Evaluation of Guaranteed Minimum Withdrawal Benefits in the Black Scholes model

INRIA
PREMIA 16
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February 2015

Premia 22

We present here the implementation of a methodology for the valuation of Guaranteed Minimum Withdrawal Benefits (GMWB) developed by Chen and Forsyth in [2]. First we describe the model, then the numerical implementation.

1 GMWB model

A GMWB contract consists of a personal sub-account and a virtual guarantee account. The funds in the sub account are invested in a reference portfolio by the insurance company. Let S denote the value of the reference portfolio of the underlying assets. We suppose that the risk-adjusted process of S follows a Black-Scholes model given by the stochastic differential equation (SDE):

$$dS = rSdt + \sigma SdZ,$$

where $r > 0$ is the risk-free interest rate, σ is the volatility, and Z is a standard Gauss-Wiener process. Let W denote the balance of the personal sub-account, and A denote the guarantee account balance. At the inception of the contract, The policyholder pays a lump-sum premium w_0 to the insurer. This amount becomes the initial sub-account and guarantee account balance. The GMWB contract allows the policyholder to withdraw funds from the sub-account at specified times (annually, semi-annually or even continuously). The withdrawals reduce the balance of the guarantee account with the same amount, therefore A is a decreasing process taking values in $[0, w_0]$. The policyholder may withdraw as long as the guarantee account is positive, even if the sub-account balance falls to zero before the maturity of the contract. The insurer also issues a proportional annual insurance rate $\alpha \geq 0$. The risk adjusted dynamics of the sub-account balance W is given by the following SDE:

$$\begin{aligned} dW &= (r - \alpha)Wdt + \sigma WdZ + dA \\ W &= 0, \text{ if } W = 0. \end{aligned}$$

Let T denote the maturity of the contract. We will note by $V(W, A, \tau)$ the no-arbitrage value of the contract at time $t = T - \tau$. Withdrawals are allowed at discrete times $t_O^i, i = 1, 2, \ldots, K$ with t_O^K. If the withdraw amount at t_O^i is higher than the threshold $G_r(t_O^i - t_O^{i-1})$ with G_r denoting the withdrawal rate, then a proportional penalty $\kappa > 0$ is imposed on the value exceeding the threshold as well as a fix cost $c \geq 0$. Therefore the cash flow $f(\gamma)$received by the policyholder when withdrawing the amount γ at t_O^i is given by:

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G_r(t_O^i - t_O^{i-1}), \\ \gamma - \kappa(\gamma - G_r(t_O^i - t_O^{i-1})) - c & \text{if } \gamma \geq G_r(t_O^i - t_O^{i-1}). \end{cases}$$

Between withdrawal times, the value process $V(W, A, \tau)$ follows the dynamics given by:

$$V_\tau = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r - \alpha)WV_W + -rV. \tag{1}$$

At a withdrawal time $\tau_O^i = T - t_O^i$, V satisfies the no-arbitrage condition:

$$V(W, A, \tau_O^{i+}) = \sup_{\gamma \in [0,A]} \left[V(\max(W - \gamma, 0), A - \gamma, \tau_O^i) + f(\gamma) \right], \quad i = 0, \ldots, K - 1,$$

where τ_O^{i+} is the time infinitesimally after τ_O^i. The terminal boundary condition takes the form:

$$V(W, A, \tau = 0) = \max(W, (1 - \kappa)A - c).$$

2 Numerical method

We define the solution domain as $(W, A, \tau) \in [0, W_{max}] \times [0, w_0] \times [0, T]$. We use an unequally spaced grid $[W_0, W_1, \ldots, W_{i_{max}}]$ with $W_0 = 0$,$W_{i_{max}} = W_{max}$ in the W direction and an equally spaced grid $[A_0, A_1, \ldots, A_{j_{max}}]$ with $A_0 = 0$,$A_{j_{max}} = w_0$ in the A direction. The discrete timesteps are noted by $\tau^n = n\Delta\tau$, $n = 0, \ldots, N$, with $\tau_N = T$. We assume that the withdrawal times τ_O^k, $k = 1, 2, \ldots, K$ coincide with some discrete timesteps. Between withdrawal times we solve the PDE (1) for each value A_j, $j = 1, \ldots, j_{max}$ using second order finite difference methods in the W direction, as the dynamics does not depend on the value of A. We use central, forward or backward differencing in the discretization to ensure the positivity of the coefficients see [3] , and we use a fully implicit timestepping scheme. The tridiagonal matrix equation is solved using a simple forward sweep method.

