Pricing European Options in Hybrid Model by Fourier-Cosine Method: Implementation in PREMIA

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Abstract

Under the hybrid model with a Heston-type volatility and stochastic interest rate, we price the European options with constant dividend by the Fourier-Cosine method. To price with the Fourier-Cosine method, it requires the characteristic function (ChF) of the logarithm of the underlying asset, but a close form of the ChF of the logarithm of the underlying is not available. According to [1], if a process is affine, its ChF can be derived in a close form. Then an approximation of the non-affine terms for the hybrid model is proposed such that the model is affine, so that an approximation of the ChF is obtained by the theory in [1]. Applying the approximation of the ChF to the Fourier-Cosine expansion, we can efficiently calculated the European option price. This method can be extended to the four-dimensional hybrid model with a stochastic dividend followed the Vasicěk type dynamics. The pricing methods in both models are implemented in PREMIA.

Premia 22

1 Introduction

We consider the pricing problem of the European option in a hybrid model with Heston-type stochastic volatility and stochastic interest rate by Vasicěk model. The implementation of this problem is based on the Fourier-Cosine expansion proposed by Grzelak and Oosterlee [3]. Then the model is extended to the four-dimensional hybrid model with stochastic dividend following the Vasicěk model. The pricing method for the four-dimensional hybrid model is modified from that of a four-dimensional model in the foreign exchange context in [2], which is also based on the Fourier-Cosine expansion. To apply the Fourier-Cosine expansion,

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it needs the ChF of the logarithm of the underlying. We will apply the theory in [1], that a closed form solution of the ChF exists, if a process is a affine diffusion process(AD), to derive the ChF.

An AD process is defined as a system of SDEs

$$\mathrm{d}\mathbf{X}_t = \mu(\mathbf{X}_t)\mathrm{d}t + \sigma(\mathbf{X}_t)\mathrm{d}W_t \tag{1}$$

satisfies:

$$\begin{aligned}
\mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t, \quad \text{for any}(a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \\
\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, \quad \text{for arbitrary}(c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \\
r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \quad \text{for}(r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n,
\end{aligned}$$

for any $i, j = 1, \cdots, n$ with $r(\mathbf{X}_t)$ being an interest rate component, where $\mathbf{X}_t$ is the n-dimensional processes.

According to the theory of affine process in [1], the discounted ChF of the AD processes (1) is of the following form:

$$\phi(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r_s ds + i\mathbf{u}^T\mathbf{X}_T\right)|\mathcal{F}(t)\right) = e^{\mathbf{A}(\mathbf{u},\tau)+\mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}_t},$$

where the expectation is taken under the risk-neutral measure $\mathbb{Q}$. For a time lag, $\tau := T - t$, the coefficients $\mathbf{A}(\mathbf{u}, \tau)$ and $\mathbf{B}^T(\mathbf{u}, \tau)$ have to satisfy the following complex-valued ordinary differential equation (ODEs):

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(\mathbf{u}, \tau) = -r_1 + a_1^T\mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1\mathbf{B}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{A}(\mathbf{u}, \tau) = -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0\mathbf{B}, \end{cases} \tag{2}$$

with $a_i, c_i, r_i, i = 0, 1$ as in (2).

But the hybrid model is not affine, to apply the theory of [1], we need to approximate the model such that it becomes affine. In the following, we will provide the details of approximation and the close form of ChF derived from the theory of [1], and the main idea of Fourier-Cosine method for pricing.

The Fourier-Cosine method can be used for the hybrid model with Vasicěk model, but for ease of presenting, we present here a constant mean of reversion for the Vasicěk model in the following. All the result provided here can be used for the Vasicěk model by changing the constant mean of reversion of the interest rate by time-varying mean of reversion.

The rest of this file is as follows: we introduce the model with stochastic volatility and stochastic interest rate but with a constant dividend and derive its discounted ChF in Section 2. Then in Section 3 we extend to 4-dimensional hybrid model with the stochastic dividend and provide its ChF as well. In Section 4, we present the pricing method of Fourier Cosine expansion using the closed form solution of ChF of the model. The program manual of the implementation in PREMIA is given in Section 5.

2 The Hybrid Model with Constant Dividend

2.1 Model Description

We assume under the risk-neutral measure $\mathbb{Q}$, the dynamics system for the underlying asset $S(t)$, the volatility of the underlying asset $\nu(t)$, the stochastic interest rate $r(t)$ are given as follows:

$$\begin{cases} \mathrm{d}S(t)/S(t) & = (r(t) - q)\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_S(t), \\ \mathrm{d}\nu(t) & = \kappa_\nu(\theta_\nu - \nu(t))\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\mathrm{d}W_\nu(t) \\ \mathrm{d}r(t) & = \kappa_r(\theta_r - r(t))\mathrm{d}t + \sigma_r\mathrm{d}W_r(t), \end{cases} \tag{3}$$

