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Contents

1 Standard European Option

Standard European Options are described in

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1.1 Localization and Discretization

We recall that the price of an European option in the Black and Scholes model

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t$$

can be formulated in terms of the solution to a Partial Differential Equation. After logarithmic transformation $X_t = \log(S_t)$ the price at time t of the option is $V_t = u(t, X_t)$ where u solves the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + (r - \delta - \frac{\sigma^2}{2})\frac{\partial u}{\partial x}(t,x) - ru(t,x) = 0 \text{ in } [0,T) \times I\!R, \\ u(T,x) = \psi(x), \forall x \in I\!R, \end{cases}$$

The notations are

- - x is the logarithm of the stock price
- - σ the volatility
- - r the interest rate
- - δ the instantaneous rate of dividend
- - ψ the pay-off
- - T the maturity
- - $I\!R$ the real line $(-\infty, +\infty)$

The canonical form for a parabolic PDE (the one found in numerical analysis books) is

$$\frac{\partial v}{\partial \tau} - \frac{\partial}{\partial x}(\alpha \frac{\partial v}{\partial x}) + b\frac{\partial v}{\partial x} + av = f \text{ in } \quad \Omega \times (0,T)$$

with initial conditions ($v(x,0)$ given) and boundary conditions on $\partial\Omega$ (for instance $v(x,t)$ given for $x \in \partial\Omega$).

For smooth data and if $\alpha > 0$ and $a \geq 0$, this equation has one and only one solution which depends continuously on the data. The condition on a is not essential but the solution v may grow exponentially in time if a is not positive.

The change of variable $v(\tau, x) = u(T - \tau, e^x)$ brings the B&S equation in canonical form with $\alpha = \sigma^2/2$, $b = r - \delta - \sigma^2/2$, and $a = r$. Let $x = \log(S_0)$. We start by limiting the integration domain in space: the problem will be solved in a finite interval $D := [x - l, x + l]$. One chooses l so that

$$I\!P\left(\exists s \in [0,T],\ |X_s^x| \geq l\right) \leq \epsilon \tag{1}$$

Once D is chosen, one discretizes in space and constructs the uniform grid $\{x_i\}$ with

$$x_i := x - l + \frac{2i}{M}, \quad \text{for } 1 \leq i \leq M - 1.$$

Let V_M denote the space generated by the indicator function of $[x_i, x_{i+1}[$ for $0 \leq i \leq M - 1$.

1.2 Finite Differences

One approximates the differential operator

$$A\phi := \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2} + (r-\delta-\frac{\sigma^2}{2})\frac{\partial\phi}{\partial x} - r\phi$$

by a discrete operator A_h acting on functions $u_h(t,\cdot)$ defined on V_M. The easiest and most natural is to take:

$$A_h u_h(t,x_i) = \frac{1}{2}\sigma^2\frac{\delta^2 u_h}{\delta x^2}(t,x_i) + (r-\delta-\frac{\sigma^2}{2})\frac{\delta u_h}{\delta x}(t,x_i) - r u_h(t,x_i)$$

with

$$\frac{\delta^2 u_h}{\delta x^2}(t,x_i) = \frac{1}{h^2}(u_h(t,x_{i+1}) - 2u_h(t,x_i) + u_h(t,x_{i-1}))$$
$$\frac{\delta u_h}{\delta x}(t,x_i) = \frac{1}{2h}(u_h(t,x_{i+1}) - u_h(t,x_{i-1})).$$

Remark 1. If $|r-\delta|/\sigma^2$ is not small then a less precise but more stable finite difference approximation for this term is

$$\frac{\delta u_h}{\delta x}(t,x_i) = \begin{cases} \frac{1}{h}(u_h(t,x_i) - u_h(t,x_{i-1})) & \text{if } r-\delta-\frac{\sigma^2}{2} < 0 \\ \frac{1}{h}(u_h(t,x_{i+1}) - u_h(t,x_i)) & \text{if } r-\delta-\frac{\sigma^2}{2} > 0 \end{cases}$$

