The generalized Cox-Ross-Rubinstein model

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February 4th 1999

Premia 22

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1 Market assumptions

We consider an idealized market in discrete time $n \in [0,..,N]$, $N \in \mathbb{N}^*$, which consists of a risky stock S and a bank account which yields an interest rate r over a period of time. We make the following assumptions:

1. There is no transaction costs on S. The stock S has a unique selling/buying price at time n which we denote by S_n. We identify in the sequel S and the process of its market price $(S_n)_{0\leq n\leq N}$.

2. There is no limit size of selling/buying of S, and no impact on the market price of a transaction.

3. Short-selling is allowed, ie it is possible to sell shares without owning them.

4. It is possible to bargain any algebric quantity of S, not only integer multiple of the stock price.

5. There is no cash constraint: a negative position on the bank account is allowed.

6. The lending rate is the same as the borrowing rate r.

These assumptions are usually summarized by "perfect market". We also assume that the rate r is constant and that S does not yield dividends.

2 Options

2.1 European options

A European option on S will be for us a pair $(p,\ H_p)$ where p denotes the maturity and H_p the payoff of the option. H_p is a positive measurable function from $\left(\mathbb{R}_+^*\right)^{p+1}$ to $\mathbb{R}$ which maps the trajectory until time p of the underlying asset S to the payoff of the option settled at time p : in other words the owner of the option has the right to ask at time p to the seller (at some time before p) of the option the amount of money H_p which may depend on what has happened in the underlying asset life before time p.

Standard options depend on the past of the underlying only through the value at maturity. For a Call option with strike K, $H_p\left(\omega\right) = \left(\omega_p - K\right)^+$ where K is the exercise price. For a Put option, $H_p\left(\omega\right) = \left(K - \omega_p\right)^+$.

Exotic or *path-dependant* depend on another quantity than ω_p. For instance LookBack options depend on the minimum or the maximum of the underlying during the life of the option, Asian options on the average of the underlying.

2.2 American options

An American option is a pair $\left(p, (H_n)_{0\leq n\leq p}\right)$ where $H_n(\omega)$ is the payoff of the option at time n : the owner of the option may ask once in the life of the option, say at time n for a payoff which depends on the life of the underlying until time n.

2.3 Option pricing

2.3.1 Pricing and dynamic hedging

The gneric problem we shall discuss is the pricing of the option, that is the astonishing fact that it is possible to find out a fair price to buy or sell the option before maturity, say at time 0. We shall take the point of view of the seller of the option.

Consider for example the case of a european Call option with maturity N. The picture is the following: we sell the option at time 0 at a price say C_0, that is we receive at time 0 this amount of money, but in turn it is mandatory for us to pay at time N the payoff $(S_N - K)^+$, which may be very high depending on the movements of the underlying.

At first glance it seems puzzling that any wise enough insitution may go in such a bargain. The idea, of course, is to go to the undelying market to buy some shares in order to "hedge" the possibility of a high increase of the underlying. This is best understood by looking at a general discrete-time model with two period with say $r = 0$ and an underlying which may move in a range:

$$S_1 \in [mS_0,\ MS_0]$$

where $m < 1 < M$.

So the very possibility to trade options, at least in a safe or quite safe manner, is closely related to the access to the underlying market, or more generally to hedging instruments. In a multi-period model the hedge may be performed dynamically to balance in a required way the option payoff.

2.3.2 Economic assumptions

A very appealing feature of our approach is that no economic assumptions is required, the spot dynamic is considered as exogeneous. The only axiom we require is :

There is NO ARBITRAGE OPPORTUNITY

in the following way: starting from nothing it's not possible to get at time N an amount of money wich is positive in any case and strictly positive on a set of strictly positive probability.

3 Stochastic model

3.1 Stochastic dynamic

Let $0 < d < u < \infty$ and $0 < p < 1$ be fixed parameters.

The stochastic dynamic of S is the following: S_0 is a (positive) constant, at each time n S evolves according to

$$S_{n+1} = \begin{array}{c} uS_n \text{ with probability } p \\ dS_n \text{ with probability } 1-p \end{array} \tag{1}$$

More precisely, under the historical probability P, the conditional law at time n of S_{n+1} given the past is given by (1).