where the parameters κ_ν, κ_r is the reversion speed of the volatility and that of the interest rate, respectively, θ_ν, θ_r determine the long term mean of the volatility and that of the interest rate, respectively, and σ_ν, σ_r is the volatility of the volatility and that of the interest rate, respectively, q is a constant representing the dividend, $W_S(t), W_\nu(t)$ and $W_r(t)$ are the Brownian motions under the risk-neutral measure $\mathbb{Q}$, with their correlations given by $\mathrm{d}W_S(t)\mathrm{d}W_\nu(t) = \rho_{S\nu}\mathrm{d}t$, $\mathrm{d}W_S(t)dW_r(t) = \rho_{Sr}\mathrm{d}t$, $\mathrm{d}W_\nu(t)\mathrm{d}W_r(t) = \rho_{\nu r}\mathrm{d}t$. Note that we assume independence between the instantaneous short rate, $r(t)$, and the volatility process $\nu(t)$, i.e. $\rho_{\nu r} = 0$.

2.2 Model Reformulate

In order to obtain a well-defined Heston hybrid model with an indirectly imposed correlation to simplify the computation of the pricing, [3] proposed to reformulate the hybrid model in the following way:

$$\begin{cases} \frac{\mathrm{d}S(t)}{S(t)} = (r(t) - q)\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}\widetilde{W}_S(t) + \Delta\sqrt{\nu(t)}S(t)\mathrm{d}\widetilde{W}_\nu(t) + \Omega(t)\mathrm{d}\widetilde{W}_r(t), \\ \mathrm{d}\nu(t) = \kappa_\nu(\theta_\nu - \nu(t))\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\mathrm{d}\widetilde{W}_\nu(t), \\ \mathrm{d}r(t) = \kappa_r(\theta_r - r(t))\mathrm{d}t + \sigma_r\sqrt{r(t)}\mathrm{d}\widetilde{W}_r(t), \end{cases}$$

$$\text{with } \mathrm{d}\widetilde{W}_S(t)\mathrm{d}\widetilde{W}_\nu(t) = \hat{\rho}_{S\nu}\mathrm{d}t, \quad \mathrm{d}\widetilde{W}_S(t)\mathrm{d}\widetilde{W}_r(t) = 0, \quad \mathrm{d}\widetilde{W}_\nu(t)\mathrm{d}\widetilde{W}_r(t) = 0. \tag{4}$$

Let $x(t) = \log(S(t))$, then we have

$$\begin{aligned} \mathrm{d}x(t) &= \left[(r(t) - q) - \frac{1}{2}\left(\Omega_t^2 r(t) + \nu(t)(1 + \Delta^2 + 2\hat{\rho}_{S\nu}\Delta)\right)\right]\mathrm{d}t \\ &\quad + \sqrt{\nu(t)}d\widetilde{W}_S(t) + \Delta\sqrt{\nu(t)}S(t)\mathrm{d}\widetilde{W}_\nu(t) + \Omega_t\sqrt{r(t)}S(t)\mathrm{d}\widetilde{W}_r(t) \\ &= \left[r(t) - q + \frac{1}{2}\nu(t)\right]\mathrm{d}t \\ &\quad + \sqrt{\nu(t)}d\widetilde{W}_S(t) + \Delta\sqrt{\nu(t)}S(t)\mathrm{d}\widetilde{W}_\nu(t) + \Omega_t\sqrt{r(t)}S(t)\mathrm{d}\widetilde{W}_r(t). \end{aligned}$$

The pricing method will be derived based on the SDE system $\mathbf{X}^*(t) := [r(t), \nu(t), x(t)]^T$, since the SDE system $\mathbf{X}^*(t)$ is coordinated with that of $\mathbf{X}(t) := [S(t), \nu(t), r(t)]^T$ in the sense that

$$\Omega_t = \rho_{Sr}\sqrt{\nu(t)}, \quad \hat{\rho}^2_{S\nu} = \rho^2_{S\nu} + \rho^2_{Sr}, \quad \Delta = \rho_{S\nu} - \hat{\rho}_{S\nu}. \tag{5}$$

For details about the proof of the coordination of the reformulated model (4) and the original model (3), please refer to [3].

2.3 Close Form of Characteristic Function

For the hybrid model (4), the system instantaneous covariance matrix in (2) for the SDE system $\mathbf{X}_t^*$ is given by:

$$\Sigma := \sigma(\mathbf{X}_\mathbf{t}^*)\sigma(\mathbf{X}_\mathbf{t}^*)^T = \begin{bmatrix} \sigma_r^2 r(t) & 0 & \sigma_r\Omega_t \\ * & \sigma_\nu^2\nu(t) & \sigma_\nu\hat{\rho}_{S\nu}\nu(t) + \sigma_\nu\Delta\nu(t) \\ * & * & \Omega_t^2 + \nu(t)(1+\Delta^2+2\hat{\rho}_{S\nu}\Delta) \end{bmatrix}. \quad (6)$$

From the definition of AD process, we know the model (4) is not affine. To make the hybrid model (4) affine we need to approximate the non-affine terms $\sqrt{\nu(t)}$ in $\Sigma_{(1,3)} = \sigma_r\Omega_t = \sigma_r\rho_{x,r}\sqrt{\nu(t)}$ of the instantaneous covariance matrix. Note that $\Sigma_{(3,3)}$ does not seems to be of the affine form, but in fact by (5), it equals $\Sigma_{(3,3)} = \nu(t)$.