One then seeks the vector $(u_h(t,x_i), 0 \le i \le M)$ such that, there holds

- In the case of natural Dirichlet boundary conditions,

$$\begin{cases} \text{for } 0 \le t \le T,\ 1 \le i \le M-1, \\ \frac{d}{dt}u_h(t,x_i) + A_h u_h(t,x_i) = 0 \\ u_h(T,x_i) = \psi(x_i) \\ u_h(t,x-l) = \psi(x-l), \\ u_h(t,x+l) = \psi(x+l). \end{cases} \tag{2}$$

- In the case of Neumann boundary conditions,

$$\begin{cases} \text{for } 0 \le t \le T,\ 1 \le i \le M-1, \\ \frac{d}{dt}u_h(t,x_i) + A_h u_h(t,x_i) = 0, \\ u_h(T,x_i) = \psi(x_i), \\ u_h(t,x_1) = u_h(t,x-l) + h\frac{\partial\psi}{\partial x}(x-l), \\ u_h(t,x_{M-1}) = u_h(t,x+l) - h\frac{\partial\psi}{\partial x}(x+l). \end{cases} \tag{3}$$

Set $u_h(t) := (u_h(t, x_1), \ldots, u_h(t, x_{M-1}))$ and

$$\begin{aligned}
\alpha &:= \frac{\sigma^2}{2h^2} - \frac{1}{2h}(r - \frac{\sigma^2}{2}), \\
\beta &:= -\frac{\sigma^2}{h^2} - r, \\
\gamma &:= \frac{\sigma^2}{2h^2} + \frac{1}{2h}(r - \frac{\sigma^2}{2}).
\end{aligned}$$

According to this new notation, the operator A_h applied to $u_h(t, \cdot)$ can be described as follows:

$$A_h u_h(t, \cdot) = M^h u_h(t) + v^h,$$

with

- In the case of natural Dirichlet boundary conditions,

$$M^h = \begin{bmatrix} \beta & \gamma & 0 & \cdots & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \cdots & \alpha & \beta \end{bmatrix} \tag{4}$$

  and

$$v^h = \begin{bmatrix} \psi(x-l)\alpha \\ 0 \\ \vdots \\ 0 \\ \psi(x+l)\gamma \end{bmatrix};$$

- In the case of artificial Neumann boundary conditions,

$$M^h = \begin{bmatrix} \beta+\alpha & \gamma & 0 & \cdots & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \cdots & \alpha & \beta+\gamma \end{bmatrix} \tag{5}$$

and

$$v^h = \begin{bmatrix} -\alpha h \frac{\partial \psi}{\partial x}(x-l) \\ 0 \\ \vdots \\ 0 \\ \gamma h \frac{\partial \psi}{\partial x}(x+l) \end{bmatrix}.$$

REMARK 2. When $M = 2p+1$, x don't belong to $\{x_i\,;\, 1 \leq i \leq M-1\}$. Thus, at each time step, we use linear interpolation to compute the option value corresponding to the initial stock price $u_h(t,x)$ which is then approximated by $\frac{1}{2}\left(u_h(t,x_p) + u_h(t,x_{p+1})\right)$.

Now, let us discuss the discretization in time.

1.3 The "θ-scheme"

The standard θ-scheme ($\theta \in [0,1]$) of the parabolic equation (1.1) may be summarize as follows: fix a discretization step k such that $T = Nk$ and construct an approximation

$$u_{h,k}(t,x) = \sum_{n=0}^{N} u_h^n(x) \mathbf{1}_{[nk,(n+1)k[}(t)$$

where $u_h^0, \ldots, u_h^N$ are the elements of V_M satisfying

$$\begin{cases} u_h^N = \psi_h \\ \text{for } 0 \leq n \leq N-1 \\ \frac{u_h^{n+1} - u_h^n}{k} + A_h(u_h^{n+1} + \theta(u_h^n - u_h^{n+1})) = 0. \end{cases} \tag{6}$$

Besides, one must add the appropriate boundary conditions:

$$\begin{cases} u_h^n(x-l) &=& \psi(x-l), \\ u_h^n(x+l) &=& \psi(x+l), \end{cases}$$

for Dirichlet boundary conditions and

$$\begin{cases} u_h^n(x_1) &=& u_h^n(x-l) + \frac{\partial \psi}{\partial x}(x-l)h \\ u_h^n(x_{N-1}) &=& u_h^n(x,+l) - \frac{\partial \psi}{\partial x}(x+l)h. \end{cases}$$

for Neumann boundary conditions.
For $\theta = 0$, we recover the Euler explicit scheme. Similarly, for $\theta = 1$ the scheme is the "fully implicit" Euler scheme, and for $\theta = \frac{1}{2}$ it is the Crank-Nicholson scheme.

