Pricing Convertible Bonds with Call Protection

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The following method computes the price of convertible bonds with call protection. It is based on the paper [2], which is a follow-up paper to [1].

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1 Introduction

A convertible bond pays coupons from time 0 until a terminal payoff

$$\mathbb{1}_{\{\zeta=\tau<T\}} l(\tau, S_\tau) + \mathbb{1}_{\{\theta=\tau\}} h(\theta, S_\theta) + \mathbb{1}_{\{\zeta=T\}} g(S_T) \tag{1}$$

occurs at the minimum $\zeta = \tau \wedge \theta$ of two $[0,T]$-valued stopping times θ and τ. Here the *put time* τ and the *call time* θ are $[0,T]$-valued stopping times under the control of the holder and the issuer of the bond respectively, and

- $g(S_T)$ corresponds to a *terminal payoff* that is paid by the issuer to the holder at time T if the contract was not exercised before the maturity time T;
- $l(\tau, S_\tau)$, respectively $h(\theta, S_\theta)$, corresponds to an *early put payoff*, respectively *early call payoff*, that is paid by the issuer to the holder of the claim in the event of early termination of the contract at the initiative of the holder, respectively issuer.

Convertible bond call times θ are subjects to constraints, called *called protections*, preventing the issuer from calling the bond on certain random time intervals. From the mathematical point of view, the study of such products leads to doubly reflected backward stochastic differential equations with an upper barrier which is only active on random time intervals (called RIBSDE in the literature).

2 The Model

For the numerical experiments, we consider the local drift and volatility model of a non-negative underlying process S:

$$dS_t = S_t(b(t, S_t)dt + \sigma(t, S_t)dW_t), \; S_0 = x$$

where W is a standard univariate Brownian motion, and

$$b(t,S) = r - q + \eta\gamma_0 \left(\frac{S_0}{S}\right)^{\alpha}, \; \sigma(t,S) = \sigma.$$

r is the riskless short interest rate, q is the equity dividend yield, $\gamma_0 \left(\frac{S_0}{S}\right)^{\alpha}$ represents the local defaults intensity of the firm issuing the bond, and $\eta(\leq 1)$ is a real constant which represents the fractional loss on S in case of a default of the firm issuing the bond.
Concerning the payoff function defined in (1), we set

$$l(t,S_t) = l(S_t) = \overline{P} \vee S_t, h(t,S_t) = h(S_t) = \overline{C} \vee S_t, g(S_T) = \overline{N} \vee S_T = \xi.$$

for non-negative constants $\overline{P} \leq \overline{N} \leq \overline{C}$.
The convertible bond continuously pays coupons $c(t,S_t)dt$, from time 0 onwards, until the terminal payoff (1) is paid at time $\tau \wedge \theta$. Accounting for credit risk and recovery on the bond upon default, one assumes the following form of the coupon rate function c

$$c(t,S) = c + \gamma_0 \left(\frac{S_0}{S}\right)^{\alpha} ((1-\eta)S \vee R),$$

where c is the *nominal coupon rate* function and R is the *nominal recovery on the bond upon default*.
Call times θ are modelized by a non-decreasing sequence $(\theta_n)_n$ of stopping times, which represent times of switching of call protection. We assume that issuer calls are forbidden on the even time intervals $[\theta_{2l}, \theta_{2l+1})$.
We finally denote by $\beta_t = e^{-\int_0^t \mu(s,S_s)ds}$ a risk-neutral *credit risk adjusted discount factor*, where $\mu(t,S) = r + \gamma_0 \left(\frac{S_0}{S}\right)^{\alpha}$.

