Computing Credit Valuation Adjustment solving coupled PIDEs in the Bates model

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Abstract

Credit value adjustment (CVA) is the charge applied by financial institutions to the counterparty to cover the risk of losses on a counterpart default event. In this paper we estimate such a premium under the Bates stochastic model (Bates [?]), which considers an underlying affected by both stochastic volatility and random jumps. We propose an efficient method which improves the finite-difference Monte Carlo (FDMC) approach introduced by de Graaf et al. [?]. In particular, the method we propose consists in replacing the Monte Carlo step of the FDMC approach with a finite difference step and the whole method relies on the efficient solution of two coupled partial integro-differential equations (PIDE) which is done by employing the Hybrid Tree-Finite Difference method developed by Briani et al. [?, ?, ?]. Moreover, the direct application of the hybrid techniques in the original FDMC approach is also considered for comparison purposes. Several numerical tests prove the effectiveness and the reliability of the proposed approach when both European and American options are considered.

Keywords: Credit Value Adjustment, Hybrid methods, PIDE, Monte Carlo, Bates model.

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1 Introduction

Financial institutions suffer several risks, one of them is the counterparty credit risk (CCR). This risk arises from the possibility that the counterparty of a financial contract may default. This risk was often overlooked, but in the last decades, after the financial crisis of 2007 and the Lehman Brothers failure in 2008, it gained more and more interest by practitioners and academics. In particular, according to the Basel III framework of 2010, financial institutions must charge a premium to their counterparty according to its credit reliability in order to compensate for a possible counterparty default. Also IFRS 13 in 2013 requires the fair value of financial products to be measured based on counterparty credit risk. For these reasons, financial institutions charge to the counterparty a premium called Credit Valuation Adjustment (CVA), which is the difference between the risky value and the current risk-free value of derivatives contract. Estimating the good value of the CVA can be a demanding effort due to its complicated definition and to its dependence by the underlying stochastic model. As no specific method is prescribed in the accounting literature, various approaches are used in practice by derivatives dealers and by end users to estimate the effect of credit risk on the fair value of financial derivatives. The common approach is to price the CVA through the so called expected exposure, which is the mean exposure distribution at a future date. Usually, this exposure is calculated by means of Monte Carlo approaches which are computationally expensive. In particular, nested Monte Carlo simulations or least squares techniques are employed by Joshi and Kwon [?]. These techniques have been improved through the use of stochastic grid bundling method by Jain and Oosterlee [?] and by Karlson et al. [?]. An interesting approach to compute the CVA - when the Heston model is assumed - is the so called finite-difference Monte Carlo (FDMC) method, proposed by de Graaf et al. [?]. Such an approach combines the finite-difference method and the Monte Carlo method to solve a partial differential equation (PDE) and to estimate the mean exposure respectively. Feng [?] adapted the FDMC approach to deal with the case of an underlying evolving according to the Bates model: in this particular case, the PDE to be solved is replaced by a partial integral differential equation (PIDE), which implies an additional computational effort. Other recently introduced alternative approaches consist in employing the fast Fourier transform (Borovykh et al. [?]) or marked branching diffusions (Henry-LabordÃšre [?]).

In this paper we focus on the computation of the CVA when the Bates model is considered and we propose an efficient method which improves the results of the FDMC method. Specifically, the Bates model considers an underlying affected by both stochastic volatility and random jumps: the dynamics of the underlying asset price is driven by both a Heston stochastic volatility [?] and a compound Poisson jump process of the type originally introduced by Merton [?]. Such a model was introduced by Bates [?] in the foreign exchange option market in order to tackle the well-known phenomenon of the volatility smile behavior. In the case of plain vanilla European options, Fourier inversion methods, employed by Carr and Madan [?], lead to closed-form formulas to compute the price under the Bates model. Two innovative and efficient approaches to price derivatives when the Bates model is considered are proposed by Briani et al. [?], the so called Hybrid Tree-Finite Difference method and Hybrid MC method.