Once we have computed $u_{h,k}$, we recover the delta-hedging $\Delta = \dfrac{1}{e^x}\dfrac{\partial u(t,x)}{\partial x}$ by its approximation on $[nk,(n+1)k[\times]x-l,x+l[$ given by

$$\Delta_h = \frac{1}{e^x}\frac{u_h^n(x) - u_h^n(x-h)}{2h}.$$

1.4 Explicit Method

First, let us discuss the case $\theta = 0$. Using the definition of A_h, the approximating scheme (6) is reduced to

$$\begin{cases} u_h^N = \psi \\ \text{for } 1 \leq n \leq M-1 \\ u_h^n(x_i) = p_1 u_h^{n+1}(x_{i-1}) + p_2 u_h^{n+1}(x_i) + p_3 u_h^{n+1}(x_{i+1}) \end{cases}$$

where

$$p_1 = k(\frac{\sigma^2}{2h^2} - \frac{b}{2h}) p_2 = 1 - k(r + \frac{\sigma^2}{h^2}) \quad p_3 = k(\frac{\sigma^2}{2h^2} + \frac{b}{2h})$$

with $b = r - \delta - \frac{1}{2}\sigma^2$.
This scheme is stable if $k \leq \frac{h^2}{\sigma^2 + rh^2}$.

1.5 Implicit Methods

When we choose $1 \geq \theta > 0$, we have to solve at each time step, a linear system of the type

$$\mathbf{M}u_{k,h}(jk,\cdot) = \mathbf{N}u_{k,h}((j+1)k,\cdot)$$

where $\mathbf{M}$ and $\mathbf{N}$ are tridiagonal matrix of the type

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & a_M & b_M \end{pmatrix}.$$

For example in the case of natural Dirichlet boundary condition, $\mathbf{M}$ is given by

$$a_i = \theta k(\frac{b}{2h} - \frac{\sigma^2}{2h^2}), \quad b_i = 1 + \theta k(r + \frac{\sigma^2}{h^2}), \quad c_i = -\theta k(\frac{b}{2h} + \frac{\sigma^2}{2h^2})$$

for every i and $\mathbf{N}$ is given by

$$a_i = (1-\theta)k(\frac{\sigma^2}{2h^2} - \frac{b}{2h}), \quad b_i = 1-(1-\theta)k(r+\frac{\sigma^2}{h^2}), \quad c_i = (1-\theta)k(\frac{b}{2h}+\frac{\sigma^2}{2h^2}) \tag{7}$$

The fully implicit Euler, the Crank–Nicolson methods and all those with $\theta \neq 0$ require the resolution of a linear system

$$\mathbf{M}u = v,$$

where u and v are M-dimensional vectors. Let us describe two algorithms of resolution of such a linear system.

1.5.1 Gauss Factorization

To solve such a system, the following Gauss factorization is often used; it is based on the fact that a regular matrix can be factorized into

$$\mathbf{M} = \mathbf{LU}$$

where $\mathbf{L}$ is a lower triangular matrix (all elements above the diagonal are zero) and $\mathbf{U}$ is an upper triangular matrix with all ones on its diagonal. The solution of the linear system $\mathbf{LU}z = v$ is decomposed into $\mathbf{L}y = v, \mathbf{U}z = y$; the first one is solved by a loop from 1 to M and the second one by a loop from M to 1. It is easy to see that $\mathbf{M}$ triangular implies that $\mathbf{L}, \mathbf{U}$ are also triangular and so only the upper diagonal of $\mathbf{U}$ and the two diagonals of $\mathbf{L}$ need to be found.
The computation of $\mathbf{L}, \mathbf{U}, v'$ are done in the same downsweep:

$$\left|\begin{array}{l} b'_M := b_M, \qquad y_M := v_M \\ \text{For } 1 \leq i \leq M-1, \ i \text{ increasing :} \\ \qquad b'_i = b_i - c_i a_{i+1}/b'_{i+1}, \\ \qquad y_i = v_i - c_i y_{i+1}/b'_{i+1}. \end{array}\right.$$

$$\left|\begin{array}{l} u_1 = y_1/b'_1 \\ \text{For } 2 \leq i \leq M, \ i \text{ decreasing} \\ \qquad z_i = (y_i - a_i u_{i-1})/b'_i. \end{array}\right.$$

Remark 3. Note that it is necessary that all the b_i (called the pivots) be not 0.

1.5.2 SOR Iterative Methods

An alternative, which in the case of tridiagonal systems is justified only by its programming simplicity, is to uses the Successive Over-Relaxation scheme for solving the linear system

$$\mathbf{M}u = v$$

The solution is computed as the limit of a converging sequence, $u = \lim_{p \longrightarrow \infty} u^p$. The basic steps are:

- ***Step 0 :*** *Choose* $u^0 \geq 0$. *Choose* $\epsilon > 0$, $1 < \omega < 2$. *Set* $p = 0$.
- ***Step 1 :*** *Form an intermediate vector* $h^{p+1} = (h_i^{p+1})_{1 \leq i \leq N}$ *by*

$$h_i^{p+1} = -\frac{1}{m_{ii}}(v_i - \sum_{j=1}^{i-1} M_{ij} u_j^{p+1} - \sum_{j=i}^{m} M_{ij} u_j^p).$$

- ***Step 2*** Define u^{p+1} *by :*

$$u_i^{p+1} = u_i^p + \omega(h_i^{p+1} - u_i^p))$$

- ***Step 3*** *Set* $p = p + 1$ *and repeat until* $|u^{p+1} - u^p| < \varepsilon$ *where* ε *is the prescribed precision.*

In practice one stores all the u^p in the same computer memory, so the exponent p does not appear in the computer program except as a loop index.

2 Standard American Option

Standard American Options are described in

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2.1 Variational inequality in finite dimension

Consider the following approximating obstacle problem on $Q_l = [0, T] \times \Omega_l$ where $\Omega_l =]x - l, x + l[$.

$$\begin{cases} \max(\frac{\partial u}{\partial t} + Au, \psi - u) = 0 \\ \quad u(T, .) = \psi \end{cases} \tag{8}$$

with a Dirichlet boundary condition $u = \psi$ on $]0, T[\times \partial\Omega_l$.
In order to make the numerical analysis of obstacle problem (8), we introduce a finite difference grid in space similar to the European case and construct an approximation:

$$u_{h,k}(t, x) = \sum_{n=0}^{N} u_h^n(x) \mathbf{1}_{[nk,(n+1)k[}(t)$$

where $u_h^0, \ldots, u_h^N$ are the elements of V_M satisfying

$$\begin{cases} u_h^N = \psi_h \\ (\frac{u_h^{n+1} - u_h^n}{k} + \left(A_h(u_h^{n+1} + \theta(u_h^n - u_h^{n+1})), v_h - u_h^n\right)_l \leq 0 \quad \forall v_h \geq \psi_h \end{cases} \tag{9}$$

Let us describe in the two next sections the computational treatment of variational inequalities in finite dimension (9). For a better understanding, we refer to [6] for a detailed presentation of the numerical analysis of variational inequalities.

