Hull and White and CIR ++ Models

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March 4, 2020

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1 Hull and White

Hull and White method aim here at pricing zero-coupon bond, european and american options on bond, cap and floor, coupon bearing, payer and receiver swaptions and also δ for hedging, with tree or EDP technics.
Hull and white models are defined by an EDS which describes the evolution of the spot rate $r(t)$:

$$\begin{cases} dx(t) = -a\,x(t)\,dt + \sigma\,dW(t), \quad x(0) = 0 \\ r(t) = x(t) + \phi(t). \end{cases}$$

Where the function ϕ is a deterministic function totally given by the market values of the zero coupon bonds.
Let us denote by $B_M(0,T)$ the market zero coupon bond value maturing at

time T and $f_M(t) = -\frac{\partial log(B(0,t))}{\partial t}$ the market present instantaneous forward rate, then with

$$\phi(t) = f_M(t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

the model exactly fits the market bonds curve and we have several analytical formulas:
Zero coupon bond at time t :

$$B(t,T) = A_1(t,T)e^{-A_2(t,T)\,r(t)}.$$

Explicite formulations for A_1 and A_2 can be found in [?]. Option at time t :

$$E_t\left[e^{-\int_t^T r(s)ds}(B(T,S) - K)_+\right] = B(t,S)\Phi(h + \delta h) - KB(t,T)\Phi(h).$$

Where Φ is the cumulative function of the normal law, $h = \frac{1}{\delta h} log\left(\frac{B(t,S)}{B(t,T)K}\right) - \frac{\delta h}{2}$ and $\delta h = \sigma\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}A_2(T,S)$. This closed formula for european option on bond also leads to closed formula for cap and floor and for coupon bearing and sawption.

2 CIR ++

CIR++ methods aim here at pricing zero-coupon bond, european and american options on bond, cap and floor, coupon bearing, payer and receiver swaptions and also δ for hedging, with tree or EDP technics.
CIR++ models are defined by an EDS which describes the evolution of the spot rate $r(t)$:

$$\begin{cases} dx(t) = a(b - x(t))\,dt + \sigma\,\sqrt{x(t)}\,dW(t), \quad x(0) = x_0 \\ r(t) = x(t) + \phi(t). \end{cases}$$

Where the function ϕ is a deterministic function totally given by the market values of the zero coupon bonds.
Let us denote by $B_M(0,T)$ the market zero coupon bond value maturing at time T and $f_M(t) = -\frac{\partial log(B(0,t))}{\partial t}$ the market present instantaneous forward rate, with $k = \sqrt{a^2 + 2\sigma^2}$ and

$$\phi(t) = f_M(t) - \frac{2ab\left(e^{kt} - 1\right)}{2k + (a+k)\left(e^{kt} - 1\right)} - x_0\frac{4k^2e^{kt}}{2k + (a+k)\left(e^{kt} - 1\right)}$$

the model exactly fits the market bonds curve and we have several analytical formulas:
Zero coupon bond at time t :

$$B(t,T) = A_1(t,T)e^{-A_2(t,T)\,r(t)}$$

Explicite formulations for A_1 and A_2 can be found in [?]. Option at time t :

$$E_t\left[e^{-\int_t^T r(s)ds}(B(T,S)-K)_+\right] = B(t,S)\chi(h+\delta h) - KB(t,T)\chi(h).$$

Where $\chi =$ is the cumulative function of the chi2 law with $\frac{4ab}{\sigma^2}$ degree of freedom and certain non central parameter (see [1] for the details of these analytical formulas). This closed formula for european option on bond also leads to closed formula for cap and floor and for coupon bearing and sawption.

3 Trinomial Tree method

It is possible to simulate de spot rate diffusion r through a trinomial tree for a general positive shift model of the form :

$$\begin{cases} dx(t) = \mu_x(t)dt + \sigma(t)\,dW(t), \quad x(0) = x_0 \\ r(t) = x(t) + \phi(t). \end{cases}$$

It is important that the volatility σ is independant of x so that the trinomial tree converges. The Hull and White model satifies this form, but not the CIR++ model since

$$dx = a(b - x(t))\,dt + \sigma\,\sqrt{x(t)}\,dW(t).$$

Hoverver setting $y = \sqrt{x}$ then the equation on y is

$$dy = \left[\frac{\gamma}{y} - \frac{ay}{2}\right]dt + \frac{\sigma}{2}\,dW(t)$$

wiht $\gamma = \left(\frac{ab}{2} - \frac{1}{8\sigma^2}\right)$. Then y can be computed in a trinomial tree. For a very usual log normal diffusion of a random variable x, the variable y simulated in the tree will be $y = log(x)$.