3 Link with BSDEs

By application of [1], pricing convertible bond can be linked to the following reflected BSDE

$$\Pi_T = g(S_T), \text{ and for all } t \in [0,T]$$
$$\begin{cases} -d\Pi_t = (c(t,S_t) - \mu(t,S_t)\Pi_t)dt + dA_t - \Delta_t \sigma(t,S_t) S_t dW_t \\ L_t \leq \Pi_t \leq U_t, \; (\Pi_t - L_t)dA_t^+ = (U_{t^-} - \Pi_{t^-})dA_t^- = 0. \end{cases}$$

to be solved in (Π, Δ, A) in the 'usual spaces' of square integrable processes. The upper barrier U_t is defined by $U_t = \mathbb{1}_{\{l_t \text{ is even}\}}\infty + \mathbb{1}_{\{l_t \text{ is odd}\}} h(S_t)$ with l_t s.t. $\theta_{l_t} \leq t < \theta_{l_t+1}$, and the lower barrier $L_t = l(S_t)$. Π represents the price of the convertible bond, and Δ represents the derivative of the price w.r.t. the asset value.

4 Different clauses on call times

In this Section we present different causes on call times. The asset S is discretized in n time steps with an Euler Scheme

$$S^n_{i+1} = S^n_i (1 + b(t_i, S^n_{t_i})h + \sigma(t_i, S^n_{t_i})(W_{t_{i+1}} - W_{t_i}), \; S^n_0 = x,$$

where $h := \frac{T}{n}$ and $t_i = ih$, for all $i = 0, \cdots, n$.

4.1 No call protection

One considers the case of no call protection, i.e. the issuer can call the bond at any time between 0 and the maturity T. ($\theta_1 = 0$, $\theta_2 = T$). In this case, the discounted price process $\beta\Pi$ of the bond is given by

$$\beta_t\Pi_t = \text{essinf}_{\theta\in\mathcal{T}_t}\text{esssup}_{\tau\in\mathcal{T}_t}\mathbb{E}_t\left\{\int_t^\zeta \beta_s c_s ds + \beta_\zeta(\mathbb{1}_{\{\tau\leq\theta,\tau<T\}}l(S_\tau) + \mathbb{1}_{\{\theta<\tau\}}h(S_\theta) + \mathbb{1}_{\{\tau=\theta=T\}}\xi)\right\}$$

where $\zeta = \tau\wedge\theta$, $\beta_t = \exp^{-\int_0^t \mu(s,S_s)ds}$ and $\mathcal{T}_t$ represents the set of stopping times taking values in $[t,T]$.
The simulation scheme is done in two steps: first, we compute the continuation value computed by iteration on the values (backward estimates): $v_n^j = g(S_n^j)$ for $j = 1\cdots M$, and for $i = n-1\cdots 0$ for $j = 1\cdots M$,

$$v_i^j = \min\left(h(S_i^j), \max\left(l(S_i^j), e^{-\mu_i^j h}\mathbb{E}_i^j(v_{i+1} + hc_{i+1})\right)\right).$$

The conditional expectations are computed by non-linear regression of $(v_{i+1}^j + hc_{i+1}^j)_{j=1\cdots M}$ against $(S_i^j)_{1\leq j\leq M}$.
Then, we recover the put and call regions and optimal put and call policies:

$$\mathcal{E}^p = \{(i,S_i^j); v_i^j = l(S_i^j)\}, \tau_j = \inf\{i\in\{0,\cdots,n\}; S^j\in\mathcal{E}^p\}\wedge n$$
$$\mathcal{E}^c = \{(i,S_i^j); v_i^j = h(S_i^j)\}, \theta_j = \inf\{i\in\{0,\cdots,n\}; S^j\in\mathcal{E}^c\}\wedge n.$$

We then have the following policy iteration estimates (forward estimates) for the option price at time 0, with $\zeta^j = \tau^j\wedge\theta^j$.

$$\tilde{v}_0 = \frac{1}{M}\sum_{j=1}^M \beta_i^j c(S_i^j) + \beta_{\zeta^j}^j\left(\mathbb{1}_{\{\zeta^j=\tau^j<n\}}l(S_{\tau_j}^j) + \mathbb{1}_{\{\theta^j<\tau^j\}}h(S_{\theta^j}^j) + \mathbb{1}_{\{\zeta^j=n\}}g(S_n^j)\right).$$