2.2 Linear complementarity problem

It is well-known that the variational inequality in finite dimension (9) can be expressed as a linear complementarity problem.
At each time step n, we have to solve

$$\begin{cases} MX \geq G \\ X \geq \Phi \\ (MX - G, X - \Phi) = 0 \end{cases} \tag{10}$$

with

$$\begin{cases} M = I - k\theta A_h \\ X = u^n \\ G = (I + k(1 - \theta)A_h)u^{n+1} \\ \Phi = \psi_h \end{cases}$$

Remark that M is a tridiagonal matrix:

$$M = \begin{pmatrix} b & c & \cdots & \cdots & 0 \\ a & b & c & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & a & b & c \\ 0 & \cdots & \cdots & a & b \end{pmatrix}$$

with

$$\begin{cases} a = \theta k \left(-\frac{\sigma^2}{2h^2} + \frac{1}{2h}(r - \delta - \frac{\sigma^2}{2})\right) \\ b = 1 + \theta k(\frac{\sigma^2}{h^2} + r) \\ c = -\theta k \left(\frac{\sigma^2}{2h^2} + \frac{1}{2h}(r - \delta - \frac{\sigma^2}{2})\right) \end{cases}$$

We close this section by the description of three algorithms which solve the linear complementarity problem (10)

2.2.1 Brennan-Schwartz method

Consider the tridiagonal matrix $\tilde{M}$ defined as follows:

$$\tilde{M} = \begin{pmatrix} b_1 & 0 & \cdots & \cdots & 0 \\ a & b_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & a & b_{d-1} & 0 \\ 0 & \cdots & \cdots & a & b_d \end{pmatrix}$$

where $b_d = b$ and $b_{i-1} = b - \dfrac{ac}{b_i}$ for $i = -M, \ldots, M$.

Brennan and Schwartz ([4])have developed the following algorithm for pricing American option related to the resolution of a new linear complementarity problem

$$\begin{cases} \tilde{M}X \geq \tilde{G} \\ X \geq \Phi \\ (\tilde{M}X - \tilde{G}, X - \Phi) = 0 \end{cases} \tag{11}$$

where $\tilde{G} = (\tilde{g}_i)$,$(i = -M, \ldots, M)$, with $\tilde{g}_M = g_M$ and $\tilde{g}_{i-1} = g_{i-1} - \dfrac{c\tilde{g}_i}{b_i}$.

Their method can be summarized as follows:

$x_{-M} = \max(\dfrac{\tilde{g}_{-M}}{b_{-M}}, f_{-M})$.

And,

for $-M + 1 \leq i \leq M$, set

$$x_i = \max\left(\frac{\tilde{g}_i - a x_{i-1}}{b_i}, f_i\right). \tag{12}$$

We refer to Jaillet et al. ([1])for a rigorous justication of the convergence of this algorithm in the case of American Put.

2.2.2 PSOR Method

The linear complementarity problem (10) can be written as follows: find vectors $W = (w_i)_{0\geq i\geq M-1}$ and $Z = (z_i)_{0\geq i\geq M-1}$ in $I\!R^M$ such that

$$\begin{cases} W = MZ + V & (13.1) \\ W \geq 0, \quad Z \geq 0 & (13.2) \\ (W, Z) = 0 & (13.3) \end{cases} \tag{13}$$

where we have set $Z = X - \Phi$ and $V = M\Phi - G$.

Such a linear complementarity problem can be solved with a Projected-SOR scheme in the following manner:

- ***Step 0:*** *Choose* $z^0 \geq 0, 1 < \omega < 2$. *Then, set* $p = 0$.
- ***Step 1:*** *Form:*

$$y_i^{p+1} = -\frac{1}{M_{ii}}(v_i + \sum_{j=1}^{i-1} M_{ij} z_j^{p+1} + \sum_{j=i}^{m} M_{ij} z_j^p)$$

- ***Step 2:*** *Define the new vector* z^{p+1} *by:*

$$z_i^{p+1} = \max(\psi(x_i), \quad z_i^p + \omega(y_i^{p+1} - z_i^p))$$

- ***Step 3:****Set* $p = p + 1$ *and repeat until* $|z^{p+1} - z^p| < \varepsilon$ *where* ε *is the prescribed precision.*

Convergence has been established by Cryer ([2]).

2.2.3 The Algorithm of Cryer

This algorithm is based on a direct method and is a modification of Saigal's (1970) algorithm. The main difference is that the data are scanned in alternating forward and backward passes.