[3] proposed two ways to approximate the non-affine terms, one is deterministic approximation, which approximates $\sqrt{\nu(t)}$ its expectations and the other is a stochastic approximation by a normal distributed random variable. Here we only apply the deterministic approximations and we provide only the approximation result, the proof can be refered to [3].

Lemma 2.1. *The expectation, $\mathbb{E}(\sqrt{\nu(t)})$, with stochastic process given by equation (3) can be approximated by*

$$\sqrt{\nu(t)} \approx \mathbb{E}\left[\sqrt{\nu(t)}\right] \approx \beta_1 + \beta_2 e^{-\beta_3 t} := \varphi(t), \quad (7)$$

where $\beta_1 = \sqrt{\theta_\nu - \sigma_\nu^2/8\kappa_\nu}$, $\beta_2 = \sqrt{\nu(0)} - \beta_1$, $\beta_3 = -\log[\beta_2^{-1}(\Lambda(1)-\beta_1)]$, and

$$\begin{aligned} \Lambda(t) &= \sqrt{c(t) - [\lambda(t)-1] + c(t)d + \frac{c(t)d}{2[d+\lambda(t)]}}, \\ c(t) &= \frac{1}{4\kappa_\nu}\sigma_\nu^2(1-e^{-\kappa_\nu t}), \quad d = \frac{4\kappa_\nu\theta_\nu}{\sigma_\nu^2}, \quad \lambda(t) = \frac{4\kappa_\nu\nu(0)e^{-\kappa_\nu t}}{\sigma_\nu^2(1-e^{-\kappa_\nu t})}. \end{aligned}$$

By the approximation given above, the hybrid model (4) can be fitted into the AD class, thus we have a closed form solution of the discount ChF $\phi(u, \mathbf{X}_t, t, T)$ which is defined as

$$\phi(u, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)\mathrm{d}s + iux(T)\right)|\mathcal{F}(t)\right).$$

For $\tau := T - t$, we denote $\phi(u, \mathbf{X}_t, t, T)$ by $\phi(u, \mathbf{X}_t, \tau)$.

Theorem 2.2. *The discount ChF of X_t^* is given by*

$$\phi(u, \mathbf{X}_t, \tau) = \exp(A(u,\tau) + B_x(u,\tau)x(t) + B_\nu(u,\tau)\nu(t) + B_r(u,\tau)r(t)), \quad (8)$$

where

$$
\begin{aligned}
B_x(u,\tau) &= iu, && (9)\\
B_r(u,\tau) &= (iu-1)\kappa_r^{-1}\left(1-e^{-\kappa_r\tau}\right), && (10)\\
B_\nu(u,\tau) &= \frac{1-e^{-D\tau}}{\sigma_\nu^2(1-Ge^{-D\tau})}(\kappa_\nu-\sigma_\nu\zeta iu-D), && (11)
\end{aligned}
$$

and

$$A(u,\tau)=\kappa_r\theta_r I_1(\tau)+\kappa_\nu\theta_\nu I_2(\tau)+\frac{1}{2}\sigma_r^2 I_3(\tau)+\sigma_r\rho_{Sr}I_4(\tau),$$

with $\zeta=\hat{\rho}_{S\nu}+\Delta, D=\sqrt{(\sigma_\nu\zeta iu-\kappa_\nu)^2-(iu-1)iu\sigma_\nu^2}$, *and* $G=\frac{\kappa_\nu-\sigma_\nu\zeta iu-D}{\kappa_\nu-\sigma_\nu\zeta iu+D}$ *The Integrals* $I_1(\tau), I_2(\tau)$ *and* $I_3(\tau)$ *admit analytic solution and* $I_4(\tau)$ *a semi-analytic solution:*

$$
\begin{aligned}
I_1(\tau) &= \frac{1}{\kappa_r}(iu-1)\left[\tau+\frac{1}{\kappa_r}\left(e^{-\kappa_r\tau}-1\right)\right],\\
I_2(\tau) &= \frac{\tau}{\sigma_\nu^2}\left(\kappa_\nu-\sigma_\nu\zeta iu-D\right)-\frac{2}{\sigma_\nu^2}\log\left(\frac{1-Ge^{-D\tau}}{1-G}\right),\\
I_3(\tau) &= \frac{1}{2\kappa_r^3}(i+u)^3\left(3+e^{-2\kappa_r\tau}-4e^{-\kappa_r\tau}-2\kappa_r\tau\right),\\
I_4(\tau) &= iu\int_0^\tau \mathbb{E}(\sqrt{\nu(T-s)})B_r(u,s)\,ds\\
&= -\frac{1}{\kappa_r}(iu+u^2)\int_0^\tau \mathbb{E}(\sqrt{\nu(T-s)})\left(1-e^{-\kappa_r s}\right)ds.
\end{aligned}
$$

The proof can be found in Appendix B of[3].