4.2 Standard Call protection

We consider the case where the issuer can call the bond on $[\theta_1, T]$, where

$$\theta_1 = \inf\{t\in\mathbb{R}_+ : S_t\geq\overline{S}\}\wedge T.$$

for some trigger level $\overline{S} > S_0$.
The simulation scheme is the following : given a stochastically generated mesh $(S_i^j)_{0\leq i\leq n}^{0\leq j\leq M}$ and setting

$$\theta_1^j = \inf\{i\in\{0,\cdots,n\} : S_i^j\geq\overline{S}\}\wedge n,$$

a simulation algorithm for estimating the price at the points $(t_i, S_{t_i^j})_{0\leq i\leq n, 0\leq j\leq M}$ writes as follows : $u_n = v_n = g$ and then do, for $i = n-1,\cdots,0$, $j = 0,\cdots,M$

- $v_i^j = \min\left(h(S_i^j), \max(l(S_i^j), e^{-\mu_i^j h}\mathbb{E}_i^j(v_{i+1} + hc_{i+1}))\right)$,

- If $i \geq v_1^j$, $u_i^j = v_i^j$, else

$$u_i^j = \max\left(l(S_i^j), e^{-\mu_i^j h}\mathbb{E}_i^j(u_{i+1} + hc_{i+1})\right).$$

v_i^j is the no call protection price of the previous section.
We can recover from the protection pricing function u above the following estimates of the protection put region and of the optimal protection put policy

$$\tilde{\mathcal{E}_p} = \{(i, S_i^j) : u_i^j = l(S_i^j)\}, \ \tilde{\tau}^j = \inf\{i \in \{0, \cdots, \theta_1^j\} : S_i^j \in \tilde{\mathcal{E}_p}\} \wedge n.$$

One then have the following policy iteration estimate for the option price at time 0, with $\zeta^j = \tilde{\tau}^j \wedge \theta_1^j$

$$\tilde{u_0} = \frac{1}{M}\sum_{j=1}^{M}\left\{h\sum_{i=1}^{\tilde{\zeta}^j}\beta_i^j c(S_i^j) + \beta_{\tilde{\zeta}^j}^j\left(\mathbb{1}_{\{\tilde{\tau}^j<\theta_1^j\}}l(S_{\tilde{\tau}^j}^j) + \mathbb{1}_{\{\theta_1^j\leq\tilde{\tau}^j\}}v_{\theta_1^j}^j\right)\right\}.$$

4.3 Path dependent Call Protection

Given $\overline{S} > 0$ and a fixed increasing sequence of monitoring times $\mathcal{I} = \{T_0 = 0, T_1, \cdots, T_N = T\}$ (pratically includes in set of discretization times $i * h$, $i = 0, \cdots, n$), let $(H_t)_{t\in[0,T]}$ stands for the number of consecutive monitoring dates T_Is with $S_{T_I} \geq \overline{S}$ from time t backwards. In particular, at any given time t, one has $H_t = 0$ is S is smaller than $\overline{S}$ at the last monitoring date before t. We still consider a call protection for $t \geq \theta_1$, but we define

$$\theta_1 = \inf\{t \in \mathbb{R}_+ : H_t \geq l\} \wedge T$$

The lifting time of the call protection θ_1 is thus given as the first time, capped at T, that S_t has been $\geq \overline{S}$ at the last l monitoring dates.

The simulation scheme is the same as the one in case of standard call protection, except for the definition of θ_1^j. Given a time mesh $(t_i)_{0\leq i\leq n}$ refining the monitoring time grid $\mathcal{I}$, we generate a stochastic grid $(S_i^j, H_i^j)_{0\leq i\leq n}^{1\leq j\leq M}$ by an Euler scheme for S, using past values of S to fill H. For all $j = 1, \cdots, M$, we define

$$\theta_1^j = \inf\{i \in \{0, \cdots, n\} : H_i^j \geq l\} \wedge n.$$

In the numerical experiments, we take $T_{I+1} - T_I$ =one day, l can varies from 1 to 180, if the maturity $T = 180/365 = 0.4932$.

