Summary of Option prices under stochastic volatility

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Abstract

It is well known that the reality of stock prices movement can be captured by the Heston model introduced in [4]. A exact solution for the model is also presented in [4]. However, the solution is an integrals in the complex plane, posing significant difficulties in numerical evaluation. Hence, in [2], Zhang et al. present a closed form solutions for option prices and implied volatilities which is accurate when compared with the exact solutions. In this summary, we present the main results of [2] and also provide an approximation for the Delta.

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1 Introduction

The Heston model [3] has the following dynamics:

$$\begin{aligned} \mathrm{d}S_t &= rS_t\mathrm{d}t + \sqrt{v_t}S_t\mathrm{d}W_t^s, \\ \mathrm{d}v_t &= \kappa\left(\theta - v_t\right)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}W_t^v, \end{aligned} \tag{1}$$

where r is the rate of return of the stock, θ is the long run average price volatility, κ is the rate of v_t reversion to θ and ξ is the volatility of the volatility. W_t^s and W_t^v are 2 Brownian motions satisfying $\langle W^s, W^v\rangle_t = \rho t$. We set the interest rate r and dividend rate q to 0 in this paper as one can always introduce a discounted financial instruments and a discount factor e^{-qt} in the stock price process to eliminate the effect or r and q respectively. We introduce a parameter ϵ which is the speed of fast mean reversion and replace κ with $\frac{\kappa}{\epsilon}$ and ξ with $\frac{\xi}{\sqrt{\epsilon}}$ to obtain a fast mean-reverting process when $0 < \epsilon << 1$ (see [1]). It can be show that the option price $f(t,s,v)$ satisfy the following PDE:

$$f_t + \left(\frac{\kappa}{\epsilon}\left(\theta - v\right) - \frac{\xi}{\sqrt{\epsilon}}\lambda v\right)f_v + \frac{1}{2}vs^2 f_{ss} + \frac{\xi}{\sqrt{\epsilon}}s\rho v f_{sv} + \frac{1}{2}\frac{\xi^2}{\epsilon}v f_{vv} = 0 \tag{2}$$

where $\lambda(t,s,v) = \lambda v^{\frac{1}{2}}$ is the price of volatility risk. We also have the payoff function to be $f(T,s,v) = (s-K)^+$ for a call and $f(T,s,v) = (K-s)^+$ for a put where K is the strike price. We will set $\lambda = 0$ as we can always introduce a change of variables $\tilde{\kappa} = \kappa + \sqrt{\epsilon}\xi\lambda$ and $\tilde{\theta} = \frac{\kappa\theta}{\tilde{\kappa}}$ for $\lambda \neq 0$. Thus, Equation (2) becomes

$$f_t + \frac{\kappa}{\epsilon}\left(\theta - v\right)f_v + \frac{1}{2}vs^2 f_{ss} + \frac{\xi}{\sqrt{\epsilon}}s\rho v f_{sv} + \frac{1}{2}\frac{\xi^2}{\epsilon}v f_{vv} = 0. \tag{3}$$

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We have in [3] the exact solution of Equation (3) expressed as 2 integrals of a complex function. However, in [5] and [6], we see that there are significant difficulties in the numerical evaluation of these integrals. Therefore, in [1], Fouque et al. developed a method with the following price expansion for a European call:

$$C = C_0 + \sqrt{\epsilon} C_1 + \ldots, \tag{4}$$

where $C_0 = sN(d_1) - KN(d_2)$ and $C_1 = -\frac{1}{2}\rho\frac{\xi}{\kappa}d_2 K\phi(d_2)$ with $d_1 = \frac{\ln(s/K)+\theta(T-t)/2}{\sqrt{\theta(T-t)}}$ and $d_2 = d_1 - \sqrt{\theta(T-t)}$ with the following implied volatility expansion

$$\tilde{\sigma}^i = \tilde{\sigma}_0 + \sqrt{\epsilon}\tilde{\sigma}_1 + \ldots, \tag{5}$$

where $\tilde{\sigma}_0 = \sqrt{\theta}$ and $\tilde{\sigma}_1 = -\rho\xi\frac{d_2}{2\kappa}\sqrt{T-t}$. Here, we see that the approximate solution does not depend on the stochastic volatility, so in this summary, we present the method in [2] by Zhang et al. which provide a more accurate analytical approximate solution that do depends on the stochastic volatility.

