Andersen & Brodie's approach for pricing Bermudan options

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1 Dual algorithm for multi-dimensional Bermudan options

1.1 Theoretical aspects

In this part, we describe the dual approach for pricing American options as independently introduced by Rogers [1] and Haugh & Kogan [2].

Proposition 1. *In the dual approach, the following bound holds for the option price Q_t at initial time $t = 0$:*

$$Q_0 = \inf_{\pi \epsilon H} \left\{ \pi_{t_0} + E\left[\max_{t \epsilon \{t_0, \ldots, t_N\}} \left(B\left(0,t\right) \varphi\left(S_t\right) - \pi_t \right) \right] \right\}, \tag{1}$$

where H is the set of martingale processes that satisfy $\left\{ E\left[\sup_{t \epsilon \{t_0, \ldots, t_N\}} |\pi_t| \right] < \infty \right\}$.

Like previously,

$$Q_0 = \sup_{\tau \epsilon \Im_{0,N}} E\left[B\left(0,\tau\right) \varphi\left(S_\tau\right) \right].$$

For any martingale $\pi \epsilon H$,

$$\begin{aligned} Q_0 &= \pi_{t_0} + \sup_{\tau \epsilon \Im_{0,N}} E\left[B\left(0,\tau\right) \varphi\left(S_\tau\right) - \pi_\tau \right] \\ &\leqslant \pi_{t_0} + E\left[\sup_{t \epsilon \{t_0, \ldots, t_N\}} \left(B\left(0,t\right) \varphi\left(S_t\right) - \pi_t \right) \right]. \end{aligned}$$

Taking the infimum over all $\pi \epsilon H$ proves that Q_0 is bounded above by the right-hand side of (1).

The dual problem is then equivalent to finding the best martingale process in H that minimizes the duality gap to the option's true price Q_0.

To find a possible candidate, we can use the Doob-Meyer decomposition of the Snell envelope process $\left(B\left(0,t\right) Q_t\right)_{t \geqslant 0}$, which is the smallest supermartingale that dominates the actualized payoff $\left(B\left(0,t\right) \varphi\left(S_t\right)\right)_{t \geqslant 0}$.

This leads to two processes $(A_t)_{t\geqslant 0}$ an increasing process such that $A_0 = 0$, and $(M_t)_{t\geqslant 0}$ a martingale process, that satisfy:

$$\forall t \geqslant 0, B(0,t) Q_t = M_t - A_t.$$

Hence, using the martingale process $(M_t)_{t\geqslant 0}$, the dual inequality verifies:

$$Q_0 \leqslant Q_0 + E\left[\max_{t\epsilon\{t_0,\ldots,t_N\}} \left(B(0,t)\varphi(S_t) - B(0,t) Q_t - A_t\right)\right] \leqslant Q_0,$$

where the second inequality is given by $\varphi(S_t) \leqslant Q_t$ and $A_t \geqslant 0$.

The equality holds consequently when we take the martingale part of the true price process $(B(0,t) Q_t)_{t\geqslant 0}$.

1.2 Basic principle

Andersen & Broadie [3] proposed a way to use a primal algorithm as a basis to build the martingale π.

Thus let's consider a sub-optimal exercice strategy obtained with one of the previous primal procedures.

According to this strategy, at each exercise date we build the indicator process $(l_t)_{t\epsilon\{t_0,\ldots,t_N\}}$ which is equal to 1 when the strategy advises to exercise at time t, 0 if it asks to continue.

We introduce the sequence of stopping times $(\tau_t)_{t\epsilon\{t_0,\ldots,t_N\}}$, defined by:

$$\tau_t = \inf\{u \in \{t_0,\ldots,t_N\} \cap [t,T] : l_u = 1\}. \tag{2}$$

Then we consider the lower bound price process $(L_t)_{t\epsilon\{t_0,\ldots,t_N\}}$ according to this strategy, such that:

$$B(0,t) L_t = E\left[B(0,\tau_t)\varphi(S_{\tau_t})|\mathcal{F}_t\right].$$

The process $(L_t)_{t\epsilon\{t_0,\ldots,t_N\}}$ is used to extract the martingale π thanks to the following steps: $\pi_{t_0} = L_{t_0}$, and for $t_1 \leqslant t_j \leqslant t_N$,

$$\begin{aligned}\pi_{t_j} &= \pi_{t_{j-1}} + B(0,t_j) L_{t_j} - B(0,t_{j-1}) L_{t_{j-1}} \\ &\quad - l_{t_{j-1}} E\left[B(0,t_j) L_{t_j} - B(0,t_{j-1}) L_{t_{j-1}}|\mathcal{F}_{t_{j-1}}\right].\end{aligned}$$

One can easily check the process $(\pi_t)_{t_0\leqslant t\leqslant t_N}$ is a martingale, as we have:

- If $l_{t_{j-1}} = 0$, i.e. the strategy indicates a continuation at time t_{j-1}, $\tau_{t_{j-1}} \geqslant t_j$ and by (2) $\tau_{t_{j-1}} = \tau_{t_j}$.

  This leads to:

$$\begin{aligned}B(0,t_{j-1}) L_{t_{j-1}} &= E\left[B\left(0,\tau_{t_{j-1}}\right)\varphi\left(S_{\tau_{t_{j-1}}}\right)|\mathcal{F}_{t_{j-1}}\right] \\ &= E\left[E\left[B\left(0,\tau_{t_{j-1}}\right)\varphi\left(S_{\tau_{t_{j-1}}}\right)|\mathcal{F}_{t_j}\right]|\mathcal{F}_{t_{j-1}}\right] \\ &= E\left[B(0,t_j) L_{t_j}|\mathcal{F}_{t_{j-1}}\right].\end{aligned}$$

  So, $\pi_{t_{j-1}} = E\left[\pi_{t_j}|F_{t_{j-1}}\right]$: π is a martingale process in the continuation zone;

- If the strategy recommends to exercise at time t_{j-1} (i.e. $l_{t_{j-1}} = 1$),

$$\begin{aligned} E\left[\pi_{t_j}|F_{t_{j-1}}\right] &= \pi_{t_{j-1}} + E\left[B\left(0,t_j\right)L_{t_j} - B\left(0,t_{j-1}\right)L_{t_{j-1}}|\mathcal{F}_{t_{j-1}}\right] \\ &\quad -E\left[B\left(0,t_j\right)L_{t_j} - B\left(0,t_{j-1}\right)L_{t_{j-1}}|\mathcal{F}_{t_{j-1}}\right] \\ &= \pi_{t_{j-1}}. \end{aligned}$$

As we previously set $\pi_{t_0} = L_{t_0}$, the martingale process π satisfies the dual inequality:

$$Q_0 \leqslant L_{t_0} + E\left[\max_{t\epsilon\{t_0,...,t_N\}}\left(B\left(0,t\right)\varphi\left(S_t\right) - \pi_t\right)\right].$$

Finally, we introduce $\Delta_0 = E\left[\max_{t\epsilon\{t_0,...,t_N\}}\left(B\left(0,t\right)\varphi\left(S_t\right) - \pi_t\right)\right]$, the duality gap associated with the martingale process π.

1.3 Pricing algorithm

To compute Δ_0, Andersen & Broadie recommend using Monte-Carlo simulations in order to build the martingale π.

We choose to estimate the lower bound $\widehat{L}_{t_0}$ of the true option's price thanks to the Lonsgtaff-Schwartz algorithm, from which we keep in memory the set of regression factors $\alpha^j = (\alpha_l^j, 1 \leqslant l \leqslant k) \in \mathbb{R}^k$ obtained for each time step t_j where $1 \leqslant j \leqslant N-1$.

