Stochastic Volatility and Local Volatility

Ludovic Goudenège

Premia 22

1 Problem formulation

We consider the following model for the stock price:

$$\left\{ \begin{array}{lcl} dS_t & = & (r-q)S_t dt + \sqrt{V_t}\sigma_{SLV}(S_t,t)dW_t^S, \\ dV_t & = & \alpha(\beta - V_t)dt + \omega\sqrt{V_t}dW_t^V, \end{array} \right. \tag{1}$$

with α, β and $\omega \in \mathbb{R}_+$, where W_t^S and W_t^V are Gaussian processes, where r is the interest rate, q is a foreign interest or dividend, and with the correlation between the two implied Gaussian processes given by

$$\langle dW_t^S, dW_t^V \rangle = \rho dt$$

The function $\sigma_{SLV}(x,t)$ is often called the leverage function.

This SLV model can be viewed as a mixture of the Local Volatility model

$$dS_t = (r-q)S_t dt + \sigma_{LV}(S_t,t)dW_t^S$$

with Dupire's Local Volatility function σ_{LV}, and the Stochastic Volatility model

$$\left\{ \begin{array}{lcl} dS_t & = & (r-q)S_t dt + \sqrt{V_t}dW_t^S, \\ dV_t & = & \alpha(\beta - V_t)dt + \omega\sqrt{V_t}dW_t^V. \end{array} \right.$$

By default, the initial values are $S = S_0 = 100$ and $V = V_0 = 0.01$; and the parameters are $r = 0.09531$, $q = 0$ $\alpha = 2.0$, $\beta = 0.01$, $\omega = 0.2$ and $\rho = 0.5$.

Accordingly, in financial practice the leverage function is calibrated by making use of a relationship between the SLV model and the LV model. It is well-known, see e.g. [1], that these models yield the same marginal distribution for the exchange rate S_t , and hence always define the same fair value for vanilla options, if the leverage function $\sigma_{SLV}(s,t)$ satisfies

$$\sigma_{LV}^2(s,t) = \mathbb{E}[\sigma_{SLV}^2(S_t,t)V_t | S_t = s] = \sigma_{SLV}^2(s,t)\mathbb{E}[V_t | S_t = s], \tag{2}$$

for all $s \in \mathbb{R}_+$ and $t \geq 0$.

The latter term, corresponding to conditional expectation, reads

$$\mathbb{E}[V_t|S_t = s] = \frac{\int_{-\infty}^{+\infty} vp(s,v,t;S_0,V_0)dv}{\int_{-\infty}^{+\infty} p(s,v,t;S_0,V_0)dv} \tag{3}$$

where $p(s,v,t;S_0,V_0)$ denotes the joint density of (S_t,V_t) given by the SLV model

Using a martingale approach for an european or an american option (call or put), we can prove that the non-discounted value $U(s,v,t)$ is given by the solution of the following partial differential equation

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{1}{2}s^2 v\sigma_{SLV}^2(s,T-t)\frac{\partial^2 U}{\partial s^2} + \frac{1}{2}\omega^2 v\frac{\partial^2 U}{\partial v^2} + \rho\omega s v\sigma_{SLV}(s,T-t)\frac{\partial^2 U}{\partial s\partial v}\\ &\quad +(r-q)s\frac{\partial U}{\partial s} + \alpha(\beta - v)\frac{\partial U}{\partial v},\end{aligned}$$

with the following boundary conditions for the call option

$$\begin{aligned} U(s,v,t) &= 0 && \text{whenever } s = 0,\\ \frac{\partial U}{\partial s}(s,v,t) &= \exp(-q\ t) && \text{whenever } s = S_{\max},\\ \frac{\partial U}{\partial v}(s,v,t) &= 0 && \text{whenever } v = V_{\max},\end{aligned}$$

and the following boundary conditions for the put option

$$\begin{aligned} U(s,v,t) &= K\exp(-r\ t) && \text{whenever } s = 0,\\ \frac{\partial U}{\partial s}(s,v,t) &= 0 && \text{whenever } s = S_{\max},\\ \frac{\partial U}{\partial v}(s,v,t) &= 0 && \text{whenever } v = V_{\max},\end{aligned}$$

and the initial condition given by $U(s,v,0) = (b(s-K))_+$ with $b = 1$ for the call option and $b = -1$ for the put option, the maturity T is one year and the strike value K is 100.