2 Main results

The approximate solutions derived in [2] for the option prices, implied volatility and delta are provided in this section.

Theorem 1 (Theorem 1, [2]). *Let $\tau = T - t$ and $z = \theta\tau + \frac{\epsilon}{\kappa}\left(1 - e^{-\frac{\kappa\tau}{\epsilon}}\right)(v - \theta)$. We have the following series expansion for the option price f:*

$$f = f_0 + \sqrt{\epsilon} f_1 + \epsilon f_2 + \ldots, \tag{6}$$

where

$$f_0(s,z) = \begin{cases} sN(d^+) - KN(d^-), & \text{for call} \\ KN(-d^-) - sN(-d^+), & \text{for put} \end{cases} \tag{7}$$

$$f_1(\tau,s,z) = g_1(\tau,z)\,G_1(s,z), \quad \text{for call and put} \tag{8}$$

$$f_2(\tau,s,z) = g_2(\tau,z)\,G_2(s,z) + h_2(\tau,z)\,H_2(s,z) + m_2(\tau,z)\,M_2(s,z), \quad \text{for call and put.} \tag{9}$$

Here, we have $d^\pm = \frac{\ln(s/K)\pm z/2}{\sqrt{z}}$, $N(x)$ is the cumulative distribution function of the standard normal distribution and

$$\begin{aligned}
&g_1(\tau,z) = A(\tau)z + \theta B(\tau), \quad g_2(\tau,z) = \frac{1}{2}\left(A(\tau)z + \theta B(\tau)\right)^2, \\
&h_2(\tau,z) = C(\tau)z + \theta D(\tau), \quad m_2(\tau,z) = E(\tau)z + \theta F(\tau), \\
&G_1(s,z) = -\frac{1}{2}\rho\frac{\xi}{\kappa}d^- z^{-1}K\phi(d^-), \quad G_2(s,z) = \frac{1}{4}\rho^2\frac{\xi^2}{\kappa^2}z^{-\frac{5}{2}}K\phi(d^-)\left(3 - 3(d^-)^2 - 3d^+d^- + d^+(d^-)^3\right), \\
&H_2(s,z) = -\frac{1}{2}\rho^2\frac{\xi^2}{\kappa^2}z^{-\frac{3}{2}}K\phi(d^-)\left(1 - (d^-)^2\right), \quad M_2(s,z) = \frac{1}{8}\frac{\xi^2}{\kappa^2}z^{-\frac{3}{2}}K\phi(d^-)\left(d^+d^- - 1\right),
\end{aligned}$$

where $\phi(x)$ is the probability density function of the standard normal distribution and

$$A(\tau) = 1 - \frac{\kappa}{\epsilon}\frac{\tau e^{-\frac{\kappa\tau}{\epsilon}}}{1 - e^{-\frac{\kappa\tau}{\epsilon}}}, \quad B(\tau) = \frac{\kappa}{\epsilon}\frac{\tau^2 e^{-\frac{\kappa\tau}{\epsilon}}}{1 - e^{-\frac{\kappa\tau}{\epsilon}}} - \frac{\epsilon}{\kappa}\left(1 - e^{-\frac{\kappa\tau}{\epsilon}}\right), \quad C(\tau) = 1 - \frac{e^{-\frac{\kappa\tau}{\epsilon}}}{1 - e^{-\frac{\kappa\tau}{\epsilon}}}\left(\frac{\kappa\tau}{\epsilon} + \frac{1}{2}\left(\frac{\kappa\tau}{\epsilon}\right)^2\right),$$