As proposed by Andersen & Broadie, for a path $\left(S_{t_j}\right)_{0\leqslant j\leqslant N}$ of the underlying assets, independent of the ones used in the Lonsgtaff-Schwartz procedure, the suboptimal strategy is then defined by:

$$\tau_t = \inf\left\{u \in \{t_0,...,t_N\} \cap [t,T] : \varphi\left(S_u\right) > \widetilde{CV}_u\right\},$$

where we choose to directly approximate at time $t_1 \leqslant t_j \leqslant t_{N-1}$ the option's continuation value in the regression basis $g^j = (g_l^j, 1 \leqslant l \leqslant k) \in \mathbb{R}^k$ by $\widetilde{CV}_{t_j} = \langle \alpha^j, g^j \rangle \left(S_{t_j}\right)$ and set $\widetilde{CV}_{t_N} = 0$.

The upper bound is computed by the following algorithm:

1. Simulate a trajectory of the d stocks:

$$S_0 = \left(S_{t_0}^1, ..., S_{t_0}^d\right), ..., S_N = \left(S_{t_N}^1, ..., S_{t_N}^d\right);$$

2. Set $\pi_{t_0} = \widehat{L}_{t_0}$, then for each exercise date $t_1 \leqslant t_j \leqslant t_N$:

   - Compute $L_{t_j} = \max\left(\widetilde{CV}_{t_j}, \varphi\left(S_{t_j}\right)\right)$ and memorize it;

- If $\varphi\left(S_{t_{j-1}}\right) \leqslant \widetilde{CV}_{t_{j-1}}$ (case $l_{t_{j-1}} = 0$), compute:

$$\pi_{t_j} = \pi_{t_{j-1}} + B\left(0, t_j\right) L_{t_j} - B\left(0, t_{j-1}\right) L_{t_{j-1}};$$

- Otherwise (case $l_{t_{j-1}} = 1$), calculate $E\left[B\left(0, t_j\right) L_{t_j} | S_{t_{j-1}}\right]$.
  Here, we sample $N_{internal}$ one step paths of S_{t_j}, starting from $S_{t_{j-1}}$.
  Then, for each inner simulation, we use again the regression factors to approximate the option's continuation value $\widetilde{CV}_{t_j}^m$ at time t_j and compute the option's lower bound value $L_{t_j}^m = \max\left(\widetilde{CV}_{t_j}^m, \varphi\left(S_{t_j}^m\right)\right)$ for each internal trajectory $1 \leqslant m \leqslant N_{internal}$.
  We finally evaluate $E\left[B\left(0, t_j\right) L_{t_j} | S_{t_{j-1}}\right]$ with its standard estimator:

$$E\left[B\left(0, t_j\right) L_{t_j} | S_{t_{j-1}}\right] \simeq \frac{1}{N_{internal}} \sum_{m=1}^{N_{internal}} B\left(0, t_j\right) L_{t_j}^m.$$

  Then compute:

$$\pi_{t_j} = \pi_{t_{j-1}} + B\left(0, t_j\right) L_{t_j} - \frac{1}{N_{internal}} \sum_{m=1}^{N_{internal}} B\left(0, t_j\right) L_{t_j}^m;$$

3. Finally, compute the dual gap for the trajectory:

$$\max_{t \epsilon \{t_0, \ldots, t_N\}} \left(B\left(0, t\right) \varphi\left(S_t\right) - \pi_t\right);$$

4. Repeat steps (1)-(2)-(3) for N_{dual} independent trajectories indexed by i and estimate Δ_0 thanks to the Monte-Carlo estimator $\widehat{\Delta}_0$:

$$\widehat{\Delta}_0 = \frac{1}{N_{dual}} \sum_{i=1}^{N_{dual}} \left[\max_{t \epsilon \{t_0, \ldots, t_N\}} \left(B\left(0, t\right) \varphi\left(S_t^i\right) - \pi_t^i\right)\right].$$

1.4 Algorithm improvements

As introduced by Braodie & Cao [4], several improvements of the dual algorithm are implementable to speed up the basic dual procedure.

1.4.1 Sub-optimality checking

In the previous section, we choose to use the regression basis generated by the Longstaff-Schwartz algorithm to compute the approximations of the option's continuation value at each time step.

To limit the impact of these approximations, which can directly imply incorrect launches of inner simulations, we use a theoretical lower bound of the option's continuation value to prevent some of these incorrect decisions.