Our main result consists in developing an efficient numerical method to estimate the CVA. In particular, the method we propose consists in computing the CVA value as the solution of two coupled PIDE – one for the risk free price and one for the risk adjusted price – which are solved by means of the Hybrid Tree-Finite Difference method. That is, we replace the Monte Carlo step in the FDMC method with the computation of the solution of a PIDE and this improves clearly the computational efficiency.

Moreover, the direct application of the hybrid techniques in the original FDMC approach is considered for comparison purposes. Specifically, we apply the two hybrid methods developed by Briani et al. to perform both the finite difference step and the Monte Carlo step of the FDMC approach.

Several numerical experiments show that the proposed method is efficient and reliable as it provides accurate approximations for the CVA value with a low computational cost.

The reminder of the paper is organized as follows. Section 2 introduces the Bates stochastic model and the partial integro-differential equation that allows one to compute option prices. Section 3 presents the CVA definition and some useful properties used in the following. Section 4 outlines the main features of the hybrid methods and how to employ them in option pricing. Section 5 describes the numerical methods for computing the CVA. Section 6 presents and discusses the results of the numerical simulations. Section 7 concludes.

2 The Model

In the Bates model the volatility is assumed to follow the CIR process (Cox et al. [?]) while the underlying asset price process contains a further noise from a jump process. In particular, the model for the stock price and its volatility is given by the following relations:

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= (r-\eta)dt + \sqrt{V_t}\,dZ_t^S + dH_t,\\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}\,dZ_t^V,\end{aligned} \tag{2.1}$$

where η denotes the continuous dividend rate, S_0, V_0 are positive values, Z^S, Z^V are correlated Brownian motions such that $\left\langle dZ_t^s, dZ_t^V \right\rangle = \rho dt$ and H is a compound Poisson process with intensity λ and i.i.d. jumps $\{J_k\}_k$, that is

$$H_t = \sum_{k=1}^{K_t} J_k, \tag{2.2}$$

K denoting a Poisson process with intensity λ. As in the model of Merton [?], we assume that the jumps are i.i.d. log-normal random variables, described by

$$\log(1+J) \sim N\left(\alpha - \frac{\beta^2}{2}, \beta^2\right), \tag{2.3}$$

with α and β real parameters.

We consider an option with maturity $T > 0$, payoff function $\psi(S)$ and we denote with $\mathcal{V}(t, S, V)$ its fair price at time t, if $S_t = S$ and $V_t = V$. Then, the price of the derivative $\mathcal{V}$ is then given by

$$\mathcal{V}(t, S, V) = \mathbb{E}\left[e^{-r(T-t)}\psi(S_T)\,|S_0 = S, V_0 = V\right]. \tag{2.4}$$

Using a martingale approach for an European option, it is possible to show that $\mathcal{V}(t, S, V)$ satisfies the following partial integro-differential Equation (PIDE) (Salmi et al [?]):

$$\begin{aligned}\frac{\partial \mathcal{V}(t,S,V)}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2\mathcal{V}(t,S,V)}{\partial S^2} + \rho\sigma VS\frac{\partial^2\mathcal{V}(t,S,V)}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2\mathcal{V}(t,S,V)}{\partial V^2} + (r - q - \lambda(e^\gamma - 1))\frac{\partial \mathcal{V}(t,S,V)}{\partial S}\\ + \kappa(\theta - V)\frac{\partial \mathcal{V}(t,S,V)}{\partial V} - (r+\lambda)\mathcal{V}(t,S,V) + \lambda\int_0^\infty \mathcal{V}(t, xS, V)\,p_J(x)\,dx = 0,\end{aligned} \tag{2.5}$$

with the terminal condition is

$$\mathcal{V}(T, S, V) = \psi(S), \tag{2.6}$$

where $p_J(x)$ is the log-normal probability density function of the jump variable J.

