Half step scheme for local and stochastic volatility models calibrated to the market price of vanilla options

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1 Calibrated local and stochastic volatility model

We consider the following class of models for the stock price S:

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t)\sigma(t,S_t)S_t dW_t^X, \\ dY_t &= b_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y. \end{aligned} \tag{1}$$

Here, r is the interest rate, assumed to be constant, $f:\mathbb{R}\to\mathbb{R}$, $\sigma:\mathbb{R}^2\to\mathbb{R}$, $b_Y:\mathbb{R}\to\mathbb{R}$, $\sigma_Y:\mathbb{R}\to\mathbb{R}$ are deterministic functions, and W^X, W^Y are unidimensional Brownian motions that have correlation $\rho\in[-1,1]$. We also assume that the market prices of all puts $P(T,K)$ with maturity $T>0$ and strike $K>0$ are encoded in a Dupire local volatility function $\sigma_{Dup}:\mathbb{R}^2\to\mathbb{R}$, i.e.:

$$\forall T,K>0, \mathbb{E}\left[e^{-rT}\left(K-S_T^{Dup}\right)_+\right] = P(T,K),$$

where

$$dS_t^{Dup} = rS_t^{Dup}dt + \sigma_{Dup}\left(t,S_t^{Dup}\right)S_t^{Dup}dW_t^X.$$

As presented in [?], with the choice $\sigma(t,x) = \frac{\sigma_{Dup}(t,x)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t=x]}}$ we have that under existence of the process S, for $T\geq 0$ and $K>0$,

$$\mathbb{E}\left[e^{-rT}\left(K-S_T\right)_+\right] = P(T,K).$$

The goal is to simulate a solution to the SDE nonlinear in the sense of McKean satisfied by the logspot $X=\log(S)$:

$$\begin{aligned} dX_t &= \left(r - \frac{1}{2}\frac{f^2(Y_t)}{\mathbb{E}\left[f^2(Y_t)|X_t\right]}\sigma_{Dup}^2(t,e^{X_t})\right)dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|X_t\right]}}\sigma_{Dup}(t,e^{X_t})dW_t^X, \\ dY_t &= b_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y. \end{aligned} \tag{2}$$

2 Half step scheme

Given a finite time horizon $T>0$, the explicit Euler scheme associated with (X,Y) using $n\in\mathbb{N}^*$ steps with constant time step $\Delta=\frac{T}{n}$ is given by

$$\begin{aligned} dX_t^n &= \left(r - \frac{1}{2}\frac{f^2(Y_{\tau_t}^n)}{\mathbb{E}\left[f^2(Y_{\tau_t}^n)|X_{\tau_t}^n\right]}\sigma_{Dup}^2(t,e^{X_{\tau_t}^n})\right)dt + \frac{f(Y_{\tau_t}^n)}{\sqrt{\mathbb{E}\left[f^2(Y_{\tau_t}^n)|X_{\tau_t}^n\right]}}\sigma_{Dup}(t,e^{X_{\tau_t}^n})dW_t^1, \\ dY_t^n &= b_Y\left(Y_{\tau_t}^n\right)dt + \sigma_Y\left(Y_{\tau_t}^n\right)dW_t^2, \\ (X_0^n,Y_0^n) &\sim \mu_0, \end{aligned} \tag{3}$$

where for $t \in [0,T]$, $\tau_t := \lfloor \frac{nt}{T} \rfloor \frac{T}{n}$ is the last discretization time before t. The presence of the conditional expectation in the diffusion X prevents the simple use of explicit Euler scheme discretizing in time the diffusion. One way of overcoming that difficulty is to use kernel approximations of the conditional expectation and introduce a particle system, as proposed in [?]. In our framework, we assume the ellipticity of the diffusion coefficient, that is the existence of a constant $\underline{\sigma}$ such that

$$\forall t, x, y \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \; \frac{f(y)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|X_t = x\right]}} \sigma_{Dup}(t, e^x) \geq \underline{\sigma} > 0.$$