To summarise let us consider generally the diffusion y:

$$dy(t) = \mu_y(t)dt + \sigma(t), dW(t)$$

and the relation $r(t) = F(y(t)) + \phi(t)$ where $F : D_1 \subset \mathbf{R}_+ \longrightarrow \subset \mathbf{R}_+$ is a bijective function. The first node is y_0 ($y_0 > 0$ in general) then each node can evolves in three nodes with a given transition probability computed as follow:

Let $0 = t_0 < t_1 < .. < t_n = T$ be a time scale for our tree in $[0, T]$ and $y_{i,j}$ the y node value at time t_i for the j^{th} space step of the tree (starting from the down). We need then :

$$\begin{cases} E_{i,j} = E\left(y(t_i)_{|y(t_{i-1})=y_{i-1,j}}\right) \\ V_{i,j} = V_i = \sqrt{Var\left(y(t_i)_{|y(t_{i-1})=y_{i-1,j}}\right)} \\ dy_i = \sqrt{3}V_i. \quad \text{space step at time } t_i. \end{cases}$$

Starting from node $(t_0 = 0, y_{0,0} = y_0)$, at time t_1 we set $y_{1,0} = E_{1,0}$ then $dy_1 = \sqrt{3}V_1$ and $j_1^{min} = -1$ and $j_1^{max} = +1$ and then $y_{1,1} = y_{1,0} + dy_1$ and $y_{1,-1} = y_{1,0} - dy_1$. Then by a forward induction we compute all the nodes till time T.

Knowing the nodes at time t_{i-1}, we compute first $y_{i,0} = E_{i,0}$ then the V_i and all the $E_{i,j\,(j=j_{i-1}^{min},..,j_{i-1}^{max})}$ and :

$$\begin{cases} dy_i = \sqrt{3}V_i \\ j_i^{min} \quad \text{such that} \quad y_{i,j_i^{min}} < E_{i,j_{i-1}^{min}} - \frac{dy_i}{2} < y_{i,j_i^{min}+1} \\ j_i^{max} \quad \text{such that} \quad y_{i,j_i^{max}-1} < E_{i,j_{i-1}^{max}} + \frac{dy_i}{2} < y_{i,j_i^{max}} \\ y_{i,j} = y_{i,0} + j\,dy_i \quad \text{for } j_i^{min} \leq j \leq j_i^{max} \end{cases}$$

and then compute the transition probilities, pu, pm and pd (for all $j_{i-1}^{min} \leq j \leq j_{i-1}^{max}$), from node $y_{i-1,j}$ to $y_{i,k+1}$, $y_{i,k}$ and $y_{i,k-1}$:

$$\begin{cases} pu_{i-1,j} = \frac{1}{6} + +\frac{\eta^2}{2dy_i^2} + \frac{\eta}{2dy_i} \quad \text{probability to go from } (i-1,j) \text{ to } (i,k+1) \\ pm_{i-1,j} = \frac{2}{3} - \frac{\eta^2}{dy_i^2} \quad \text{probability to go from } (i-1,j) \text{ to } (i,k) \\ pd_{i-1,j} = \frac{1}{6} + \frac{\eta^2}{2dy_i^2} - \frac{\eta}{2dy_i} \quad \text{probability to go from } (i-1,j) \text{ to } (i,k-1) \end{cases}$$

with $\eta = E_{i,j} - y_{i,k}$ and k the integer such that $y_{i,k}$ is the closer to $E_{i,j}$:

$$k = round\left[\frac{E_{i,j} - y_{i,0}}{dy_i}\right].$$

Then we change all the y nodes of the tree in x nodes thanks to $x = F(y)$ then we can compute directly on the tree the translation $\phi(t_i)$ to get $r_{i,j} = x_{i,j} + \phi(t_i)$ for the nodes thanks to a forward iteration on $\phi(t_i)$ and the Arrow-Debreu node prices knowing all the $B_M(0, t_j)$ (see [?] §3.3.3).