We define the following operators:

$$L_C\mathcal{V} = \frac{1}{2}VS^2\frac{\partial^2\mathcal{V}}{\partial S^2} + \rho\sigma VS\frac{\partial^2\mathcal{V}}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2\mathcal{V}}{\partial V^2} + (r - q - \lambda(e^\gamma - 1))\frac{\partial\mathcal{V}}{\partial S} + \kappa(\theta - V)\frac{\partial\mathcal{V}}{\partial V} - (r+\lambda)\mathcal{V}, \tag{2.7}$$

and

$$L_J\mathcal{V} = \lambda\int_0^\infty \mathcal{V}(t, xS, V)\,p_J(x)\,dx. \tag{2.8}$$

Therefore, by using (2.7) and (2.8), the PIDE (2.5) can be rewritten in the more compact form as follows:

$$\frac{\partial\mathcal{V}(t,S,V)}{\partial t} + L_C\mathcal{V}(t,S,V) + L_J\mathcal{V}(t,S,V) = 0. \tag{2.9}$$

3 The CVA

Let us define the exposure (or financial exposure) $E(t)$ towards a counterparty at time t as the positive side of a contract (or portfolio) value $\mathcal{V}(t, S_t, V_t)$, that is

$$E(t) = \max\left[\mathcal{V}(t, S_t, V_t), 0\right]. \tag{3.1}$$

This amount represents the maximum loss if the counterparty defaults at t: an economic loss would occur if the transactions with the counterparty has a positive economic value at the time of default (see the Basel Committee [**?**]).

Let us define the present Expected Exposure (EE) at a future time $t < T$ as

$$EE(t) = \mathbb{E}\left[E(t) \,|\mathcal{F}_0\right] \tag{3.2}$$

where $\mathcal{F}_0$ is the filtration at time $t = 0$. In particular, in the case of a long position in a Call or Put option, the price is always positive and thus the EE is equal to the future option price.

Let τ_C denote the counterparty's default time: such a time is a random variable which is supposed to be independent from the stochastic processes Z^S, Z^V and H. Moreover, we describe its cumulative distribution function as follows:

$$PD(t) = 1 - \exp\left(-\int_0^t \delta(s)\, ds\right). \tag{3.3}$$

Here, $\delta(t)$, the so called hazard rate, is a non-negative function such that $\int_0^{+\infty} \delta(s)\, ds = +\infty$ (Promislow [**?**]). If the counterparty has not defaulted yet, then $\delta(t)\, dt$ is the probability that it would default between t and $t + dt$. The possibility of counterparty default reduces the value of the option as in case of default the holder does not receive the whole value of the option. In particular, the holder can recover only a percentage of the contract value, which is called the recovery rate R. The CVA is defined as the difference between the risk free price and the risk adjusted price. More precisely, as stated by Gregory [**?**], the CVA is given by:

$$\text{CVA} = (1 - R) \int_0^T D(0, s)\, EE(s)\, dPD(s), \tag{3.4}$$

where $D(0, t)$ is the risk-free discount factor. This mean that the CVA is the expected value of the possible losses due to counterparty default. We stress out that definition requires the exposure and the counterparties default probability to be independent and the discount factor to be also independent of the exposure. Using equations (3.2) and (3.4), we obtain the following relation:

$$\text{CVA} = \mathbb{E}\left[\int_0^T D(0, s)(1 - R)\, E(s)\, dPD(s) \,|\mathcal{F}_0\right]. \tag{3.5}$$

According to (3.5), the CVA is the price (mean value of the future cash flows) of a financial derivative which pays $(1 - R)\max\left[\mathcal{V}(\tau_C, S_{\tau_C}, V_{\tau_C}), 0\right]$ at time τ_C. Therefore, we can consider the CVA as a derivative itself and we denote its financial value at time t with $\mathcal{C}(t, S_t, V_t)$, that is

$$\mathcal{C}(t, S_t, V_t) = \mathbb{E}\left[\int_t^T D(0, s)(1 - R)\, E(s)\, dPD_s |\mathcal{F}_t\right],$$

having $CVA = \mathcal{C}(0, S_0, V_0)$. It is possible to show that $\mathcal{C}(t, S_t, V_t)$ solves the following PIDE (see the Appendix for more details):

$$\frac{\partial \mathcal{C}(t, S, V)}{\partial t} + L_C \mathcal{C}(t, S, V) + L_J \mathcal{C}(t, S, V) + (1 - R)\max\left[\mathcal{V}(t, S_t, V_t), 0\right] \frac{\partial PD}{\partial t}(t) = 0, \tag{3.6}$$

with the terminal condition

$$\mathcal{C}(T, S, V) = 0. \tag{3.7}$$

We stress out that equation (3.6) depends on the value $\mathcal{V}(t, S_t, V_t)$ which has to be computed previously by solving equation (2.9).