$$D(\tau) = \tau e^{-\frac{\kappa\tau}{\epsilon}} - 2\frac{\epsilon}{\kappa}\left(1 - e^{-\frac{\kappa\tau}{\epsilon}}\right) + \frac{\tau e^{-\frac{\kappa\tau}{\epsilon}}}{1 - e^{-\frac{\kappa\tau}{\epsilon}}}\left(\frac{\kappa\tau}{\epsilon} + \frac{1}{2}\left(\frac{\kappa\tau}{\epsilon}\right)^2\right), \quad E(\tau) = 1 - \frac{e^{-\frac{\kappa\tau}{\epsilon}}}{1 - e^{-\frac{\kappa\tau}{\epsilon}}}\left(2\frac{\kappa\tau}{\epsilon} - \left(1 - e^{-\frac{\kappa\tau}{\epsilon}}\right)\right),$$
$$F(\tau) = \frac{1}{2}\frac{\epsilon}{\kappa}\left(1 - e^{-2\frac{\kappa\tau}{\epsilon}}\right) - 2\frac{\epsilon}{\kappa}\left(1 - e^{-\frac{\kappa\tau}{\epsilon}}\right) + \frac{\tau e^{-\frac{\kappa\tau}{\epsilon}}}{1 - e^{-\frac{\kappa\tau}{\epsilon}}}\left(2\frac{\kappa\tau}{\epsilon} - \left(1 - e^{-\frac{\kappa\tau}{\epsilon}}\right)\right).$$

Theorem 2 (Theorem 2, [2])**.** *The implied volatility can be approximated by*

$$\sigma^i = \sigma_0 + \sqrt{\epsilon}\sigma_1 + \epsilon\sigma_2 + \ldots, \tag{10}$$

where

$$\sigma_0 = \sqrt{\frac{z}{T-t}}, \quad \sigma_1 = -\frac{1}{2}\rho\frac{\xi}{\kappa}(Az + \theta B)\frac{d^- z^{-1}}{\sqrt{T-t}}, \quad \sigma_2 = f_2 \times \left(K\phi(d^-)\sqrt{T-t}\right)^{-1} - \frac{1}{2}\sigma_1^2\frac{d^+ d^-}{\sigma_0}.$$

Theorem 3. *The Delta, Δ, has the following expression:*

$$\Delta = \frac{\partial f_0}{\partial s} + \sqrt{\epsilon}\frac{\partial f_1}{\partial s} + \epsilon\frac{\partial f_2}{\partial s}. \tag{11}$$

Here,

$$\frac{\partial f_0}{\partial s} = \begin{cases} N(d^+) + \frac{\phi(d^+)}{\sqrt{z}} - K\frac{\phi(d^-)}{s\sqrt{z}}, & \textit{for call} \\ -N(-d^+) + \frac{\phi(-d^+)}{\sqrt{z}} - K\frac{\phi(-d^-)}{s\sqrt{z}}, & \textit{for put} \end{cases}$$
$$\frac{\partial f_1}{\partial s} = g_1\frac{\partial G_1}{\partial s}, \quad \textit{for call and put}$$
$$\frac{\partial f_2}{\partial s} = g_2\frac{\partial G_2}{\partial s} + h_2\frac{\partial H_2}{\partial s} + m_2\frac{\partial M_2}{\partial s}, \quad \textit{for call and put}$$

where

$$\frac{\partial G_1}{\partial s} = \frac{1}{2}\rho\frac{\xi}{\kappa}z^{-1}K\frac{\phi(d^-)}{s\sqrt{z}}\left((d^-)^2 - 1\right),$$
$$\frac{\partial G_2}{\partial s} = \frac{1}{4}\rho^2\frac{\xi^2}{\kappa^2}z^{-\frac{5}{2}}K\frac{\phi(d^-)}{s\sqrt{z}}\left(\left(d^+\right)^3 - 3d^+ + 6d^+(d^-)^2 - d^+(d^-)^4 + 3(d^-)^3 - 12d^-\right),$$
$$\frac{\partial H_2}{\partial s} = \frac{1}{2}\rho^2\frac{\xi^2}{\kappa^2}z^{-\frac{3}{2}}K\frac{\phi(d^-)}{s\sqrt{z}}\left(3d^- - (d^-)^3\right),$$
$$\frac{\partial M_2}{\partial s} = \frac{1}{8}\frac{\xi^2}{\kappa^2}z^{-\frac{3}{2}}K\frac{\phi(d^-)}{s\sqrt{z}}\left(d^+ + 2d^- - d^+(d^-)^2 - 1\right).$$

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