It is easy to see that the AOA axiom is granted here by the hypothesis:

$$d < 1 + r < u \tag{2}$$

3.2 Information

We assume that there is no external source of information: the information available at time n may be identified with the knowledge of the past market prices of S. Let us denote it by $\mathcal{S}_n$. Then

$$\mathcal{S}_n = (S_0, ..., S_n)$$

is a vector of $\left(\mathbb{R}_+^*\right)^{n+1}$ which belongs to the set of possible (under P) trajectories of S until n. We denote by Ω_n this set. We denote by Π_n^p the natural projection operator from $\left(\mathbb{R}_+^*\right)^p$ to $\left(\mathbb{R}_+^*\right)^n$ for $p \geq n$, π_n^p the n-th coordinate of a vector of $\left(\mathbb{R}_+^*\right)^p$ (starting from zero).

We thus work on the canonical filtered probability space

$$\left(\left(\mathbb{R}_+^*\right)^{N+1}, \mathcal{B}\left(\left(\mathbb{R}_+^*\right)^{N+1}\right), (\mathcal{F}_n)_{0 \leq n \leq N}, P\right)$$

with $\mathcal{F}_n = \sigma\left(\Pi_{n+1}^{N+1}\right)$, $\mathcal{S}_n(\omega) = \Pi_{n+1}^{N+1}(\omega)$, $S_n(\omega) = \pi_n^{N+1}(\omega)$ and $P(\mathcal{S}_n \in \Omega_n) = 1$ for any $n \leq N$.

Remark 1. *The standard Cox-Ross-Rubinstein scheme corresponds to the choice* $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$, $r = e^{\rho h} - 1$ *with* $h = \frac{T}{N}$ *which is tailored to approximate the option prices in the Black-Scholes model at volatility* σ *and instantaneous interest rate* ρ.

4 European option pricing

From now on, we shall denote by $\widetilde{A}_n$ the discounted value of A at time 0, that is we set

$$\widetilde{A}_n \overset{def}{=} (1+r)^{-n} A_n$$

4.1 The backward iterative method

Let us consider the pricing at time 0 of a general european option (N, H_N).

We shall perform a backward recursion scheme.

The hypothesis will be the following:

Assume that at time $n+1$, the option has a unique selling/buying price C_{n+1} which depends on $\mathcal{S}_{n+1}$.

This is true at time $n+1 = N$ since $C_N = H_N(\mathcal{S}_N)$without discussion.

We need temporarily an auxiliary hypothesis: there is a market for the option between time 0 and time N.

Let's place ourselves at time $n < N$. We know the information $\mathcal{S}_n$, and we look for the price of the option (if some) in this "state of the nature", C_n.

We expect that hedging come into play in some way. The argument is the following:

Assume that there is a value C_n such that for some stock quantity Δ_n KNOWN at time n we have:

$$(1+r) C_n + \Delta_n (S_{n+1} - (1+r) S_n) = C_{n+1}(\mathcal{S}_{n+1}) \tag{LocRep}$$

This means that we want for (LocRep) to be in force whatever the state of the nature at time $n+1$.

Then by arbitrage C_n is the unique buying/selling price of the option at time n.

But obviously here:

$$\mathcal{S}_{n+1} = (\mathcal{S}_n,\, S_{n+1}) = \begin{array}{c} (\mathcal{S}_n,\, uS_n) \ \text{ or} \\ (\mathcal{S}_n,\, dS_n) \end{array}$$

so that solving (LocRep) yields

$$\Delta_n = \frac{C_{n+1}((\mathcal{S}_n,\, uS_n)) - C_{n+1}((\mathcal{S}_n,\, dS_n))}{uS_n - dS_n} \tag{Delta}$$

which is known et time n, whence

$$C_n(\mathcal{S}_n) = \frac{1}{(1+r)} \left[qC_{n+1}((\mathcal{S}_n,\, uS_n)) + (1-q) C_{n+1}((\mathcal{S}_n,\, dS_n)) \right] \tag{Price}$$

with

$$q=\frac{(1+r)-d}{u-d}$$

Therefore C_n is a function of $\mathcal{S}_n$ (Δ_n also), the recursion hypothesis is in force at time $n-1$, achieving the backward induction we get the price at time 0.