Remark. **According to Basel III [?] regulatory framework, banks with an internal model method must compute the CVA in order to account for counterparty risk losses. In this framework, the CVA is defined as**

$$\mathbf{CVA} = LGD_{MKT} \cdot \sum_{i=1}^{T} Max\left(0; \exp\left(\frac{-s_{i-1}t_{i-1}}{LGD_{MKT}}\right) - \exp\left(\frac{-s_i t_i}{LGD_{MKT}}\right)\right) \cdot \frac{EE\left(t_{i-1}\right) \cdot D\left(0, t_{i-1}\right) + EE\left(t_i\right) \cdot D\left(0, t_i\right)}{2}, \tag{3.8}$$

where $t_0, t_1, \ldots, t_T$ is a discrete set of event times, s_i is the counterparty credit spread at time t_i and LGD_{MKT} is the market loss given default of the counterparty. Although this definition differs from the one employed in this paper, that is 3.4, the two are metologically equivalent. In particular, recalling the relation between the recovery ratio R and the loss given default LGD given by

$$R = 1 - LGD, \tag{3.9}$$

and the first order relation between the hazard rate $\delta(t)$ and the credit spread s_i (see Hull [?]) given by

$$\frac{\int_0^{t_i} \delta(u)\, du}{t_i} = \frac{s_i}{LGD}, \tag{3.10}$$

it is easy to show that formulation in (3.8) converges to expression in (3.4) as $\max_i (t_i - t_{i-1}) \to 0$.

4 Hybrid methods in the Bates model

The Hybrid Tree Finite Difference (HTFD) and the Hybrid Tree Monte Carlo (HTMC) methods are two innovative and efficient approaches to price derivatives when the Bates model is considered. In this Section we present the main ideas that underlie these two methods: the interested reader can find more details about the numerical procedures in Briani et al. [?].

4.1 The HTFD method

The HTFD method is a backward induction algorithm that works following a finite difference PIDE method in the direction of the share process and following a tree method in the direction of the other random sources, that is volatility in the case of Bates model. Specifically, the method is based on the following steps. First of all, a binomial tree for the CIR volatility process V is considered according to Apolloni et al. [?]. Then, a transformation which keeps the diffusion processes S and V uncorrelated is applied. Finally, a finite difference approach in the S-direction is developed.

In particular, consider a large integer value N, a time horizon $[0, T]$ and define $h = T/N$. For $n = 0, 1, \ldots, N$, define

$$V_n^h = \{v_{n,k}\}_{k=0,1,\ldots,n} \tag{4.1}$$

with

$$v_{n,k} = \left(\sqrt{V_0} + \frac{\sigma}{2}(2k-n)\sqrt{h}\right)^2 1_{\sqrt{V_0} + \frac{\sigma}{2}(2k-n)\sqrt{h} > 0}. \tag{4.2}$$

We define the multiple jumps

$$\begin{aligned} k_d^h(n,k) &= \max\left\{k^* : 0 \leq k^* \leq j \text{ and } v_{n,k} + \mu_V(v_{n,k})\, h \geq v_{n+1,k*}\right\}, \\ k_u^h(n,k) &= \min\left\{k^* : k+1 \leq k^* \leq n+1 \text{ and } v_{n,k} + \mu_V(v_{n,k})\, h \leq v_{n+1,k^*}\right\} \end{aligned}$$

in which μ_V denotes the drift coefficient of V, that is $\mu_V(v) = \kappa(\theta - v)$. Starting from the node (n,k), the discrete process can reach the up-node $\left(n+1, k_u^h(n,k)\right)$ or the down-jump node $\left(n+1, k_d^h(n,k)\right)$ with transition probability given by