3 Four-Dimensional Hybrid Model with Stochastic Dividend

3.1 Model Description

The four-dimensional hybrid model with the underlying asset $S(t)$, the stochastic volatility $\nu(t)$, the stochastic interest rate $r(t)$ and the stochastic dividend $q(t)$ is given as follows:

$$
\begin{cases}
\mathrm{d}S(t)/S(t) &= (r(t)-q(t))\mathrm{d}t+\sqrt{\nu(t)}\mathrm{d}W_S(t),\\
\mathrm{d}\nu(t) &= \kappa_\nu(\theta_\nu-\nu(t))\mathrm{d}t+\sigma_\nu\sqrt{\nu(t)}\mathrm{d}W_\nu(t)\\
\mathrm{d}r(t) &= \kappa_r(\theta_r-r(t))\mathrm{d}t+\sigma_r\mathrm{d}W_r(t)\\
\mathrm{d}q(t) &= \kappa_q(\theta_q-q(t))\mathrm{d}t+\sigma_q\mathrm{d}W_q(t),
\end{cases}
\qquad (12)
$$

where $W_S(t), W_\nu(t), W_r(t)$ and $W_q(t)$ are the Brownian motions under the risk-neutral measure $\mathbb{Q}$, the parameters $\kappa_{(\cdot)}$ is the reversion speed, $\theta_{(\cdot)}$ determine the long term mean and $\sigma_{(\cdot)}$ is the volatility, the subscription of the parameters ν, r, q corresponding to the parameters for the volatility, the interest rate and the dividend, respectively.

Under the risk-neutral measure $\mathbb{Q}$, we assume a full matrix of correlations between the Brownian motions $W_S(t), W_\nu(t), W_d(t), W_f(t)$, i.e. $\rho_{i,j} := \mathrm{d}W_i(t) \cdot \mathrm{d}W_j(t)/\mathrm{d}t \neq 0$, when $i \neq j$ and $i, j \in \{S, \nu, r, q\}$.

3.2 Change of Measure

Direct pricing the European options under the risk-neutral measure $\mathbb{Q}$ will result in solving a four-dimensional PDE, which is imfeasible and unstable. To reduce the complexity of the pricing problem, we move from the risk-neutral measure $\mathbb{Q}$ to the forward measure.

The time t price of an option is given as the expectation under the risk-neutral measure $\mathbb{Q}$, denote it by $V(t, X(t))$, where $X(t) := [S(t), \nu(t), r(t), q(t)]^T$:

$$\begin{aligned} V(t, X(t)) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s} \max(S(T) - K, 0)|\mathcal{F}(t)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M_r(t)}{M_r(T)} \max(S(T) - K, 0)|\mathcal{F}(t)\right] \end{aligned}$$

with

$$M_r(t) = \exp\left(\int_0^t r(s)\mathrm{d}s\right).$$

Note that given the information at time t, $M_r(t)$ is deterministic, then

$$V(t, X(t)) = M_r(t)\mathbb{E}^{\mathbb{Q}}\left(\frac{\max(S(T) - K, 0)}{M_r(T)}|\mathcal{F}(t)\right),$$

denote by $\Pi(t)$ the forward price

$$\Pi(t) = \mathbb{E}^{\mathbb{Q}}\left(\frac{\max(S(T) - K, 0)}{M_r(T)}|\mathcal{F}(t)\right) = \frac{V(t, X(t))}{M_r(t)}. \tag{13}$$

To reduce the complexity of computing the forward price $\Pi(t)$, we move from the risk-neutral measure $\mathbb{Q}$ to the forward measure $\mathbb{Q}^T$ where the numéraire is the zero-coupon bond

$$P_r(t, T) := \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}].$$

The forward measure $\mathbb{Q}^T$ is defined by Radon-Nikodym derivative

$$\Lambda_{\mathbb{Q}}^T := \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \frac{P_r(t, T)}{P_r(0, T)M_r(t)}. \tag{14}$$

Under the filtration up to time t,

$$\Lambda_{\mathbb{Q}}^T(t) := \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}|\mathcal{F}(t) = \frac{M_r(t)}{M_r(T)P_r(t, T)}. \tag{15}$$

By switching from the risk-neutral measure $\mathbb{Q}$ to the T-forward measure $\mathbb{Q}^T$, the discounting will be decoupled from taking the expectation, i.e.

$$\Pi(t) = P_r(t,T)\mathbb{E}^T\left[\max(FX^T(T)-K,0)|\mathcal{F}(t).\right] \tag{16}$$

where $FX^T(t)$ is the forward underlying asset under the T-forward measure is given by

$$FX^T(t) = \frac{S(t)P_q(t,T)}{P_r(t,T)}, \quad P_q(t,T) := \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T q(s)\mathrm{d}s}].$$