Remark 2. *In case H_N depends only on S_N we get by induction that C_n depends on $\mathcal{S}_n$ only through S_n. In this case we get a 2-dimensional representation of our scheme, which is the generalized "Cox-Ross-Rubinstein tree". This is a re-combining tree with $N+1$ leaves, whereas the general scheme yields a tree with 2^N leaves. These trees are $2-$trees in the sense that a node has strictly two sons.*

Remark 3. *The quantity Δ_n is called the hedge ratio at time n.*

4.2 Independence from p and perfect replication

A remarkable feature of C_0 is that it does depend on u, d, r, also N, H_N yet not on the probability p. We explain this puzzling fact here, where we also show that the assumption of the existence of a market for the option may be relaxed.

Let re-start from (LocRep on the previous page) and re-write (LocRep on the preceding page) as:

$$\Delta_n\left(\mathcal{S}_n\right)\left(\widetilde{S}_{n+1}-\widetilde{S}_n\right)=\widetilde{C}_{n+1}-\widetilde{C}_n$$

in monetary unit of time 0. Summing over n we get:

$$\widetilde{C}_N\left(\mathcal{S}_N\right)=C_0+\sum_{n=0}^{N-1}\Delta_n\left(\mathcal{S}_n\right)\left(\widetilde{S}_{n+1}-\widetilde{S}_n\right)$$

where $\mathcal{S}_n=\Pi_{n+1}^{N+1}\mathcal{S}_N$ or

$$\widetilde{H}_N\left(\mathcal{S}_N\right)=C_0+\sum_{n=0}^{N-1}\Delta_n\left(\Pi_{n+1}^{N+1}\mathcal{S}_N\right)\left(\widetilde{\pi_{n+1}^{N+1}}\mathcal{S}_N-\widetilde{\pi_n^{N+1}}\mathcal{S}_N\right) \tag{Rep}$$

which means that we design the amount of money $\widetilde{H}_N$ at time N out of a cash amount C_0 and a sequence of buying/selling (hedging) of the underlying asset between time 0 and N, with no option deals any longer between time 0 and N.

We have "perfectly replicated (or duplicated)" the option.

Note that (Rep) is in fact a system of $|\Omega_N|=2^N$ equalities.

Before stating a fundamental unicity result about (Rep), let us set:

Definition 4. *A process A on the probability space $\left(\left(\mathbb{R}_+^*\right)^{N+1}, \mathcal{B}\left(\left(\mathbb{R}_+^*\right)^{N+1}\right), (\mathcal{F}_n)_{0\leq n\leq N}, P\right)$ is said to be adapted if and only if whatever $n \leq N$, A_n is $\mathcal{F}_n$ measurable.*

Then in (Rep on the previous page) Δ is an adapted process (the measurability of each Δ_n is shown by backward induction). Now we can state:

Theorem 5. *Whatever the measurable function $H_N : \left(\mathbb{R}_+^*\right)^{N+1} \to \mathbb{R}$ there is a unique pair (C_0, Δ) where C_0 is a real number and Δ an adapted process up to time $N-1$ such that (Rep on the preceding page) holds.*

The existence and unicity proof is easy by backward induction: it is the same as above. Note that we drop the assumption of positivity of H_N, it doesn't come into play as far as the replication is concerned.

Let's observe now that (Rep on the previous page) is a purely pathwise expression, ie an equality between quantity on every path of the underlying asset S under P, whatever the probability weight of this path. In other words, (Rep on the preceding page) may be re-stated:

$\forall\, \mathcal{S}_N \in \Omega_N,\ \widetilde{H}_N(\mathcal{S}_N) = C_0 + \sum_{n=0}^{N-1} \Delta_n \left(\Pi_n^{N+1}\mathcal{S}_N\right)\left(\widetilde{S}_{n+1} - \widetilde{S}_n\right)$

Here obviously: THE PATH SPACE Ω_N DOES NOT DEPEND ON p.

Yet another formulation: P comes into play only through its support which is defined here without ambiguity as the space Ω_N.

We'll come back to this fundamental idea later on.

Remark 6. *The fact that every european option is replicable is taken as a definition of the completeness of the market.*

4.3 Risk-neutral "probability"

Let's look at the functional:

$$H_N \mapsto \Psi(H_N) \overset{def}{=} (1+r)^N C_0$$

It is obviously: positive (by induction), linear (by existence and uniqueness in (Rep on the previous page)) and $\Psi(1) = 1$ (by induction or uniqueness in (Rep on the preceding page)).