$$\text{up-jump: } p^h_{k_u^h(n,k)} = 0 \vee \frac{\mu_V\left(v_{n,k}\right)h + v_{n,k} - v_{n+1,k_d^h(n,k)}}{v_{n+1,k_u^h(n,k)} - v_{n+1,k_d^h(n,k)}} \wedge 1, \tag{4.3}$$

$$\text{down-jump : } p^h_{k_d^h(n,k)} = 1 - p^h_{k_u^h(n,k)}. \tag{4.4}$$

Multiple jumps and jump probabilities are set in order to match the first local moment of the tree and of the process V up to order one with respect to h. As a consequence, as h approaches to 0, the weak convergence on the path space is guaranteed. Moreover, in order to obtain the convergence, the Feller condition (Albrecher et al. [**?**]) is not required.

Let us consider the diffusion pair (Y, V), where Y is a stochastic process defined by

$$Y_t = \log\left(S_t\right) - \frac{\rho}{\sigma}V_t. \tag{4.5}$$

Clearly the couple (S, V) can be retraced by (Y, V) by inverting relation (4.5). We set $\bar{\rho} = \sqrt{1-\rho^2}$ and we consider (W, Z) as a standard Brownian motion in $\mathbb{R}^2$. Then, the dynamics of the couple (Y, V) is given by

$$dY_t = \left(r - \eta - \frac{1}{2}V_t - \frac{\rho}{\sigma}\kappa\left(\theta - V_t\right)\right)dt + \bar{\rho}\sqrt{V_t}dZ_t + dN_t, \tag{4.6}$$

$$dV_t = \kappa\left(\theta - V_t\right)dt + \sigma\sqrt{V_t}dW_t, \tag{4.7}$$

with $Y_0 = \log S_0 - \frac{\rho}{\sigma}V_0$. Here, N_t is the compound Poisson process written through the Poisson process K, with intensity λ, and the i.i.d. jumps $\{\log\left(1+J_k\right)\}$, that is

$$N_t = \sum_{k=1}^{K_t}\log\left(1+J_k\right). \tag{4.8}$$

We set

$$\mu_Y\left(v\right) = r - \eta - \frac{1}{2}v - \frac{\rho}{\sigma}\kappa\left(\theta - v\right) \tag{4.9}$$

and

$$\mu_V\left(v\right) = \kappa\left(\theta - v\right). \tag{4.10}$$

Let $\bar{V}^h = \left(\bar{V}^h_n\right)_{n=0,\dots,N}$ denote the tree process approximating V and set $V^h_t = \bar{V}^h_{\lfloor t/h \rfloor}$, $t \in [0, T]$, the associated piecewise constant and cÃ dlÃ g approximation path. In order to approximate Y, Briani et al. construct a Markov chain from the finite difference method. Starting from the Euler scheme $Y^h_0 = Y_0$ and for $t \in (nh, (n+1)h]$, $n = 0, \dots, N$, it is possible to set

$$Y^h_t = Y^h_{nh} + \mu_Y\left(V^h_{nh}\right)\left(t - nh\right) + \bar{\rho}\sqrt{V^h_{nh}}\left(Z_t - Z_{nh}\right) + \left(N_t - N_{nh}\right), \tag{4.11}$$

Z being independent of the noise driving $\bar{V}^h$. Now, let $\mathcal{W}\left(t, Y, V\right) = \mathcal{V}\left(t, \exp\left(Y + \frac{\rho}{\sigma}V\right), V\right)$ the function which gives the price of the financial derivative at time t in terms of the couple (Y, V) . Then,

$$\mathbb{E}\left(\mathcal{W}\left(\left(n+1\right)h, Y_{(n+1)h}, V_{(n+1)h}\right)|Y_{nh} = y, V_{nh} = v\right) \approx \mathbb{E}\left(\mathcal{W}\left(\left(n+1\right)h, Y^h_{(n+1)h}, V^h_{(n+1)h}\right)|Y_{nh} = y, V_{nh} = v\right) \tag{4.12}$$

$$= \mathbb{E}\left(u^h\left(nh, y; v, V^h_{(n+1)h}\right)|V^h_{nh} = v\right) \tag{4.13}$$