We consider logically the corresponding option's European price E_t as a lower bound of the continuation value. If the European price is luckily given by a close formula or by close form approximations, the computation cost will also be negligible in comparason to the calculation time saved.

Moreover, one can generally use the value of any option which is dominated by the American option or the maximum among the values of all dominated options.

We now introduce the adjusted approximate continuation value, defined by:

$$\underline{\widetilde{CV}}_t = \max\left(\widetilde{CV}_t, E_t\right), \forall t \epsilon \{t_0, ..., t_N\},$$

where E_t is the European lower bound at time t and $\widetilde{CV}_t$ the approximation of the option's continuation value as introduced in the previous section.

As a consequence, the indicator process $(l_t)_{t\epsilon\{t_0,...,t_N\}}$ in the basic dual algorithm is now turned into:

$\forall t \in \{t_0, ..., t_N\}$:

- $l_t = 1$ if $\varphi(S_t) > \underline{\widetilde{CV}}_t$;
- $l_t = 0$ otherwise.

Even if this strategy is simple and generates sub-optimal exercise rules, it limits the exercise strategy misunderstandings due to errors of approximations. This brings strong computational improvements especially for OTM and ATM options.

For example, two kinds of payoff are available in PREMIA for American options on a basket of stocks:

- Call or Put on the average of a basket: we use the approximation of the corresponding European price generated by the formulas of Carmona & Durrleman;
- Call on the maximum of a basket: as proposed by Braodie & Cao, we find a lower bound to the continuation value by averaging the prices of the corresponding single stock European vanilla calls of the basket.

1.4.2 Boundary distance grouping

Definition of an alternative dual estimator

Broadie & Cao proposed another procedure to reduce the computation time of the dual algorithm.

They try to quantify the distance of an option to the "exercise boundary". We define this boundary as the surface in the state space where the option holder, based on the exercise policy, is indifferent between holding and exercising the option. Mathematically spreaking, this is the set of states at which the adjusted approximate continuation value is equal to the exercise payoff $\left\{\omega_t, \underline{\widetilde{CV}}_t = \varphi(S_t)\right\}$.

We introduce the distance measure $d_t = \left|\underline{\widetilde{CV}}_t - \varphi(S_t)\right|$, which determines how the option is close to the "exercise boundary".

In the boundary distance grouping procedure, Broadie & Cao separate the simulated paths in two groups, according to the distance of each path to the "exercise boundary". Intuitively, simulations such that the distance to the boundary gets smaller than a threshold during the option's life are placed into the group $\overline{Z}$ called 'non-zero', because one can hope their contribution to the upper bound increment is not null. All other simulated paths are placed into the 'zero' group Z.

The two groups are consequently defined by:

$$\begin{cases} Z = \left\{\omega : \forall t \in \Gamma, d_t(\omega) \geqslant \delta \text{ or } \varphi(S_t) \leqslant \widetilde{\underline{CV}_t}\right\} \\ \overline{Z} = \left\{\omega : \exists t \in \Gamma, d_t(\omega) < \delta \text{ and } \varphi(S_t) > \widetilde{\underline{CV}_t}\right\} \end{cases}.$$

We separate the sample paths between the two groups:

$$\omega_1, ..., \omega_{n_{\overline{Z}}} \in \overline{Z} \text{ and } \omega_{n_{\overline{Z}}+1}, ..., \omega_{N_{Dual}} \in Z.$$

And we also introduce $p_{\overline{Z}}$, the probability that a sample path belongs to group 'non-zero',

$$p_{\overline{Z}} = P\left(\omega \in \overline{Z}\right) = P\left(\left\{\omega : \exists t \in \Gamma, d_t(w) < \delta \text{ and } \varphi(S_t) > \widetilde{\underline{CV}_t}\right\}\right),$$

and $\mu_Z = E[\Delta_i|Z], \sigma_Z^2 = Var[\Delta_i|Z], \mu_{\overline{Z}} = E\left[\Delta_i|\overline{Z}\right], \sigma_{\overline{Z}}^2 = Var\left[\Delta_i|\overline{Z}\right]$.
Braodie & Cao propose an alternative estimator to evaluate the dual increment Δ_0:

$$\widetilde{\Delta}_0 = \frac{1}{N_{dual}}\left(\sum_{i=1}^{n_{\overline{Z}}}\Delta_i + \frac{N_{dual} - n_{\overline{Z}}}{l_Z}\sum_{i=n_{\overline{Z}}+1}^{n_{\overline{Z}}+l_Z}\Delta_i\right),$$

where $n_{\overline{Z}}$ is the number of trajectories in the group $\overline{Z}$ and l_Z a random number of paths chosen in "zero" group such that $l_Z \ll N_{dual} - n_{\overline{Z}}$ (for example, we consider the l_Z first paths belonging to group Z).

The simulation procedure for the dual algorithm, taking into account the boundary distance grouping procedure, is defined as follows:

1. **Preliminairy step:** generate N_{test} sample paths and compute dual increments thanks to the original dual algorithm. For each test distance δ, estimate $\mu_Z, \sigma_Z^2, \mu_{\overline{Z}}, \sigma^2{}_{\overline{Z}}$ and $p_{\overline{Z}}$ to get l_Z^*, the optimal number of trajectories in groupe Z to be included in the calculation of $\widetilde{\Delta}_0$; then choose the distance δ^* that optimizes the efficient measure;

2. Simulate a trajectory of the d stocks:

$$S_0 = \left(S_{t_0}^1, ..., S_{t_0}^d\right), ..., S_N = \left(S_{t_N}^1, ..., S_{t_N}^d\right);$$

3. Estimate the boundary distance d_t at each exercise opportunity, if $\exists t \epsilon \{t_0, ..., t_N\}$ such that $d_t < \delta^*$ and $\varphi(S_t) > \widetilde{\underline{CV}_t}$, the path is assigned to group $\overline{Z}$, otherwise put it in group Z;

4. If the considered path belongs to group $\overline{Z}$ or among the first l_Z^* trajectories of group Z, estimate the increment Δ_i at the upper bound with the dual algorithm, otherwise skip the simulation;

5. Repeat steps (2)-(3)-(4) for N_{dual} sample trajectories and estimate $\widetilde{\Delta}_0$ using the new formula.

Even if this new algorithm asks a prelimary step that can be very time consuming as we don't have an a priori knowledge about the set of test distances δ, it can bring dramatic computational improvements for ITM options, especially when the preliminary step reaches a proper distance threshold rapidly.

Preliminary algorithm

To measure the efficiency of their new algorithm, Braodie & Cao introduce an effective saving factor for the boundary distance grouping procedure. It can be calculated as the ratio of the efficiency before and after improvement, where the efficiency of simulation is defined as the product of sample variance and computational time.

We set T_P, the expected time spent for generating one simulated path, T_I is the expected time to identify the group the path belongs to, T_{Δ_Z} and $T_{\Delta_{\overline{Z}}}$ are the expected times to estimate dual increment Δ_i from group Z or $\overline{Z}$. One can typically suppose that T_I, $T_P \ll T_{\Delta_Z}$, $T_{\Delta_{\overline{Z}}}$.

The total expected time for estimating $\widetilde{\Delta}_0$ is equal to:

$$\begin{aligned} T_{\widetilde{\Delta}_0} &\approx N_{dual}T_P + N_{dual}T_I + N_{dual}p_{\overline{Z}}T_{\Delta_{\overline{Z}}} + l_Z T_{\Delta_Z} \\ &= N_{dual}\left[T_P + T_I + p_{\overline{Z}}T_{\Delta_{\overline{Z}}}\right] + l_Z T_{\Delta_Z}. \end{aligned}$$

For a fixed boundary distance threshold $\delta > 0$, parameters $\mu_Z, \sigma_Z^2, \mu_{\overline{Z}}, \sigma^2{}_{\overline{Z}}$ et $p_{\overline{Z}}$ can be estimated by simulation.

