

## Premia 22

# Calibration of Stochastic Volatility model with Jumps

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The evolution process of the Heston model, for the stochastic volatility, and Merton model, for the jumps, is:

$$\begin{cases} \frac{dS_t}{S_t} &= (r - d)dt + \sqrt{V_t}dW_t^1 + (e^J - 1)dN_t \\ dV_t &= \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^2 \\ S(t=0) &= S_0 \\ V(t=0) &= V_0 \end{cases}$$

where  $d < W^1, W^2 >_t = \rho dt$  and  $J \sim \mathcal{N}(m, v)$ .

For European options, two pricing formula are giving based on the Fourier transform method [1]. In this document, we use the following notation:  $\varphi = +1$  for a call and  $\varphi = -1$  for a put;  $\tau = T - t$ ;  $x_t = \ln(S_t)$  and  $X = \ln(S_t/K) + (r - d)\tau$ .

## 1 The characteristic formula

The price  $F(x_t, t)$  is given by:

$$F(x_t, t) = \frac{1 + \varphi}{2} e^{x_t - (T-t)d} + \frac{1 - \varphi}{2} e^{1 - (T-t)r} K - e^{-(T-t)r} f(x, V, \lambda, t)$$

where

$$f(x, V, \lambda, \tau) = \frac{K}{\pi} \int_0^\infty \Re \left[ \frac{Q(k, x, V, \lambda, \tau)}{k^2 + 1/4} \right] dk$$

with

$$Q(k, x, V, \lambda, \tau) = e^{(-ik + 1/2)X + A(k, \tau) + B(k, \tau)V_0 + C(k, \tau) + D(k, \tau)\lambda}$$

The hedge  $\delta$  is given by:

$$\delta = \frac{1 + \varphi}{2} e^{-(T-t)d} - e^{-(T-t)r} \frac{K}{\pi} \int_0^\infty \Re \left[ \frac{(1/2 - ik)}{S_t} \cdot \frac{Q(k, x, V, \lambda, \tau)}{k^2 + 1/4} \right] dk$$

The coefficients  $A(k, \tau)$ ,  $B(k, \tau)$ ,  $C(k, \tau)$  and  $D(k, \tau)$  are specified as follows:

1. Volatility:

- Constant volatility :  
 $A(k, \tau) = 0$ ,  $B(k, \tau) = -1/2(k^2 + 1/4)\tau$ .
- Stochastic volatility (Heston) :  
 $A(k, \tau) = -\frac{\kappa\theta}{\sigma_v^2} \left[ \psi_+ \tau + 2 \ln \left( \frac{\psi_- + \psi_+ e^{-\tau\zeta}}{2\zeta} \right) \right]$   
 $B(k, \tau) = -(k^2 + 1/4) \frac{1 - e^{-\tau\zeta}}{\psi_- + \psi_+ e^{-\tau\zeta}}$   
where  $\psi_{\pm} = \mp(u + ik\rho\sigma_v) + \zeta$ ,  $\zeta = \sqrt{k^2\sigma_v^2(1 - \rho^2) + 2ik\rho\sigma_v u + u^2 + \sigma_v^2/4}$   
and  $u = \kappa - \rho\sigma_v/2$ .

2. Jumps :

- Merton model : constant jump rate intensity and log-normal jump size distribution  
 $C(k, \tau) = 0$ ,  $D(k, \tau) = \tau\Lambda(k)$  where  
 $\Lambda(k) = e^{-ik(m+v^2/2) - (k^2-1/4)v^2/2 + 1/2m} - 1 - (-ik+1/2)(e^{m+v^2/2} - 1)$ .

## 2 The Black-Scholes-style fomula

The price  $F(x_t, t)$  is given by

$$F(x_t, t) = \varphi \left( e^{-d(T-t)} S_t P_1(\varphi) - e^{-r(T-t)} K P_2(\varphi) \right)$$

and the hedge  $\delta$  by

$$\delta = \varphi e^{-d(T-t)} P_1(\varphi)$$

where  $P_j(\varphi) = \frac{1-\varphi}{2} + \varphi \Pi_j$  for  $j \in \{1, 2\}$  with

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{\phi_j(k)}{ik} \right] dk$$

where the characteristic functions  $\phi_j$ , for  $j \in \{1, 2\}$ , are given by:

$$\phi_j(k) = e^{ikX + A(k, \tau) + B(k, \tau)V_0 + C(k, \tau) + D(k, \tau)\lambda}$$

Using the notations :

$$\begin{cases} u = +1, & I = 1, & b = \kappa - \rho\sigma_v & \text{if } j = 1 \\ u = -1, & I = 0, & b = \kappa & \text{if } j = 2 \end{cases}$$

the coefficients  $A(k, \tau)$ ,  $B(k, \tau)$ ,  $C(k, \tau)$  and  $D(k, \tau)$  are given as follows:

### 1. Volatility:

- Constant volatility :  
 $A(k, \tau) = 0, B(k, \tau) = -1/2(k^2 - uik)\tau.$
- Stochastic volatility (Heston) :

$$A(k, \tau) = -\frac{\kappa\theta}{\sigma_v^2} \left[ \psi_+ \tau + 2 \ln \left( \frac{\psi_- + \psi_+ e^{-\tau\zeta}}{2\zeta} \right) \right]$$

$$B(k, \tau) = -(k^2 - uik) \frac{1 - e^{-\tau\zeta}}{\psi_- + \psi_+ e^{-\tau\zeta}}$$

where  $\psi_{\pm} = \mp(b - \rho\sigma_v ik) + \zeta$  and  $\zeta = \sqrt{(b - \rho\sigma_v ik)^2 + \sigma_v^2(k^2 - iuk)}.$