On the other side, the density $p(s,v,t)$ satisfies the Kolmogorov forward equation

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial^2}{\partial s^2}\left[\frac{1}{2}s^2 v\sigma_{SLV}^2(s,T-t)p\right] + \frac{\partial^2}{\partial v^2}\left[\frac{1}{2}\omega^2 vp\right]\\ &\quad +\frac{\partial^2}{\partial s\partial v}\left[\rho\omega s v\sigma_{SLV}(s,T-t)p\right] - \frac{\partial}{\partial s}\left[(r-q)sp\right] - \frac{\partial}{\partial v}\left[\alpha(\beta-v)p\right],\end{aligned}$$

with the Neumann homogeneous boundary condition (whatever call or put option is considered)

$$\begin{aligned}\frac{\partial p}{\partial s}(s,v,t) &= 0 && \text{whenever } s = 0,\\ \frac{\partial p}{\partial s}(s,v,t) &= 0 && \text{whenever } s = S_{\max},\\ \frac{\partial p}{\partial v}(s,v,t) &= 0 && \text{whenever } v = V_{\max},\end{aligned}$$

and the initial condition is $p(s, v, 0) = \delta_{s=S_0}\delta_{v=V_0}$ the Dirac distribution at point (S_0, V_0).

2 ADI finite difference scheme

We refer to [3] where a similar method is described to solve the partial differential equations. We have used the same grids whose sizes are given respectively for time, S-space, V-space by N_t, N_s and N_v. The default values are 500, 100 and 40. This choice ensures very good estimations for the prices of call or put options in a large variety of parameters in less than 1 second.

The Douglas scheme described in [3] has been implemented, but the methods for all the others schemes are potentially already in the code, since all the necessary functions are already implemented.

First we will solve the forward Kolmogorov equation by ADI procedure. At each time step, we compute the quantity

Actually, the backward Kolmogorov equation implemented in the code is the PDE for the discounted price, since we need a non-trivial rate to ensure the good behavior of the ADI scheme. Indeed, the rate is half distributed between the matrix in the S-direction and the matrix in the V-direction to ensure well behavior of these matrix involved in linear systems to solve.

3 Implementation

The main program fixes the variables and compute the grid in space and variance variables. It defines a Dupire's Loval Volatility as

$$\sigma_{LV}(s, t) = 0.01 + 0.1 \exp(-s/S_0) + 0.01t$$

It calls the function `compute_vol_then_price` which first computes the density p for all time t in the time grid, for all asset value S in the asset grid, and for all variance value V in the variance grid. It uses the ADI procedure described in [2] returning a three-dimensional array $p[Nt \times Ns \times Nv]$.

This density permits to compute the conditional expectation given by the formula (3), and deduce the function σ_{SLV} from (2) returning a two-dimensional array `TimeVolSto`$[Nt \times Ns]$.

It uses this array to build matrix of a new ADI procedure for the backward Kolmogorov equation for the same time, space and variance grids.

Finally it computes the price given by the dynamic (1).

References

[1] Itsvan, Gyöngy Mimicking the one-dimensional marginal distributions of processes having an Ito differential. Probab. Th. Rel. Fields, 71, 501-516 (1986). 1

[2] Tinne Haentjens and Karel J. in't Hout. ADI finite difference schemes for the Heston-Hull-White PDE. J. Comp. Finan. 16, 83-110 (2012). 3

[3] Maarten Wyns and Karel J. in't Hout. An adjoint method for the exact calibration of Stochastic Local Volatility models. (2016). 3