This enables us to introduce a half-step algorithm. Let $\left(Z_k^1, Z_{k+\frac{1}{2}}^1\right)_{k\geq 0}$, $\left(Z_k^2\right)_{k\geq 0}$ be two families of i.i.d. standard centered normal variables. The half-step algorithm is initialized with $\left(\hat{X}_0, \hat{Y}_0\right)$ deterministic and evolves inductively according to

$$\begin{aligned}
\hat{X}^n_{t_{k+\frac{1}{2}}} &= \hat{X}^n_{t_k} + \hat{b}^n_{X,t_k}\Delta + \left(\hat{a}^n_{X,t_k} - \underline{\sigma}^2 I_{d_1}\right)^{\frac{1}{2}} \sqrt{\Delta} Z_k^1, \\
\hat{X}^n_{t_{k+1}} &= \hat{X}^n_{t_{k+\frac{1}{2}}} + \underline{\sigma}\sqrt{\Delta} Z^1_{k+\frac{1}{2}}, \\
\hat{Y}^n_{t_{k+1}} &= \hat{Y}^n_{t_k} + b_Y\left(Y^n_{t_k}\right)\Delta + \sigma_Y\left(\hat{Y}^n_{t_k}\right)\sqrt{\Delta} Z_k^2,
\end{aligned} \tag{4}$$

where we define for $n \geq 1$ and $0 \leq k \leq n-1$, $\hat{b}^n_{X,t_k} = r - \frac{1}{2}\frac{f^2(\hat{Y}^n_{t_k})}{\mathbb{E}\left[f^2(\hat{Y}^n_{t_k})|\hat{X}^n_{t_k}\right]}\sigma^2_{Dup}(t, e^{\hat{X}^n_{t_k}})$ and $\hat{a}^n_{X,k} = \frac{f^2(\hat{Y}^n_{t_k})}{\mathbb{E}\left[f^2(\hat{Y}^n_{t_k})|\hat{X}^n_{t_k}\right]}\sigma^2_{Dup}(t, e^{\hat{X}^n_{t_k}})$. As $\left(Z_k^1, Z^1_{k+\frac{1}{2}}, Z_k^2\right)$ and $\left(\hat{X}^n_{t_k}, \hat{Y}^n_{t_k}\right)$ are independent, the conditional law of the term

$$\left(\hat{a}^n_{X,t_k} - \underline{\sigma}^2 I_{d_1}\right)^{\frac{1}{2}} \sqrt{\Delta} Z_k^1 + \underline{\sigma}\sqrt{\Delta} Z^1_{k+\frac{1}{2}},$$

w.r.t. $\left(\hat{X}^n_{t_k}, \hat{Y}^n_{t_k}\right)$ is the normal centered distribution with variance matrix $\Delta \hat{a}^n_{X,t_k}$, so the half step scheme (4) and the explicit Euler scheme (3) are equivalent in the sense that the vectors $\left(X^n_{t_k}, Y^n_{t_k}\right)_{0\leq k\leq n}$ and $\left(\hat{X}^n_{t_k}, \hat{Y}^n_{t_k}\right)_{0\leq k\leq n}$ have the same law.

3 Associated particles system

The advantage of the half step scheme is that in the elliptic case, it is possible to obtain a representation of the conditional expectation as a ratio of convolution against gaussian kernels. That is expressed in the following result [?, Proposition 3.3.1], where G_v is the one dimensional centered gaussian density with variance $v > 0$.

Proposition. *Let (ξ, Z, γ) be a random variable with values in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Let us assume that $Z \sim \mathcal{N}(0,1)$ and is independent of (ξ, γ). Let $\alpha > 0$ and let us define $\chi := \xi + \alpha Z$. The following assertions hold:*

(i) for any measurable and bounded function $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, we have that

$$\mathbb{E}\left[\psi(\chi,\gamma)\right] = \int_{\mathbb{R}} \mathbb{E}\left[\psi(x,\gamma) G_{\alpha^2}(x-\xi)\right] dx.$$

