# A finite dimensional approximation for pricing moving average options.

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The following method computes the price of American Options whose payoff depends on the moving average of the underlying asset price. It is based on the paper [1].

## 1 Introduction

The method computes the price of American Options whose payoff depends on the moving average $X$ defined by

$$X_t = \int_{t-\delta}^{t} S_u du, \quad \forall t \geq \delta$$

where $S$ represents the underlying asset and $\delta$ is a fixed time window. The process $(S, X)$ is not Markovian, and in a continuous time framework it is not possible to find $n$ processes $(X^1, \cdots, X^n)$ such that $(S, X, X^1, \cdots, X^n)$ are jointly Markovian. In a discrete time framework (Bermudan options), $n$ would be equal to the number of time steps within the average window.
The paper proposes a method for pricing moving average American options based on a finite dimensional approximation of the infinite-dimensional dynamics of the moving average process. The approximation is based on a truncated expansion of the weighting measure used for averaging in a series involving Laguerre polynomials, truncated at $n$ terms, which leads to $(n+1)$-dimensional Markovian approximation to the initial infinite dimensional problem.

## 2 Theoretical framework

We consider a more general moving average process of the form

$$M_t = \int_0^{\infty} S_{t-u}\mu(du),$$

where $\mu$ is a finite possibly signed measure on $[0, \infty[$. The paper proposes a finite-dimensional approximation to $M$, that is $n$ processes $X^{p,1}, \cdots, X^{p,n}$ such that $(S, X^{p,1}, \cdots, X^{p,n})$ are jointly Markov, and $M_t$ is approximated by $M_t^{n,p}$, which depends deterministically on $(S_t, X_t^{p,1}, \cdots, X_t^{p,n})$ (see [1, Proposition 2.2]). More precisely, we have

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \sum_{k=0}^{n-1} a_k^p X_t^{p,k},$$

where

$$X_t^{p,k} = \int_0^\infty L_k^p(v) S_{t-v} dv,\ \forall k = 0, \cdots, n-1$$

and $L_k^p(t) = \sqrt{2p} P_k(2pt) e^{-pt}, k \geq 0$ are the scaled Laguerre functions, in which $(P_k)_{k\geq 0}$ is the family of Laguerre polynomials (see [1, (9)]) and $p$ is a parameter to be fixed. The functions $H$ and $H_n^p$ are defined by

$$H(x) = \mu([x, \infty[), \quad H_n^p(x) = \sum_{k=0}^{n-1} A_k^p L_k^p(x), \quad \text{where } A_k^p = \langle H, L_k^p \rangle.$$

Coefficients $(a_k^p)_k$ are defined in [1, (11)].

## 2.1 Uniformly weighted measure

When $\mu(dx) = \frac{1}{\delta} \mathbb{1}_{\{[0,\delta]\}} dx$, the Laguerre coefficients $A_k^{\delta,p} = \langle H, L_k^p \rangle$ are defined by

$$A_k^{\delta,p} = (-1)^k \frac{\sqrt{2p}}{p} - \frac{1}{p} c_k^{\delta,p} - \frac{2}{p} \sum_{i=0}^{k-1} (-1)^{k-i} c_i^{\delta,p},$$

where

$$c_n^{\delta,p} = \frac{\sqrt{2p}}{\delta p} \left[ 1 - e^{-p\delta} P_n(2p\delta) + 2 \sum_{k=1}^{n} (-1)^k (1 - e^{-p\delta} P_{n-k}(2p\delta)) \right].$$

The optimal scale parameter $p_{opt}(\delta, n)$ can be computed using the relation $p_{opt}(\delta, n) = \frac{p_{opt}(1,n)}{\delta}$ and [1, Table 1].

# 3 Monte Carlo-based numerical method

We consider a uniformly weighted moving average. The following method computes the price of the discrete time version of the American option $\sup_\tau \mathbb{E}[\phi(S_\tau, M_\tau)]$, in which the moving average $X$ has been replaced by its approximation $M^{n,p_{opt}}$, and the exercise is possible on an equidistant time grid $\pi$ with $N$ time steps $\Delta t = \frac{T}{N}$. The approach corresponds to the one of Longstaff and Schwartz, and the computation of conditional expectations is done with a regression based approach. $N_\delta$ denotes the number of time steps within the average window of length $\delta$ : $N_\delta = \frac{\delta}{T} N$. The spot price is discretised on $\pi$ and is written $S^\pi$, the discretised version of $X$ is given by

$$X_{t_i}^\pi = \frac{1}{N_\delta} \sum_{j=i-N_\delta+1}^{i} S_{t_j}^\pi, \forall t_i \in \pi.$$