where

$$u^h\left(nh, y; v, z\right) = \mathbb{E}\left(\mathcal{W}\left(\left(n+1\right)h, Y^h_{(n+1)h}, z\right)|Y^h_{nh} = y, V^h_{nh} = v\right) \tag{4.14}$$

and

$$u^h(nh, y; v, z) = u^h(s, y; v, z)\,|_{s=nh}. \tag{4.15}$$

The key point of the HTFD method is that the function $(s, y) \mapsto u^h(s, y; v, z)$ solves the following PIDE in the time interval $nh < s < (n+1)h$:

$$\frac{\partial u^h}{\partial s} + \mu_Y(v)\frac{\partial u^h}{\partial y} + \frac{1}{2}\bar{\rho}^2 v \frac{\partial^2 u^h}{\partial y^2} + \int_{-\infty}^{+\infty}\left[u^h(s, y+x; v, z) - u^h(s, y; v, z)\right] p_J(x)\,dx = 0, \tag{4.16}$$

for $y \in \mathbb{R}$, and with terminal condition given by

$$u^h((n+1)h, y; v, z) = \mathcal{W}((n+1)h, y, v). \tag{4.17}$$

In order to numerically solve the PIDE (4.16) by using a finite difference scheme, we first localize the variables and the integral term to bound the computational domains. For this purpose, we use the estimates for the localization domain and the truncation of large jumps given by Voltchkova and Tankov [?]. Then, the derivatives of the solution are replaced by finite differences and the integral terms are approximated using the trapezoidal rule. Finally, the problem is solved by using an explicit-implicit scheme.

We observe that the above algorithm is referred to a European option, can easily be adapted to consider an American option. Specifically, we approximate the American option with a Bermudan option with exercise dates given by the set $\{nh\}_{n=0,\dots,N}$ and replacing equation (4.17) with the following one:

$$u^h((n+1)h, y; v, z) = \max\left[\mathcal{W}((n+1)h, y, v), \psi\left(\exp\left(y + \frac{\rho}{\sigma}v\right)\right)\right] \tag{4.18}$$

We stress out that the computation of u^h as the solution of (4.16) is a one dimensional problem with constant coefficients, thus it can be solved in a very efficient way, with a low computational cost.

4.2 The HTMC method

The HTMC method is a Monte Carlo algorithm which is based on the approximations (4.3) and (4.11). In particular, the HTMC method consists in simulating a continuous process in space (the component Y) starting from a discrete process in space (the 1-dimensional tree for V). In particular, we set $\hat{Y}_0^h = Y_0$ for $t \in [nh, (n+1)h]$ with $n = 0, 1, \dots, N-1$. Then, we compute $\hat{Y}_{n+1}^h$ recursively by the following relation:

$$\hat{Y}_{n+1}^h = \hat{Y}_n^h + \mu_Y\left(\hat{V}_n^h\right)h + \hat{\rho}\sqrt{h\hat{V}_n^h}\Delta_{n+1} + \left(N_{(n+1)h} - N_{nh}\right) \tag{4.19}$$

where $\Delta_1, \dots, \Delta_N$ are i.i.d. standard normal random variables which are independent of the noise driving the Markov chain $\hat{V}$ and $\left(N_{(n+1)h} - N_{nh}\right)$ is the compound Poisson increment. Roughly speaking, one let the pair (Y, V) evolve on the tree and simulate the process Y at time nh by using relation (4.19).

5 Numerical method for computing the CVA

In this section we present the proposed approach to compute the CVA, namely the Coupled-Hybrid Tree Finite Differences (C-HTFD), which is based on the resolution of two coupled PIDE. Moreover, we start presenting the Hybrid Tree Finite Difference-Hybrid Monte Carlo (HTFD-HTMC) method which is based on the direct application of the hybrid techniques in the original FDMC (we consider this method for comparison purposes mainly). The two approaches are both based on the application of the hybrid algorithms of Briani et al. [?], presented in Section 4.