The basic idea of this kind of algorithm is : Choose an initial value which satisfies both (13.1) and (13.2), maintain the two conditions during all steps and make satisfy gradually the non-negative condition given in (13.3). The solution of the problem (13) is then obtained.

Let us introduce the following notations : $N = \{1, 2, \cdots, m\}$, if $J \subset N$, $|J|$ denotes the number of elements in J; $N\backslash J$ denotes the complement of

J with respect to N; $(MZ+V)|J \geq 0$ denotes the system of $|J|$ inequalities obtained by deleting column and row indices not belonging to J. We recall that if M is a Minkowski matrix (M has positive principal minors, positive diagonal entries and non-positive off-diagonal entries), so is $M|J$, and $M^{-1} \geq 0$. With these notations, the basic steps of this algorithm can be described as follows :

step 0 : Choose $Z^0 \geq 0$, such that $(MZ^0+V)|J^0 = 0$, where $J^0 = \{i \in N \ : Z_i^0 > 0\}$. We can assume that $J^0 \supset Q = \{i \in N \ : V_i^0 < 0\}$. Then, set $p = 0$.

step 1 : Denote $W^p = MZ^p + V$, and $I^p = \{i \in N| \ W_i^p < 0\}$

- If $I^p = \phi$, stop. (W^p, Z^p) is the solution of problem.
- Otherwise, choose $i^p \in I^p$ and set $J^{p+1} = J^p \bigcup \{i^p\}$. Compute Z^{p+1} such that $(MZ^{p+1}+V)|J^{p+1} = 0$, and $Z^{p+1}|N \backslash J^{p+1} = 0$.

Set $p = p+1$ and repeat until $I^p = \phi$ or $p = m$.

The above basic algorithm is valid for all Minkowski matrix. In the particular case where M is a tridiagonal Minkowski matrix, an implementation of this basic method which minimizes the amount of computation can be found in [3].

2.3 Splitting methods

We will give an alternate method to solve variational inequalities in finite dimension (9) which is not related to linear complementarity problems.
The splitting methods can be viewed as an analytic version of dynamic programming. The idea contained in such a scheme is to split the American problem in two steps: we construct recursively the approximate solution $u_{h,k}$ starting from $u^N = \psi$ and computing u_h^n for $0 \leq n \leq N$ in two steps as follows:

- Step 1 We solve the following Cauchy problem on $[nk,(n+1)k[\times]x-l,x+l[$ with Dirichlet or Neumann boundary conditions

$$\begin{cases} \frac{\partial w}{\partial t} + Aw = 0, & (t,x) \in [nk,(n+1)k[\times,]x-l,x+l[\\ w(j+1,\cdot) = u_h^{n+1}(.) \end{cases}$$

  Denote by $S_k[u_h^{n+1}](\cdot)$ the solution w.

- Step 2

$$u_h^n(\cdot) = \max\Big(\psi_h(\cdot), S_k[u_h^{n+1}](\cdot)\Big)$$

Barles-Daher-Romano ([5]) prove the convergence of this scheme. As described for the European case, we solve the first step using θ-schemes. For instance, we can use explicit scheme and it can be shown that this scheme is stable if $k \leq \frac{h^2}{2}$.
Moreover, we are able to prove that the approximate solution obtained by splitting methods are bounded above by those obtained by methods related to linear complementarity problem.

3 Exotic Options

3.1 Barrier Options

We gather in the generic term *barrier* every option which value solves the following parabolic partial linear equation,

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \delta - \frac{\sigma^2}{2})\frac{\partial u}{\partial x} - ru = 0 \text{ in } [0,T) \times \Omega, \\ u(T,y) = \psi(y), \forall y \in \Omega, \\ u(t,y) = R(t,y), \forall (t,y) \in [0,T] \times \partial\Omega, \end{cases} \tag{14}$$

Let us give some examples:

- Out Options
  We consider only the case of a down barrier L, the discussion for an upper barrier U is similar.
  For this option, $\Omega =]L, x+l[$ and $R(t,L) = R$.