The discrete time version of the Laguerre processes are defined by

$$X_{t_i}^{p,k,\pi} = \sum_{j=1}^{i} (S_{t_j}^\pi - S_{t_{j-1}}^\pi)(i-j+1)\Delta t c_k^{(i-j+1)\Delta t, p} + S_0 (-1)^k \frac{\sqrt{2p}}{p}, \forall t_i \in \pi.$$

### 3.1 The $Lag - LS^*$ algorithm

The backward algorithm works as follows

1. Initialization : $\tau_N^{\pi,(m)} = T, m = 1, \cdots, M$
2. Backward induction for $i = N-1, \cdots, N_\delta$, $m = 1, \cdots, M$

$$\begin{aligned} \tau_i^{\pi,(m)} &= t_i \mathbb{1}_{\{A_i^{(m)}\}} + \tau_{i+1}^{\pi,(m)} \mathbb{1}_{\{(A_i^{(m)})^c\}}, \\ A_i^{(m)} &= \left\{ \Phi(S_{t_i}^{\pi,(m)}, X_{t_i}^{\pi,(m)}) \geq \mathbb{E}_{t_i}[\Phi(S^\pi_{\tau^\pi_{i+1}}, X^\pi_{\tau^\pi_{i+1}})] \right\} \end{aligned}$$

where $\mathbb{E}_{t_i}[\cdot] = \mathbb{E}[\cdot|(S_{t_i}^\pi, X_{t_i}^{p_{opt},0,\pi}, \cdots, X_{t_i}^{p_{opt},n-1,\pi})]$.

Estimators of the conditional expectations are constructed with a Monte Carlo based technique.

### 3.2 The $NM - LS$ algorithm

The following algorithm is a non-markovian approximation of the previous algorithm. Let $(\theta_i^\pi)_{i=N_\delta,\cdots,N}$ denote the discrete time sequence of the estimated optimal exercies times. The algorithm works as follows

1. Initialization : $\theta_N^\pi = T$
2. Backward induction for $i = N-1, \cdots, N_\delta$,

$$\begin{aligned} \theta_i^\pi &= t_i \mathbb{1}_{\{A_i\}} + \theta_{i+1}^\pi \mathbb{1}_{\{(A_i)^c\}}, \\ A_i &= \left\{ \Phi(S_{t_i}^\pi, X_{t_i}^\pi) \geq \mathbb{E}[\Phi(S^\pi_{\theta^\pi_{i+1}}, X^\pi_{\theta^\pi_{i+1}})|(S^\pi_{t_i, X^\pi_{t_i}})] \right\} \end{aligned}$$

3. Estimation of the option price at time 0 $U_0^\pi = \mathbb{E}[\Phi(S^\pi_{\theta^\pi_{N_\delta}}, X^\pi_{\theta^\pi_{N_\delta}})]$.

# 4 Numerical experiments

## 4.1 Moving average without time delay

In the case of an American moving average Call, the numerical data used by default are the following

| $S_0$ | $T$ | $r$ | $\sigma$ | $r$ | $\delta$ |
|---|---|---|---|---|---|
| 100 | 0.2 | 0.05 | 0.3 | 0.05 | 0.02 |

The number of trajectories of $S$ is $M = 10^5$. The number of discretization time steps used for the discretization of $S$ is $N = 50$ (then $N_\delta = 5$). The number of polynomial basis functions is 4. When using the $Lag - LS^*$ algorithm, $n = 3$.

### 4.2 Moving average with time delay

We consider a moving average American option with time delay $l \geq 0$ whose value at time 0 is

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}[\Phi(S_\tau, X_\tau)], \quad X_\tau = \frac{1}{\delta} \int_{\tau-l-\delta}^{\tau-l} S_u du.$$

In this case,

$$X_{t_i}^\pi = \frac{1}{N_\delta} \sum_{j=i-N_\delta-N_l+1}^{i-N_l} S_{t_j}^\pi, \forall t_i \in \pi,$$

where $N_l = l\frac{N}{T}$. The numerical data used by default are the following

| $S_0$ | $T$ | $r$ | $\sigma$ | $r$ | $\delta$ | $l$ |
|---|---|---|---|---|---|---|
| 100 | 0.2 | 0.05 | 0.3 | 0.05 | 0.02 | 0.1 |

The number of trajectories of $S$ is $M = 10^5$. The number of discretization time steps used for the discretization of $S$ is $N = 50$ (then $N_\delta = 5$ and $N_l = 25$). The number of polynomial basis functions is 4. When using the $Lag - LS^*$ algorithm, $n = 3$.

## References

[1] M. Bernhart, P.Tankov, and X. Warin. A finite dimensional approximation for pricing moving average options. *SIAM Journal of Financial Mathematics*, 2:989–1013, 2011. 1, 2