In the general case this is not enough (a monotonicity condition is required) to get a representation by a positive measure. In our situation this is obvious: indeed

$$\begin{aligned}\Psi(H_N) &= \Psi\left(\sum_{\mathcal{S}_N^i \in \Omega_N} \delta_{\mathcal{S}_N^i}(\mathcal{S}_N)\, H_N(\mathcal{S}_N)\right) \\ &= \sum_{\mathcal{S}_N^i \in \Omega_N} H_N\left(\mathcal{S}_N^i\right) \Psi\left(\delta_{\mathcal{S}_N^i}(\mathcal{S}_N)\right)\end{aligned}$$

where $\delta_x(y) = 1$ if $x = y$ and 0 otherwise, since Ω_N is a finite set, and the result follows.

Therefore it may be represented by a positive measure of mass 1 on $\mathcal{S}_N$ ie a probability measure in the mathematical sense. Let Q design this probability. Then by construction $Q << P$, whereas $Q >> P$ follows from the fact that for any point $\mathcal{S}_N^i$, $\Psi\left(\delta_{\mathcal{S}_N^i}(\mathcal{S}_N)\right) > 0$ which can be shown by backward induction.

Let $E^Q[\ \mid \mathcal{S}_n]$ the conditional expectation operator given $\mathcal{S}_n$.

Let's for any A_n depending on $\mathcal{S}_n$ look at the option with payoff:

$$H_N = (1+r)^N A_n\left(\widetilde{S}_p - \widetilde{S}_n\right)$$

where $0 \leq n < p \leq N$. Then by uniqueness in (Rep on page 6):

$$0 = C_0 = (1+r)^{-N} E^Q[H_N]$$

ie

$$E^Q\left[A_n\left(\widetilde{S}_p - \widetilde{S}_n\right)\right] = 0$$

By definition of the conditional expectation this means:

$$E^Q\left[\widetilde{S}_p \mid \mathcal{S}_n\right] = \widetilde{S}_n$$

therefore $\left(\widetilde{S}_n\right)_{0\leq n\leq N}$ is a $Q-$martingale with respect to its natural filtration (which is also that of S).

In more general contexts, a measure which makes the discounted underlying price a martingale is called a "risk-neutral" measure.

Conversely, if $\left(\widetilde{S}_n\right)_{0\leq n\leq N}$ is a $R-$martingale for some probability R then from (Rep on page 6):

$$E^Q\left[(1+r)^{-N} H_N\right] = E^R\left[(1+r)^{-N} H_N\right]$$

for any H_N and therefore $R = Q$.

We may now state:

The price of an european option is the expectation of its discounted payoff with respect to the unique probability measure which makes the discounted underlying asset a martingale.

Remark 7. *The existence and uniqueness of a risk neutral measure was in fact obvious in our case. In a general discrete-time finite-horizon setting, it may be proved that the existence of a risk-neutral measure is equivalent to the absence of arbitrage opportunities (suitably defined-one implication is obvious, the other one difficult). The uniqueness of a risk-neutral measure* *is in the same context equivalent to the completeness of the market (same remark).*

In continuous time, some care is needed but the situation is roughly speaking the same.

4.4 A robustness property, dynamic model and pricing functionals

From the independance on the probability p, it follows that the price (and hedge ratios) of an option are the same for the whole family of dynamic models:

$$S_0 > 0,\ S_{n+1} = \begin{array}{l} uS_n \text{ with probability } p_n \\ dS_n \text{ with probability } 1 - p_n \end{array}$$

where p_n might even be stochastic, ie might be path-dependant: $p_n = p_n(\mathcal{S}_n)$, as long as $0 < p_n(\mathcal{S}_n) < 1$ for any path $\mathcal{S}_n$. All these dynamic models have the same path space, and we re-say that a perfect-replication option price depends only on the path space, not on the probabilities of the paths.

This remark is fundamental: for different p_n, the statistical features of S may be very different, whereas the option "pricing functional" will be the same. So the right point of view is to distinguish between: on one hand, the *dynamic model* which in fact comes into play only through the set of its trajectories, and on the other hand the *pricing functionals* which stands for a "map" of prices $C_n(\mathcal{S}_n)$ and hedge ratios $\Delta_n(\mathcal{S}_n)$ defined in a deterministic manner regardless of any probability along the trajectories:

$$\mathcal{S}_n \mapsto C_n(\mathcal{S}_n), \Delta_n(\mathcal{S}_n)$$

In fact, in order to price an option the true probability p_n may even be unknown: this is one step further, we don't have even to specify a dynamic model, but only a set of trajectories.

