor $\phi^+_{-\psi(\xi)}$ can be efficiently computed and a justification of the contour transformation is done.

Note that in the program implemented to Premia one can manage by the parameter N of the algorithm. To improve the truncation error one should increase N. The accuracy parameter ϵ is fixed inside the program.

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