### 2. Jumps :

- Merton model : constant jump rate intensity and log-normal jump size distribution  
 $C(k, \tau) = 0, D(k, \tau) = \tau\Lambda(k)$  where  
 $\Lambda(k) = e^{(m+Iv^2)ik - v^2 k^2/2 + I(m+v^2/2)} - 1 + (ik + I)(e^{m+v^2/2} - 1).$

## 3 Numerical integration

In order to compute the infinite integrals, needed in the pricing formulas, we use the approximation:

$$\int_0^\infty f(x)dx \simeq \sum_{j=0}^N \int_{jh}^{(j+1)h} f(x)dx.$$

The number  $N$  of the sub-integrals used is determined when the contribution of the last strip  $[jh, (j+1)h]$  is smaller than a given tolerance  $\epsilon$ . Each sub-integral  $\int_{jh}^{(j+1)h} f(x)dx$  is computed using a Gaussian quadrature.

## 4 Implementation of the pricing routines

### 4.1 The `svj.c` file

This file contains the pricing routines, giving the price of an european call or put in the Merton/Heston/Merton+Heston models. Any of the two pricing formulas presented in Sections 1 and 2 can be used.

## 4.2 The `ft_Opt_Model.c` files

*Opt* is the option type, *call* or *put*. *Model* is the model used, *merton* for the Merton model, *heston* for the Heston model and *hestmert* for the combined model Heston+Merton. In each file, we set the option type and the model parameters, next, we call the `calc_price_svj` routine from `svj.c` file. The default pricing method used is the Black-Scholes like formula given in 2.

## 5 Computing the gradients

For each model (Merton/Heston/Merton+Heston) we need to compute the gradient of an option (call/put) price with respect of the different parameters of the model.

The parameters are:  $V_0, \lambda, m$  and  $v$  for Merton model,  $V_0, \kappa, \theta, \sigma_v$  and  $\rho$  for Heston model, and  $V_0, \kappa, \theta, \sigma_v, \rho, V_0, \lambda, m$  and  $v$  for Heston+Merton model.

These gradients can be called from `grad_ft_Opt_Model.c` files which sets the option type, and the model parameters then calls the `calc_grad_svj` routine from `grad_svj.c`. This routine analytically computes the gradient of the pricing formulas (Sections 1 and 2) with respect of each parameter of the given model. The integration procedure is similaire to the one used from the pricing formulas.

## 6 Calibration

The goal is to find a parameter set  $\alpha = (x_1, \dots, x_n)$  of a model (Heston/Merton/Heston+Merton) fitting a given observed market data (call/put prices or implied volatility surface). For an option with a stike  $K_i$  and a maturity  $T_i$  we notice  $P_i^{obs}$  (resp.  $\sigma_i^{imp,obs}$ ) the observed option price (resp. implied volatility) and  $P_i$  (resp.  $\sigma_i$ ) the model price (resp. implied volatility). In order to measure the distance between the model and the market prices, we define the following norms :

- prices norm:  $f_i^1 = ||P_i^{obs} - P_i||^2$
- relative prices norm:  $f_i^2 = ||\frac{P_i^{obs} - P_i}{P_i^{obs}}||^2$
- implied volatility norm:  $f_i^3 = ||\sigma_i^{imp,obs} - \sigma_i||^2$

The calibration is then the minimization of one of these norms for the considered options:

$$\min_x f^j(x) = \min_x \sum_{i=1}^N w_i f_i^j$$

where  $w_i$  is a ponderation weight option  $i$ . The calibration strategies described in the next section are developed and tested on synthetic data. Even if the case of real market data is different, some conclusions on the behaviour of the models are still valid, and the ideas proposed can be useful for the elaboration of a robust calibration on real market data.

The synthetic market data are generated by pricing different options of strikes  $K_i$  and maturities  $T_i$  with a model having a set of parameters  $x^* = (x_1^*, \dots, x_n^*)$ . Then, starting with a random initial guess  $x^0 = (x_1^0, \dots, x_n^0)$ , the calibration is considered as successful if it can find the parameter  $x^*$ .

A first general remark is that the choice of the norm could be crucial for the success of the calibration. In fact, the form of the surfaces we are optimizing ( $f^1$ ,  $f^2$  or  $f^3$ ) is different from a norm to another: local minimum or flat surface in one case and convex in one other. As expected, the norm  $f^1$  is not appropriate as it gives very different weights for options out of the money or in the money for example. The choice is then to alternate the norms  $f^2$  and  $f^3$ .

One model can have different sensitivities to each one of its parameters. The Heston model, for example, is less sensitive to a variation of  $\kappa$  than to a variation of another parameter. Some parameters can have a conjugate effect: a simultaneous variation of two parameters has less effect than a single variation. This is the case for parameters  $m_0$  and  $v$  in the Merton model. On the strategy proposed is to perform search in subspace of the search space: for example by locking, in the Heston model, all the parameters except  $\kappa$  and then searching only in that direction.

We notice that for Heston model and especially the Heston+Merton model, we can find a parameter  $x \neq x^*$  where  $f^2(x) \simeq f^3(x) \simeq 0$ . In other words, we can find two different sets of parameters giving very close (or even the same) smile surface. This point is important for the stability of the calibration on real market data.

## References

- [1] Artur Sepp. *Pricing European-Style Options under Jump Diffusion Processes with Stochastic Volatility: Applications of Fourier Transform*, Proceedings of the 7th Tartu Conference on Multivariate Statistics, (2004). <http://www.hot.ee/seppar/papers/stochjumpvols.pdf>

## References

- [1] A.Sepp. Pricing european-style options under jump diffusion processes with stochastic volatility: Applications of fourier transform. *Proceedings of the 7th Tartu Conference on Multivariate Statistics*, 2004. 2