Thus it's essential to distinguish between the Cox-Ross-Rubinstein *dynamic model*, which is very restrictive, and the Cox-Ross-Rubinstein *pricing functionals* which may be pertaining to a much wider family of dynamic models.

Remark 8. *We'll see later on another type of robustness property which turns the Cox-Ross-Rubinstein* pricing functionals *into a real-life trading tool.*

Remark 9. *The same line of reasoning may be followed almost step by step starting from the Black-Scholes* dynamic model:

$$dS_t = \sigma S_t dB_t + \mu S_t dt$$

The property matching the independance of the option price on p is in this context the independance with respect to μ. The explanation lies in the fact that the path space of S doesn't depend on μ in fact (Girsanov's theorem), so that the Black-Scholes pricing functionals *corresponds to the whole family of models:*

$$dS_t = \sigma S_t dB_t + \mu_t S_t dt$$

even with stochastic μ_t or here again unknwon μ_t.

This is important since the conditional laws $S_{t+h} \mid \mathcal{S}_t$ of such dynamics are not log-normal in general and might exhibit some statistical feature like fat tails or sharp peak which are often advocated against the Black-Scholes pricing functionals *whereas they're directed in fact against the Black-Scholes* dynamic model.

Remark 10. *The exposition of the american option pricing in the generalized CRR model is easy.*

Let us now turn to the dependance of C_0 in u and d.

5 Model risk for convex standard options

In this section, we give an important property of the dependance of C_0 in the stochastic model, that is in u and d, in case of standard options with convex payoffs.

To make things more attractive we show along the way how this property can be used for real-life (from the point of view of stochastic assumptions at least) markets.

Consider an european option with maturity T written on a stock S with payoff $\varphi(S_T)$ under the somewhat idealized above market assumptions, with a continuous-time interest rate $e^{\rho t}$ over a time period t.

Assume that you don't know much about the stock dynamic, except the following:

For some N, u and d :

$$S_{(k+1)h} \in [dS_{kh},\, uS_{kh}]$$

with probability one, where $h = \frac{T}{N}$.

Assuming no arbitrage:

$$d < 1 + r = e^{\rho h} < u$$

We assume: φ CONVEX (e.g.: call or put option).

Suppose you SELL the option and hedge according to the CRR pricing scheme with parameters d, u and r. What about your Profit&Loss?

$$P\&L = C_0 + \sum_{n=0}^{N-1} \Delta_n (S_{nh}) \left(\widetilde{S}_{(n+1)h} - \widetilde{S}_{nh} \right) - \widetilde{\varphi} (S_T)$$

where $\Delta_n = \Delta_n (S_{nh})$, $C_0 = C_0 (S_0)$ where $\Delta_n (x)$ is the hedge ratio and $C_0 (x)$ the price given at step n and 0 by the Cox pricing scheme for a value x of the underlying asset. Here S_{nh} is the real stock price.

We may rewrite:

$$P\&L = \sum_{n=0}^{N-1} \widetilde{C}_n (S_{nh}) + \Delta_n \left(\widetilde{S}_{(n+1)h} - \widetilde{S}_{nh} \right) - \widetilde{C}_{n+1} \left(S_{(n+1)h} \right)$$

And we claim: for any n,

$$\widetilde{C}_n (S_{nh}) + \Delta_n \left(\widetilde{S}_{(n+1)h} - \widetilde{S}_{nh} \right) - \widetilde{C}_{n+1} \left(S_{(n+1)h} \right) \geq 0$$

This is a consequence of:

For any $x > 0$, $y \in [dx, ux]$

$$(1 + r) C_n (x) + \Delta_n (x) (y - (1 + r) x) \geq C_{n+1} (y)$$

which follows from the fact that the LHS, as a function of y, is the chord of the function C_{n+1} between dx and ux, making use of (Delta on page 5). Using (Price on page 5) again it's obvious that the function $y \mapsto C_{n+1} (y)$ is a convex function, whence the result.

Therefore:

$$P\&L \geq 0 \text{ a.s.}$$

Moreover, it's easy to see that the selling price C_0 is the lowest price which grants $P\&L \geq 0$ a.s. under our dynamic assumptions.

Remark 11. *The case of non-convex payoffs is another question.*

Corollary 12. *For convex payoffs, the generalized CRR price increases with the interval* $[d, u]$.

Intuitively, this corresponds to the notion of volatility in continuous-time.