At a withdrawal time τ_O^k, we need to compute the optimal withdrawal amount γ_k. This γ_k is the solution of the local optimization problem:

$$V(W_i, A_j, \tau_O^{k+}) = \sup_{\gamma_k \in [0,A_j]} \left[V(\max(W_i - \gamma_k, 0), A - \gamma_k, \tau_O^k) + f(\gamma_k) \right].$$

As it is computationally expensive to find the exact value of γ_k, we select a handful of control values and use the best value amongst them as an estimator for γ_k. The control values $\hat{\gamma_{k,\ell}}$, $\ell = 1, \ldots, L$ are homogeneously spread out on the interval $[0, A_j]$, and include the values 0, $G_r(\tau_O^{k+1} - \tau_O^k)$ and A_j. Since in most cases the true value corresponds to one of these three values, the magnitude of L does not play a significant role in the quality of this estimator. Therefore choosing a low value for L (around one fourth of the number of discretization points used for A), will provide satisfactory results.

Continuous withdrawal is approximated by letting every timestep become a withdrawal time, giving $K = N$, $\tau_O^k = \tau^k$ for $k = 0, \ldots, K - 1$.

The pricing problem now reduces to find the fair fee α such that

$$V_\alpha(S = w_0, A = w_0, t = 0) = w_0 \tag{2}$$

Viewing V_α as being parametrized by the rider fee α_g, we solve the equation (2) using a classical secant method. Typically, around 10 iterations are necessary to obtain convergence of the algorithm under a fixed tolerance of 10^{-8}.

2.1 Extension with additional variable modelling

In [1] the Authors consider the same GMBW contract taking into account additional structural features such as the separation of the rider fee into mutual fund management fee and the guarantee fee, time-dependent parameters, etc. In particular, if $\alpha = \alpha_m + \alpha_g$ where α_m denotes the mutual fund fee and α_g denotes the guarantee fee, the dynamics between withdrawal times given by (1) changes to:

$$V_\tau = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r - \alpha)WV_W + -rV + \alpha_m W.$$

The variable α_m is exogenously given, therefore in the pricing problem we are looking for the no-arbitrage insurance fee α_g.

2.2 Numerical examples

There is multiple choices for parameters. Most of them can be treated by modifying correct lines in the program. Table 1 gives an overview of the parameters with their default values in the last column.

Table 1: Parameters implemented in the program for pricing GMWB contracts

Model	`r`	risk free interest rate	0.05
	`sigma`	volatility	0.2
	`t`	maturity	10
Product	`w0`	initial lump-sum premium	100
	`gr`	maximum withdrawal rate	10/year
	`k`	withdrawal penalty	0.1
	`c`	fix cost of withdrawal	10^{-8}

Table 2: Time-dependent penalty charge ($\kappa(t)$)

Year	Penalty $\kappa(t)$
$0 \le t < 2$	0.08
$2 \le t < 3$	0.07
$3 \le t < 4$	0.06
$4 \le t < 5$	0.05
$5 \le t < 6$	0.04
$6 \le t < 7$	0.03
$7 \le t$	0

- One can choose between the continuous model and the discrete model by setting the variable `is_cont` to 1 or 0, respectively. In the case of discrete withdrawal, the difference between consecutive withdrawal times can be set by the variable `t_with`. In [2], the Authors consider 0.5 and 1 year for `t_with`.
- In [1], the Authors introduce a time-dependent withdrawal penalty, which is a typical feature of many real GMWB contracts. The penalty is usually decreasing as we approach the maturity, the values applied in the article are given in Table 2.
- Also in [1], the default value of the mutual fund fee `alpha_m` is set to 0.01.

3 Conclusion

The method is relatively easy to implement, the main difficulty is to understand the specifications of the contract. I hope that this documentation is easier to read than the original article.

References

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