4.4 Highly Path Dependent Call Protection

Given an integer d s.t. $l \leq d \leq n$, let now $(H_t)_{t\leq T}$ represents the vector of the indicator functions od the events $S_{T_I} \geq \overline{S}$ at the last d monitoring dates preceding time t. We now consider the case

$$\theta_1 = \inf\{t \in \mathbb{R}_+ : |H_t| \geq l\} \wedge T,$$

with $|H_t| = \sum_{1\leq k\leq d} H_t^k$, H_t^k represents the indicator function of the event $S_{T_I} \geq \overline{S}$ at the last k monitoring dates before t. So θ_1 represents the first time, capped at T, such that $S \geq \overline{S}$ on at least l among the last d monitoring dates. (For $l = 0$, resp. $l = d$, we are back to the no call protection case, resp. to the l consecutive monitoring dates call protection.)

To solve this system by simulation, given a time mesh $(t_i)_{0\le i\le n}$ refining the tenor $\mathcal{I}$, we generate a stochastic grid $(S_i^j, H_i^j)_{0\le i\le n}^{1\le j\le M}$ in the obvious way, using past values of S to fill H. Then, for all $j = 1, \cdots, M$, we define

$$\theta_1^j = \inf\{i \in \{0, \cdots, n\} : |H_i^j| \ge l\} \wedge n.$$

As above, the last part of the algorithm is the same as in the standard case.

4.5 Intermittent Standard Call Protection

We now come to truly intermittent protection with call payoff processes of the following form, given a non-decreasing sequence $[0, T]$-valued stopping times $\theta = (\theta_l)_{l\ge 0}$, defined by $\theta_0 = 0$ and for every $l \ge 0$

$$\theta_{2l+1} = \inf\{t > \theta_{2l} : S_t \ge \overline{S}\} \wedge T,\ \theta_{2l+2} = \inf\{t > \theta_{2l+1} : S_t \le \overline{S}\} \wedge T.$$

Then, we have

$$U_t = \Omega_t^c \infty + \Omega_t h(S_t),$$

with $\Omega_t = \mathbb{1}_{\{l_t \text{ odd}\}}$ for l_t defined by $\theta_{l_t} \le t < \theta_{l_t+1}$. Ω_t is of the form $\Omega(t, X_t)$ where X represents (S, H) and Ω is a boolean function of X, so

$$U_t = U(X_t) = \Omega^c(t, X_t)\infty + \Omega(t, X_t)h(S_t).$$

The simulation scheme works as follows: given a stochastically generated mesh $(X_i^j)_{0\le i\le n}^{1\le j\le M}$, the generic simulation pricing scheme for estimating the price u writes ; $u_n = g$ and for $i = n-1, \cdots, 0$ for $j = 1, \cdots, M$

$$u_i^j = \min\left(U(X_i^j), \max(l(S_i^j), e^{-\mu_i^j h}\mathbb{E}_i^j(u_{i+1} + hc_{i+1}))\right).$$

The minimum in the above equation plays no role outside the support of Ω_i, where $U_i(x)$ equals ∞.

5 Numerical experiments

The numerical data used by default are the following

$\overline{P}$	$\overline{N}$	$\overline{C}$	η	σ	r	q	γ_0	α	R
0	100	103	1	0.2	0.05	0	0.02	1.2	0

The number of trajectories of S which are used to compute the conditional expectations are $M = 10^4$. The constant time step used for the discretization of S by the Euler scheme is $h = 6$ hours. The conditional expectations are computed using regression methods, with a basis of 4 polynomials.

In the case of no call protection, the maturity by default is 125 days (i.e. 0.3425 years), the nominal coupon rate $c = 0$, and the spot is 100.55.

In the other cases, the maturity by default is 180 days (i.e. 0.4932 years), the nominal coupon is $c = 14.4$ (per year), (i.e. 1.2 per month).

In the standard call protection case, the trigger level $\overline{S} = 103$.

References

[1] J.F. Chassagneux and S. Crépey. Doubly reflected BSDEs with Call Protection and their Approximation. available at http://grozny.maths.univ-evry.fr/pages_perso/crepey/, 2010. 1, 2

[2] S. Crépey and A. Rahal. Pricing convertible bonds with call protection. available at http://grozny.maths.univ-evry.fr/pages_perso/crepey/, 2010.

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