- In Options
  For this option, $\Omega =]L, x+l[$, $\psi(y) = R$ and $R(t,L) = C(T-t,L)$ where C is the price of a standard European call with maturity $T-t$

- Double Barrier Out Options For this option, $\Omega =]L, U[$ and $R(t,L) = R(t,U) = R$.

- Double Barrier In Options For this option, $\Omega =]L, U[$, $\psi(y) = R$, $R(t,L) = C(T-t,L)$ and $R(t,U) = C(T-t,U)$.

3.1.1 Algorithm

For barrier options, one also discretizes in space and time with a θ-scheme and one solves with Gauss method (cf. there) in the case of european option and with Psor method (cf. there) in the american case. To obtain accurate prices the grid points is located on the barrier, where we impose Dirichlet boundary conditions. One uses linear interpolation to find the price value and delta value corresponding to the initial stock price. If the initial stock price is close to barrier one uses for delta one-sided second-order difference approximation.

3.2 Asian Options

3.2.1 Rogers-Shi Fixed Asian Options

European style Asian option may be valued using one-dimensional PDEs based on a scaling property of geometric brownian motion.[7]
Let $y = \frac{K}{x}$ and $b = r - \delta$. The price of a Asian call fixed option

$$C^a(0,x) = \mathbb{E}\left(e^{-rT}\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+\right),$$

can be formulated as

$$C^a(0,x) = e^{-\delta T} x u(0,y)$$

where u is the solution of the following PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - \left(\frac{1}{T} + bx\right)\frac{\partial u}{\partial x} = 0 \text{ in } [0,T) \times R^+ \\ u(t,0) = \frac{1-e^{-b(T-t)}}{bT} \\ u(T,x) = 0 \text{ in } \ R^+. \quad u(t,\infty) = 0 \text{ in } \ R^+. \end{cases}$$

The delta of fixed-strike Asian call option is given by:

$$\Delta^a = e^{-\delta T}(u(0,y) - y\frac{\partial u(0,y)}{\partial y})$$

The price of a Asian put fixed option

$$P^a(0,x) = \mathbb{E}\left(e^{-rT}\left(K - \frac{1}{T}\int_0^T S_s ds\right)_+\right),$$

can be formulated as

$$P^a(0,x) = e^{-\delta T} x u(0,\frac{K}{x})$$

with u solution of the PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - \left(\frac{1}{T} + bx\right)\frac{\partial u}{\partial x} = 0 \text{ in } [0,T) \times R^+ \\ u(T,x) = x_+ \text{ in } R^+. \end{cases}$$

The delta of fixed-strike Asian put option is given by:

$$\Delta^a = e^{-\delta T}(u(0,y) - y\frac{\partial u(0,y)}{\partial y})$$

The PDEs are solved with a finite difference time-implicit scheme. One discretizes in space and time and one solves the linear system with Gauss' method (there). If necessary one uses linear interpolation to find the option value corresponding to the initial stock price.

References

[1] P.JAILLET D.LAMBERTON B.LAPEYRE. Variational inequalities and the pricing of American options. *Acta Applicandae Mathematicae*, 21:263–289, 1990. 11

[2] C.W.CRYER. The solution of a quadratic programming problem using systematic overrelaxation. *SIAM J. Control*, 9:385–392, 1971. 11

[3] C.W.CRYER. The efficient solution of linear complementarity problems for tridiagonal minkowski matrices. *ACM Trans. Math. Softwave*, 9:199–214, 1983. 12

[4] M.J.BRENNAN E.S.SCHWARTZ. The valuation of the American put option. *J. of Finance*, 32:449–462, 1977. 10

[5] G.BARLES C.DAHER M.ROMANO. Convergence of numerical schemes for problems arising in finance theory. *Math. Models and Meth. in Appl. Sciences*, 5:125–143, 1995. 13

[6] R.GLOWINSKI J-L.LIONS R.TREMOLIERES. *Analyse Numérique des Inéquations Variationnelles*. Dunod, 1976. 9

[7] L.C.G.ROGERS Z.SHI. The value of an asian option. *J. Appl. Probab.*, 32(4):1077–1088, 1995. 14