By Itô's formula, the dynamics of $FX^T(t)$ under measure $\mathbb{Q}$ is

$$\begin{aligned}
\mathrm{d}FX^T(t) &= \frac{P_q(t,T)}{P_r(t,T)}\mathrm{d}S(t) + \frac{S(t)}{P_r(t,T)}\mathrm{d}P_q(t,T) - S(t)\frac{P_q(t,T)}{P_r^2(t,T)}\mathrm{d}P_r(t,T) \\
&\quad +S(t)\frac{P_q(t,T)}{P_r^3(t,T)}(\mathrm{d}P_r(t,T))^2 - \frac{P_q(t,T)}{P_r^2(t,T)}(\mathrm{d}P_r(t,T)\mathrm{d}S(t)) \qquad (17)\\
&\quad -\frac{S(t)}{P_r^2(t,T)}(\mathrm{d}P_r(t,T)\mathrm{d}P_q(t,T)) + \frac{1}{P_r(t,T)}(\mathrm{d}P_q(t,T)\mathrm{d}S(t)).
\end{aligned}$$

The dynamics of the zero-coupon bond under the risk-neutral measure $\mathbb{Q}$ is

$$\mathrm{d}P_r(t,T)/P_r(t,T) = r(t)\mathrm{d}t + \sigma_r B_r(t,T)\mathrm{d}W_r^{\mathbb{Q}}(t), \tag{18}$$

where

$$B_r(t,T) = \frac{1}{\kappa_r}\left[\mathrm{e}^{-\kappa_r(T-t)} - 1\right].$$

And the dynamics of the $P_q(t,T)$ under the risk-neutral measure $\mathbb{Q}$ is

$$\mathrm{d}P_q(t,T)/P_q(t,T) = q(t)\mathrm{d}t + \sigma_q B_q(t,T)\mathrm{d}W_q^{\mathbb{Q}}(t), \tag{19}$$

where

$$B_q(t,T) = \frac{1}{\kappa_q}\left[\mathrm{e}^{-\kappa_q(T-t)} - 1\right].$$

Then substitute the SDEs of $S(t), P_r(t,T)$ and $P_q(t,T)$ as given by (12), (18) and (19) into (17), we have

$$\begin{aligned}
\frac{\mathrm{d}FX^T(t)}{FX^T(t)} &= \sigma_r B_r(t,T)\left[\sigma_r B_r(t,T) - \rho_{S,r}\sqrt{\nu(t)} - \rho_{r,q}\sigma_q B(t,T)\right]\mathrm{d}t \qquad (20)\\
&\quad +\sigma_q B_q(t,T)\sqrt{\nu(t)}\rho_{S,q}\mathrm{d}t \\
&\quad +\sqrt{\nu(t)}\mathrm{d}W_S(t) - \sigma_r B_r(t,T)\mathrm{d}W_r(t) + \sigma_q B(t,T)\mathrm{d}W_q(t)
\end{aligned}$$

By change of measure, we have the dynamics of $FX^T(t)$ under the forward measure $\mathbb{Q}^T$:

$$\begin{aligned}
\frac{\mathrm{d}FX^T(t)}{FX^T(t)} &= \sigma_q B_q(t,T)\sqrt{\nu(t)}\rho_{S,q}\mathrm{d}t \\
&\quad +\sqrt{\nu(t)}\mathrm{d}W_S^T(t) - \sigma_r B_r(t,T)\mathrm{d}W_r^T(t) + \sigma_q B(t,T)\mathrm{d}W_q^T(t).
\end{aligned}$$

Define the log-transform of the forward price $FX^T(t)$ by $x^T(t)$, i.e. $x^T(t) := \log FX^T(t)$, its dynamics is

$$\begin{aligned}\mathrm{d}x^T(t) &= \left[\zeta(t,\sqrt{\nu(t)}) - \frac{1}{2}\nu(t) + \sigma_q B_q\sqrt{\nu(t)}\rho_{S,q}\right]\mathrm{d}t \\ &\quad + \sqrt{\nu(t)}\mathrm{d}W_S^T(t) - \sigma_r B_r \mathrm{d}W_d^T(t) + \sigma_q B_q \mathrm{d}W_f^T(t),\end{aligned}$$

where $B_r := B_r(t,T)$ and $B_q = B_q(t,T)$,

$$\zeta(t,\sqrt{\nu(t)}) = \left[\rho_{S,r}\sigma_r B_r - \rho_{S,q}\sigma_q B_q\right]\sqrt{\nu(t)} + \rho_{r,q}\sigma_r\sigma_q B_r B_q - \frac{1}{2}\left(\sigma_r^2 B_r^2 + \sigma_q^2 B_q^2\right).$$