(ii) Let $\left(\tilde{\xi}, \tilde{\gamma}\right)$ be a copy of (ξ,γ) independent of χ. For $\psi : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ a measurable function such that $\psi(\chi,\gamma)$ is integrable, we have almost surely that

$$\mathbb{E}\left[\psi(\chi,\gamma)|\,\chi\right] = \frac{\mathbb{E}\left[\psi(\chi,\tilde{\gamma})\, G_{\alpha^2}\left(\chi - \tilde{\xi}\right)\big|\,\chi\right]}{\mathbb{E}\left[G_{\alpha^2}\left(\chi-\tilde{\xi}\right)\big|\,\chi\right]}.$$

One natural implementation would be to introduce a particle system associated with the half-step scheme (4). Let $N \in \mathbb{N}^*$ be the number of particles and let the function F^N be defined for $z, x_1, ..., x_N \in \mathbb{R}$ and $y_1, ..., y_N \in \mathbb{R}$ by

$$F^N(z, x_1, ..., x_N, y_1, ..., y_N) = \frac{\frac{1}{N}\sum_{j=1}^N f^2(y_j)\, G_{\underline{\sigma}^2\Delta}(z - x_j)}{\frac{1}{N}\sum_{j=1}^N G_{\underline{\sigma}^2\Delta}(z-x_j)}. \tag{5}$$

Let $\left(\mathbf{X}_0^{i,N},\mathbf{Y}_0^{i,N}\right)_{1\leq i\leq N}$ be i.i.d. random variables with law μ_0 (dirac) and which are independent of $\left(Z_k^{1,i},Z_{k+\frac{1}{2}}^{1,i},Z_k^{2,i}\right)_{k\geq 0,1\leq i\leq N}$. The dynamics is given by

$$\begin{aligned}
\mathbf{X}_{t_{k+\frac{1}{2}}}^{n,i,N} &= \mathbf{X}_{t_k}^{n,i,N}+\mathbf{b}_{X,k}^{n,i,N}\Delta+\left(\mathbf{a}_{X,t_k}^{n,i,N}-\underline{\sigma}^2 I_{d_1}\right)^{\frac{1}{2}}\sqrt{\Delta}Z_k^{1,i},\\
\mathbf{X}_{t_{k+1}}^{n,i,N} &= \mathbf{X}_{t_{k+\frac{1}{2}}}^{n,i,N}+\underline{\sigma}^2\sqrt{\Delta}Z_{k+\frac{1}{2}}^{1,i},\\
\mathbf{Y}_{t_{k+1}}^{n,i,N} &= \mathbf{Y}_{t_k}^{n,i,N}+b_Y\left(\mathbf{Y}_{t_k}^{n,i,N}\right)\Delta+\sigma_Y\left(t_k,\mathbf{Y}_{t_k}^{n,i,N}\right)\sqrt{\Delta}Z_k^{2,i}.
\end{aligned}\tag{6}$$

Here, for $n\geq 1$, $0\leq k\leq n-1$, $N\geq 1$, $1\leq i\leq N$, $\mathbf{b}_{X,k}^{n,i,N}=r-\frac{1}{2}\frac{f^2(\mathbf{Y}_{t_k}^{n,i,N})}{\mathbf{E}_k^N\left(\mathbf{X}_{t_k}^{n,i,N}\right)}\sigma_{Dup}^2(t_k,e^{\mathbf{X}_{t_k}^{n,i,N}})$, $\mathbf{a}_{X,k}^{n,i,N}=\frac{f^2(\mathbf{Y}_{t_k}^{n,i,N})}{\mathbf{E}_k^N\left(\mathbf{X}_{t_k}^{n,i,N}\right)}\sigma_{Dup}^2(t_k,e^{\mathbf{X}_{t_k}^{n,i,N}})$ and

$$\mathbf{E}_k^N\left(\mathbf{X}_{t_k}^{n,i,N}\right)=F_N\left(\mathbf{X}_{t_k}^{n,i,N},\mathbf{X}_{t_{k-\frac{1}{2}}}^{n,1,N},...,\mathbf{X}_{t_{k-\frac{1}{2}}}^{n,N,N},\mathbf{Y}_{t_k}^{n,1,N},...,\mathbf{Y}_{t_k}^{n,N,N}\right).$$