5.1 The HTFD-HTMC approach

This method is based on the direct application of the hybrid techniques to the Finite Difference Monte Carlo (FDMC) method, which was first developed by de Graaf et al. [?] for the Heston model and then adapted for the Bates model by Feng [?]. First of all, an estimation of the value function $\mathcal{V}(t, S, V)$ is computed through a grid of values by solving equation (2.5) by employing the Alternating Direction Implicit (ADI) method and specifically the scheme proposed by Haentjens and In't Hout [?]. Then a Monte Carlo simulation is employed to estimate the expectation in (3.4) and thus estimate the CVA.

In the HTFD-HTMC approach we employ the HTFD method to solve (2.5) and the HTMC method to estimate (3.4). Specifically, let $h = T/N$ as in Subsection 4.1. First of all, the HTFD method is used to compute the risk free price $\{\mathcal{V}(nh, S_{nh}, V_{nh})\}_{n=0,\ldots,N}$ at discrete times $\{0, h, \ldots, Nh\}$. Then, we estimate the expected exposure (3.2) via a set of N_{MC} Monte Carlo simulations $\left\{\left(S^j_{nh}, V^j_{nh}\right), n = 0, \ldots, N \text{ and } j = 1, \ldots, N_{MC}\right\}$ generated via the HTMC method, that is

$$EE(nh) \approx \frac{1}{N_{MC}} \sum_{j=1}^{N_{MC}} \max\left[\mathcal{V}\left(nh, S^j_{nh}, V^j_{nh}\right)\right]. \tag{5.1}$$

Finally, we compute the CVA by approximating the integral in (3.4) using the uniform partition $\{0, h, \ldots, Nh\}$ of $[0, T]$, the estimated value $\{EE(nh), n = 0, \ldots, N\}$ and the trapezoidal rule.

5.2 The C-HTFD approach

First of all, an estimation of the value function $\mathcal{V}(t, S, V)$ is computed through a grid of values by solving equation (2.5) by means of the HTFD method. Then, the CVA is computed as the solution of the PIDE (3.6), which is done again by employing the HTFD method. We stress that C-HTFD method and the HTFD-HTMC method share the first step, but in the C-HTFD method the Monte Carlo step is replaced with a second finite difference like step.

6 Numerical Results

In this section we propose the results of some numerical experiments, which aim to compare the goodness of the proposed methods. In particular, we compare the standard FDMC method employed by Feng [?], the HTFD-HTMC approach and the C-HTFD approach. We employ the numerical methods according to 4 configurations (A, B, C, D), each of them with an increasing number of steps, determined in order to achieve approximately these run times[1] : (A) 0.25 s, (B) 1 s, (C) 4 s, (D) 16 s. The employed mesh and configuration parameters are reported in Table 1. Specifically, for the FDMC method with the sequence (time steps - points for S - points for V - MC simulations) and for the hybrid methods with the sequence (time steps - points for Y - MC simulations). Moreover, we reported the 95% confidence intervals for the FDMC method and for the HTFD-HTMC. Furthermore, we employ the FDMC method with a huge number of steps and paths $\left(250; 500; 200; 10^6\right)$ as a benchmark (BM).

We consider the following parameters for the Bates model: $S_0 = 80,\ 100,\ 120$, $K = 100$, $T = 1$, $r = 0.03$, $\eta = 0.00$, $V_0 = 0.01$ $\kappa = 2$, $\theta = 0.01$, $\sigma = 0.2$, $\lambda = 0.1$, $\alpha = 0.1$, $\beta^2 = 0.1$ and $\rho = 0.5$. According to the default parameters, we consider $\delta(t) = \delta = 0.03$ and $R = 0.4$. We compute the CVA for a European and an American a Put option with strike equal to $K = 100$.

Results are available in Tables 2 and 3, which show that all the implemented methods give similar results. Values calculated via HTFD-HTMC and values calculated via FDMC have similar accuracy. We emphasize that, despite the similarity in numerical results of these two methods, we prefer the HTFD-HTMC method because of its simplicity of implementation.

[1]We performed the numerical tests using a personal computer with the following features. CPU: Intel(R) Core(TM) i5-7200 2.50 GHz; RAM: 8GB, DDR4

The values returned by C-HTFD are the most accurate since they are very close to the benchmark for the whole parameter configurations.