From the definition of Radon-Nikodym derivative (14), we can redefine the driven Brownian motions for $\nu(t), r(t,T)$ and $q(t,T)$, then we have the dynamics of $x^T(t), \nu(t), r(t,T)$ and $q(t,T)$, under the T-forward measure $\mathbb{Q}^T$, the system under forward measure $\mathbb{Q}^T$ is as follows:

$$\begin{cases}\mathrm{d}x^T(t) = & \left[\zeta(t,\sqrt{\nu(t)}) - \frac{1}{2}\nu(t) + \sigma_q B_q\sqrt{\nu(t)}\rho_{S,q}\right]\mathrm{d}t \\ & +\sqrt{\nu(t)}\mathrm{d}W_S^T(t) - \sigma_r B_r\mathrm{d}W_d^T(t) + \sigma_q B_q \mathrm{d}W_f^T(t), \\ \mathrm{d}\nu(t) = & \left[\kappa_\nu(\theta_\nu - \nu(t)) + \sigma_\nu\rho_{\nu,r}\sigma_r B_r(t,T)\sqrt{\nu(t)}\right]\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\mathrm{d}W_\nu^T(t), \\ \mathrm{d}r(t) = & \left[\kappa_r(\theta_r(t) - r(t)) + \sigma_r^2 B_r(t,T)\right]\mathrm{d}t + \sigma_r \mathrm{d}W_r^T(t), \\ \mathrm{d}q(t) = & \left[\kappa_q(\theta_q(t) - q(t)) + \sigma_r\sigma_q\rho_{r,q}B_r(t,T)\right]\mathrm{d}t + \sigma_q\mathrm{d}W_q^T(t).\end{cases}$$

Details on the above change of measure can be found on the Appendix of [2].

3.3 Approximation of the ChF

Denote the ChF of the logarithm of the forward price $x^T(t)$ as

$$\phi^T := \phi^T(u,(x^T(t),v(t)),t,T) = \mathbb{E}^T\left[\mathrm{e}^{iux^T(T)}|\mathcal{F}(t)\right].$$

Applying the Feynman-Kac formula, we obtain the PDE for ϕ^T

$$\begin{aligned}-\frac{\partial\phi^T}{\partial t} &= \left[\zeta(t,\sqrt{\nu(t)}) - \frac{1}{2}\nu(t) + \sigma_q B_q\sqrt{\nu(t)}\rho_{S,q}\right]\frac{\partial\phi^T}{\partial x} \\ &\quad + \left[\kappa_\nu(\theta_\nu - \nu(t)) + \rho_{\nu,r}\sigma_\nu\sigma_r\sqrt{\nu(t)}B_r\right]\frac{\partial\phi^T}{\partial\nu} \\ &\quad + \left[\rho_{S,\nu}\sigma_\nu\nu(t) - \rho_{\nu,r}\sigma_\nu\sigma_r\sqrt{\nu t}B_r + \rho_{\nu,q}\sigma_\nu\sigma_q\sqrt{\nu t}B_q\right]\frac{\partial^2\phi^T}{\partial x\partial v} \\ &\quad + \left[\frac{1}{2}\nu(t) - \zeta(t,\sqrt{\nu(t)})\right]\frac{\partial^2\phi^T}{\partial x^2} + \frac{1}{2}\sigma_\nu^2\nu(t)\frac{\partial^2\phi^T}{\partial v^2}.\end{aligned}$$

Since the above PDE is not affine, it is not easy to find the solution, but we can use the approximation of the non-affine term in the PDE is proposed in (7). With the approximation of the non-affine term, the ChF ϕ^T is derived.

Theorem 3.1. *The discount ChF of $x^T(t)$ is of the following form:*

$$\phi^T(u,(x^T(t),v(t)),t,T) = \exp[A(u,\tau) + B(u,\tau)x^T(t) + C(u,\tau)\nu(t)], \tag{21}$$

where $\tau = T-t$, the functions $A(\tau) := A(u,\tau), B(\tau) := B(u,\tau), C(\tau) := C(u,\tau)$ are subject to Ordinary Differential Equations (ODE) of $A(u,\tau), B(u,\tau)$ and $C(u,\tau)$ as follows:

$$\begin{aligned}
B'(\tau) &= 0,\\
C'(\tau) &= -\kappa_\nu C(\tau) + [B^2(\tau) - B(\tau)]/2 + \rho_{S,\nu}\sigma_\nu B(\tau)C(\tau) + \sigma_\nu^2 C^2(\tau)/2,\\
A'(\tau) &= \kappa_\nu\theta_\nu C(\tau) + \rho_{\nu,r}\sigma_\nu\sigma_r\varphi(\tau)B_r(\tau)C(\tau) - \zeta(\tau,\varphi(\tau))[B^2(\tau) - B(\tau)]\\
&\quad + [-\rho_{\nu,r}\sigma_r\sigma_\nu\varphi(\tau)B_r(\tau) + \rho_{\nu,q}\sigma_\nu\sigma_q\varphi(\tau)B_q(\tau)]B(\tau)C(\tau)\\
&\quad + \rho_{S,q}\sigma_q B_q(\tau)\varphi(\tau)B(\tau),
\end{aligned}\tag{22}$$

with $B_i(\tau) = \kappa_i^{-1}[e^{-\kappa_i\tau} - 1]$ for $i = \{r,q\}$ and the initial conditions $B(0) = iu, C(0) = 0, A(0) = 0$.