Important remarks :
It is important for computation without *surprise* that the function $j \longrightarrow E_{i,j}$ is increasing so that there is no crossing pm probabilities and the number of nodes is always increasing. Morever it more easy to define j_i^{min} since the previous lowest expectation is $E_{i,j_{i-1}^{min}}$ and j_i^{max} since the previous highest expectation is $E_{i,j_{i-1}^{max}}$. For instance in CIR ++ there is a low bound for y to have this condition and we must forbid the tree to go under ; this is all the more necessary in so far as y must stay positive and the equation on y becomes totally unstable near 0 due to the term in $\frac{1}{y}$.
There also can be tricky problems because of the condition domain of the bijective function F, for CIR++ these domains are $\mathbf{R}_+ \longrightarrow \mathbf{R}_+$ and x (and y) stay positive if $2ab > \sigma^2$. We advise to chose a quite large x_0 (to have a quite large y_0) so the tree diffusion of y might not be too truncated by its low bound even if it must induce negative $\phi(t)$.
They is no particular problem dealing with Hull and White.
Now that we have a trinomial tree of the spot rate $r_{i,j}$ with their transition probabilities we can compute any payoff $h(T, r(T))$ (european, american or bermudean) thanks to a backward induction thanks to the approximation:

$$\begin{aligned} h_{i,j} = h(t_i, r_{i,j}) &= E\left[e^{-\int_{t_i}^{t_{i+1}} r(s)ds} h(t_{i+1}, r(t_{i+1}))_{|r(t_i)=r_{i,j}}\right] \\ h_{i,j} = h(t_i, r_{i,j}) &\simeq e^{-r_{i,j}(t_{i+1}-t_i)} \left[pu_{i,j} h_{i+1,k+1} + pm_{i,j} h_{i+1,k} + pd_{i,j} h_{i+1,k-1}\right] \end{aligned}$$

4 Implicite PDE method

Let us consider a general shifted model for the spot rate

$$\begin{cases} dx(t) = \mu_x(t)\,dt + \sigma_x(t)\,dW(t), \quad x(0) = x_0 \\ r(t) = x(t) + \phi(t). \end{cases}$$

Then the option price

$$V(t, r) = E\left[e^{-\int_t^T r(s)ds} h(T, r(T))_{|r(t)=r}\right]$$

can be written with respect to x, $V(t, r) = e^{-\int_t^T \phi(s)ds} U(t, r - \phi(t))$, where

$$U(t, x) = E\left[e^{-\int_t^T x(s)ds} h(T, x(T) - \phi(T))_{|x(t)=x}\right]$$

and U is the solution of the following PDE:

$$\frac{\partial U}{\partial t} + \mu_x(t)\frac{\partial U}{\partial x} + \frac{1}{2}\sigma_x^2(t)\frac{\partial^2 U}{\partial x^2} - xU(t, x) = 0$$

This transport equation is computed over a domain $[0, X_{MAX}]$. In $x = 0$, supposing $\sigma_0(t) = 0$, we have:

$$\frac{\partial U}{\partial t} + \mu_0(t)\frac{\partial U}{\partial x} = 0.$$

This equation will give us our boundary condition in $x = 0$.
Let $0 = t_0 < t_1 < .. < t_{n_T} = T$ be a time scale for our PDE on $[0, T]$ and $x_j = j\,dx$ be a space scale for $j = 0$ to n_X ($dx = round\left[\frac{X_{MAX}}{n_X}\right]$). Let us denote U^n the numerical space vector for the approximation of $U(t_n, x_j)$ for $j = 0$ to J_{MAX}.
Then dicretizing the PDE and knowing U^n, U^{n+1} is solution of the linear problem :

$$\left(\frac{1}{dt}Id - \theta M_n\right)U^{n+1} = \left(\frac{1}{dt}Id + (1-\theta)M_n\right)U^n,$$

with θ chosen in $(0, 1)$ and where M_n is the tridiagonal $nx.nx$ matrix of discretized linear differential operator of the PDE : $\forall k = 2, .., nx - 1$

$$\begin{cases} M_n[k][k-1] = \frac{\theta}{2}(\sigma^2_{x_k}(t_n)\frac{1}{dx^2} - \mu_{x_k}(t_n)\frac{1}{dx}) \\ M_n[k][k] = -\theta(\sigma^2_{x_k}(t_n)\frac{1}{dx^2} + x) \\ M_n[k][k+1] = \frac{\theta}{2}(\sigma^2_{x_k}(t_n)\frac{1}{dx^2} + \mu_{x_k}(t_n)\frac{1}{dx}). \end{cases}$$

A Neuman limit condition is taken on the right boundary to have the last line of the matrix and the previous $x = 0$ transport equation is used for the left boundary condition to have the first line of the matrix.

Resolving this equation backwardly we can compute any payoffs.
remark: For tree and PDE methods to compute an option on a zero coupon bond $B(T, S)$ maturing at time T for instance, a tree or a PDE is contruct over $[0, S]$, a first backward resolution with a payoff 1 starting at time S allows to built B(T,S) and then a second backward resolution starting at time T allows to compute the option over the payoff $B(T, S)$.

References

[1] D. Brigo and F. Mercurio. *Interest Rate Models*. Springer, 2001. 3