7 Conclusions

In this paper we have proposed a numerical method to compute the CVA of European and American options when the underlying is assumed to evolve according to the Bates model. This method is based on the the resolution of two coupled PIDE which is done by employing the Hybrid Tree-Finite Difference algorithm developed by Briani et al.. Specifically, the C-HTFD approach consist in replacing the Monte Carlo step of the FDMC method with the resolution of the PIDE followed by the CVA cost, which is done by employing the Hybrid technique for the Bates model.

Numerical results show that our method is very stable and robust. In particular, numerical tests reveals that the values returned by C-HTFD method are very accurate, much more than the results provided by other methods which involve a Monte Carlo step, since the use of a PIDE approach in place of a Monte Carlo one dramatically improves the computational efficiency. Thus, the C-HTFD is efficient and reliable and the use of the two coupled PIDE represents a relevant improvement with respect to the standard pricing techniques of the CVA when the Bates model is considered.

	FDMC	HTFD-HTMC	C-HTFD
A	$50; 80; 15; 1500$	$50; 100; 1500$	$50; 100$
B	$75; 110; 30; 2000$	$75; 150; 2000$	$75; 150$
C	$100; 200; 50; 3300$	$100; 250; 3300$	$100; 250$
D	$125; 300; 100; 6000$	$125; 350; 6000$	$125; 350$

Table 1: Configuration parameters for the numerical methods.

		FDMC	HTFD-HTMC	C-HTFD	BM
$S_0 = 80$	A	0.320071 ± 0.005172	0.327462 ± 0.006098	0.323732	0.323724 ± 0.000200
	B	0.322874 ± 0.004469	0.327165 ± 0.005107	0.323713	
	C	0.323805 ± 0.003435	0.326023 ± 0.003993	0.323707	
	D	0.324926 ± 0.002660	0.324871 ± 0.002911	0.323703	
$S_0 = 100$	A	0.058209 ± 0.003071	0.063808 ± 0.003809	0.060724	0.060359 ± 0.000125
	B	0.059096 ± 0.002635	0.062193 ± 0.003005	0.060613	
	C	0.059838 ± 0.002178	0.061673 ± 0.002385	0.060507	
	D	0.060978 ± 0.001664	0.060749 ± 0.001678	0.060467	
$S_0 = 120$	A	0.005658 ± 0.001697	0.007251 ± 0.001976	0.005633	0.005589 ± 0.000059
	B	0.005406 ± 0.001289	0.006296 ± 0.001461	0.005610	
	C	0.005334 ± 0.000954	0.006299 ± 0.001328	0.005592	
	D	0.005660 ± 0.000758	0.005616 ± 0.000852	0.005584	

Table 2: CVA for European Put options.

		FDMC	HTFD-HTMC	C-HTFD	BM
$S_0 = 80$	A	0.336835 ± 0.005078	0.342798 ± 0.006316	0.338987	0.339054 ± 0.000208
	B	0.337859 ± 0.004630	0.342703 ± 0.005295	0.339084	
	C	0.338970 ± 0.003560	0.341533 ± 0.004139	0.339134	
	D	0.340182 ± 0.002755	0.340498 ± 0.003019	0.339165	
$S_0 = 100$	A	0.059803 ± 0.003168	0.065669 ± 0.003937	0.062466	0.062145 ± 0.000130
	B	0.060742 ± 0.002723	0.063979 ± 0.003103	0.062364	
	C	0.061522 ± 0.002248	0.063446 ± 0.002458	0.062260	
	D	0.062717 ± 0.001718	0.062501 ± 0.001732	0.062221	
$S_0 = 120$	A	0.005798 ± 0.001745	0.005449 ± 0.002043	0.005782	0.005740 ± 0.000061
	B	0.005537 ± 0.001327	0.006466 ± 0.001509	0.005760	
	C	0.005473 ± 0.000983	0.006467 ± 0.001368	0.005742	
	D	0.005812 ± 0.000784	0.005763 ± 0.000877	0.005735	

Table 3: CVA for American Put options.