The ODE system (22) can be solved as

$$\begin{aligned}
B(\tau) &= iu, \qquad (23)\\
C(\tau) &= \frac{1 - \mathrm{e}^{-d\tau}}{\sigma_\nu^2(1 - g\mathrm{e}^{-d\tau})}\left(\kappa_\nu - \rho_{S,\nu}\sigma_\nu iu - d\right),\\
A(\tau) &= \int_0^\tau [\kappa_\nu\theta_\nu + \rho_{r,\nu}\sigma_\nu\sigma_r\varphi(s)B_r(s) - \rho_{\nu,r}\sigma_r\sigma_\nu\varphi(s)B_r(s)iu\\
&\quad + \rho_{\nu,q}\sigma_\nu\sigma_q\varphi(s)B_q(s)iu]\, C(s)\mathrm{d}s + (u^2 + iu)\int_0^\tau \zeta(s,\varphi(s))\mathrm{d}s\\
&\quad + iu\rho_{S,q}\sigma_q\int_0^\tau B_q(s)\varphi(s)\mathrm{d}s.
\end{aligned}$$

with $d = \sqrt{(\rho_{S,\nu}\sigma_\nu iu - \kappa_\nu)^2 - \sigma_\nu^2 iu(iu-1)}$, $\quad g = \frac{\kappa_\nu - \sigma_\nu\rho_{S,\nu}iu - d}{\kappa_\nu - \sigma_\nu\rho_{S,\nu}iu + d}$. Integration of function $A(\tau)$ will be calculated numerically by the Simpson method.

Substitute the solution of $A(u,\tau), B(u,\tau), C(u,\tau)$ into (21), we have a closed form of ChF of the forward underlying asset, then by Fourier-Cosine method, we can calculate the forward price of the option (16) and then the option price (13) efficiently.

4 Pricing Option by Fourier-Cosine Expansion

With the closed form of ChF provided for both model, the European option price can be derived by Fourier-Cosine expansion. We present a sketch of the method here, for the details about the calculation of option prices by Fourier-Cosine method, please refer to [4].

The key problem in the pricing of the European options in both model (3) and (12) can be described as the calculation of the expectation of function of the underlying:

$$v(x,t) := \mathbb{E}\left[v(y,T)|\mathcal{F}(t)\right] = \int_{\mathbb{R}} v(y,T)f(y|x)\mathrm{d}y, \tag{24}$$

For the European option price of model (3), which is defined as

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\int_t^T r(s)\mathrm{d}s}\max(S(T)-K,0)|\mathcal{F}(t)\right], \tag{25}$$

the function$v(y,T)$ is

$$v(y,T) := K\max(\mathrm{e}^y-1,0), \quad y := x(T) = \log(S(T)/K),$$

the density function $f(y|x)$ is the discounted density function of $x(t)$ under the risk-neutral measure $\mathbb{Q}$ with the initial state value $x=\log(S(t)/K)$.

To price the European option in model (12), the Fourier-Cosine method is applied to calculate the expectation part of the forward price $\Pi(t)$ as given in (16),

$$\mathbb{E}^T[\max(FX^T(T)-K,0)|\mathcal{F}(t)] = \mathbb{E}^T[v(y,T)|x] = \int_{\mathbb{R}} v(y,T)f(y|x)dy, \tag{26}$$

where

$$v(y,T) := K\max(\mathrm{e}^y-1,0),$$

and $y := \log(FX^T(T)/K)$ are states variables of the discounted forward price of the underlying asset at time T, $f(y|x)$ is the conditional probability density of $\log(FX^T(T)/K)$ given $\log(FX^T(t)/K)=x$ under the forward measure $\mathbb{Q}^T$.

To calculate (24) and (26), we truncate the infinite integration range in (24) and (26), without loosing significant accuracy to $[a,b]\in\mathbb{R}$, and we obtain its approximation v_1 :

$$v_1(x,t) = \int_a^b v(y,T)f(y|x)dy, \tag{27}$$

where

$$[a,b] := \left[c_1 - L\sqrt{c_2+\sqrt{c_4}}, c_1 + L\sqrt{c_2+\sqrt{c_4}}\right], \tag{28}$$

with $L=10$ and c_n denotes the n-th cumulant of $\log(S(t)/K)$ for (24) and that of $\log(FX^T(T)/K)$ for (26), respectively.