We now maximize the effective saving factor w.r.t. l_Z for a first fixed test distance δ and find an optimal δ^* from a set of choices selected before:

$$\delta^* = \arg\left(\min_{\delta} Var\left[\widetilde{\Delta}_0\right] \times T_{\widetilde{\Delta}_0}\right).$$

Broadie & Cao show that the variance of the alternative estimator to the dual increment $\widetilde{\Delta}_0$ satisfies:

$$\begin{aligned} Var\left[\widetilde{\Delta}_0\right] &= \frac{1}{N_{dual}}\sigma_{\overline{Z}}^2 p_{\overline{Z}} + \frac{1}{l_Z}\left(1-p_{\overline{Z}}\right)^2\sigma_Z^2 + \frac{1}{l_Z N_{dual}}p_{\overline{Z}}\left(1-p_{\overline{Z}}\right)\sigma_Z^2 \\ &\quad + \frac{1}{N_{dual}}\left(\mu_{\overline{Z}}-\mu_Z\right)^2 p_{\overline{Z}}\left(1-p_{\overline{Z}}\right), \end{aligned}$$

approximated by,

$$Var\left[\widetilde{\Delta}_0\right] \approx \frac{1}{N_{dual}}\sigma_{\overline{Z}}^2 p_{\overline{Z}} + \frac{1}{l_Z}\left(1-p_{\overline{Z}}\right)^2\sigma_Z^2 + \frac{1}{N_{dual}}\left(\mu_{\overline{Z}}-\mu_Z\right)^2 p_{\overline{Z}}\left(1-p_{\overline{Z}}\right).$$

Consequently, we obtain the following formula:

$$\begin{aligned} Var\left[\widetilde{\Delta}_0\right] \times T_{\widetilde{\Delta}_0} &= \frac{l_Z T_{\Delta_Z}}{N_{dual}}\sigma_{\overline{Z}}^2 p_{\overline{Z}} + \frac{N_{dual}}{l_Z}\left[T_P + T_I + p_{\overline{Z}}T_{\Delta_{\overline{Z}}}\right]\left(1-p_{\overline{Z}}\right)^2\sigma_Z^2 \\ &\quad + \frac{l_Z T_{\Delta_Z}}{N_{dual}}\left(\mu_{\overline{Z}}-\mu_Z\right)^2 p_{\overline{Z}}\left(1-p_{\overline{Z}}\right) + C, \end{aligned}$$

where C is a constant parameter independent of l_Z.
Let's take the partial derivative w.r.t. l_Z,

$$\begin{aligned}\frac{\partial\left(Var\left[\widetilde{\Delta}_0\right]\times T_{\widetilde{\Delta}_0}\right)}{\partial l_Z} &= \frac{T_{\Delta_Z}}{N_{dual}}\sigma^2_{\overline{Z}}p_{\overline{Z}}-\frac{N_{dual}}{l^{*2}_Z}\left[T_P+T_I+p_{\overline{Z}}T_{\Delta_{\overline{Z}}}\right]\left(1-p_{\overline{Z}}\right)^2\sigma^2_Z\\ &\quad+\frac{T_{\Delta_Z}}{N_{dual}}\left(\mu_{\overline{Z}}-\mu_Z\right)^2p_{\overline{Z}}\left(1-p_{\overline{Z}}\right),\end{aligned}$$

which achieves a unique minimum at:

$$l^*_Z=\sqrt{\frac{\left(1-p_{\overline{Z}}\right)^2\sigma^2_{\overline{Z}}\left[T_P+T_I+p_{\overline{Z}}T_{\Delta_{\overline{Z}}}\right]}{p_{\overline{Z}}T_{\Delta_Z}\left[\sigma^2_{\overline{Z}}+\left(\mu_{\overline{Z}}-\mu_Z\right)^2\left(1-p_{\overline{Z}}\right)\right]}}N_{dual}.$$

As we have already estimated the parameters $\mu_Z, \sigma^2_Z, \mu_{\overline{Z}}, \sigma^2{}_{\overline{Z}}$ et $p_{\overline{Z}}$, we have access to l^*_Z for each fixed distance threshold $\delta > 0$.

We finally choose the first distance δ for which we find the level α inferior to a choosen threshold (typically 20%) which effectively brings improvements to the dual algorithm.

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