Secondly, we replace the density by its cosine expansion in y,

$$f(y|x) = \sum_{k=0}^{+\infty}{}' A_k(x)\cos\left(k\pi\frac{y-a}{b-a}\right), \tag{29}$$

where the summation Σ here with the first term weighted by one-half and

$$\begin{aligned} A_k(x) &:= \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\ &= \frac{2}{b-a}\mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a};x\right)\exp\left(-i\frac{ka\pi}{b-a}\right)\right\}, \end{aligned} \tag{30}$$

where $\phi\left(\frac{k\pi}{b-a};x\right)$ is the discounted ChF of $\log(S(T)/K)$ for the model (3) and the conditional ChF of $\log(FX^T(T)/K)$ for model (12) given $\log(FX^T(t)/K) =$

x, the second equation in (30) is obtained by comparing the cosine coefficient A_k of $f(y|x)$ with the definition of conditional ChF $\phi\left(\frac{k\pi}{b-a};x\right)$.

Substitute (29) into (27), we have

$$v_1(x,t_0) = \frac{1}{2}(b-a)\int_a^b \frac{2}{b-a} v(y,T) \sum_{k=0}^{+\infty} A_k(x)\cos(k\pi\frac{y-a}{b-a})dy. \tag{31}$$

Then interchange the summation and integration, we have

$$v_1(x,t_0) = \frac{1}{2}(b-a)\sum_{k=0}^{+\infty} A_k(x)V_k \approx \frac{1}{2}(b-a)\sum_{k=0}^{N-1}{}' A_k(x)V_k, \tag{32}$$

with

$$V_k := \frac{2}{b-a}\int_a^b v(y,T)\cos(k\pi\frac{y-a}{b-a})dy. \tag{33}$$

Then replacing (30) of A_k in (32), we have

$$v(x,T) \approx \sum_{k=0}^{N-1} \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a};x\right)e^{-ik\pi\frac{a}{b-a}}V_k\right\}, \tag{34}$$

which is the cosine expansion formula for the price of the European option under model (3) and model (12).

At last, we just need to determine V_k in the above COS formula which can be calculate analytically. For a put option with payoff function as $v(y,T) := K\max(1-\mathrm{e}^y, 0)$ and by (33), the definition of V_k, we have

$$V_k = \frac{2}{b-a}K[\psi_k(a,0) - \chi_k(a,0)], \tag{35}$$

where

$$\begin{aligned}
\chi_k(c,d) \quad &:= \quad \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - cos\left(k\pi\frac{c-a}{b-a}\right)e^c\right. \\
&\qquad \left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right] \\
\psi_k(c,d) \quad &:= \quad \begin{cases}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi}, & k \neq 0, \\ (d-c), & k = 0.\end{cases}.
\end{aligned}$$

We refers to [4] for more details. The price of the call option can be calculated by the put-call parity and the price of the put option.

5 Program Manual

The program HAS TO work with the pnl library.

Program files:
“euro_hhw.c" for hybrid model with stochastic volatility, stochastic interest rate and constant dividend.
“hhw4d.c" for four-dimensional hybrid model with stochastic volatility, stochastic interest rate and stochastic dividend.

Model Parameters for “euro_hhw.c"
kappav: κ_ν in model (3)
thetav: θ_ν in model (3)
sigmav: σ_ν in model (3)
v0: the initial value of volatility $\nu(t)$
kappar: κ_r in model (3)
thetar: θ_r in model (3)
sigmar: σ_r in model (3)
r0: the initial value of interest rate r_t
divident: q in model(3)
rho12: $\rho_{S\nu}$ in model (3)
rho13: ρ_{Sr} in model (3)
rho23: $\rho_{\nu r}$ in model (3),
Note that the method assume $\rho_{\nu r} = 0$.

Model Parameters for “hhw4d.c"
kappav: κ_ν in model (12)
thetav: θ_ν in model (12)
sigmav: σ_ν in model (12)
v0: the initial value of volatility $\nu(t)$
kappar: κ_r in model (12)
thetar: θ_r in model (12)
sigmar: σ_r in model (12)
r0: the initial value of interest rate r_t
kappaq: κ_q in model (12)
thetaq: θ_q in model (12)
sigmaq: σ_q in model (12)
q0: the initial value of interest rate r_t
rhoSv: $\rho_{S\nu}$ in model (12)
rhoSr: ρ_{Sr} in model (12)
rhoSq: ρ_{Sq} in model (12),
rhovr: $\rho_{\nu r}$ in model (12)
rhovq: $\rho_{\nu q}$ in model (12)
rhorq: ρ_{rq} in model (12),

Parameters of the product:
S0: stock price at the initial time
K: strike of the American option
T: maturity of the American option, the expansion asymptotic works well for small maturity.

Flags to choose products:
callput_flag:callput flag: 0 for call, 1 for put

Parameters for COSINE method:
N: discrete steps in the integration range N in (34)
L: parameter in the truncate bound of $[a, b]$ as given in (28).

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[4] Fang, F., Oosterlee, C. W., 2008, A novel pricing method for European options based on Fourier-Cosine series expansions, *SIAM J. Fin. Math*, 31:826-848.

1, 2
1, 8