# A dynamic approach to the modelling of correlation credit derivatives using Markov chains

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Premia 22

## 1 Model specificatoin

We consider a portfolio of $N$ defaultable securities and assume that there exists a continuous time finite-state irreducible Markov chain $(\xi_t)_{t\geq 0}$ with infintesimal generator matrix $Q$, generating a filtration $\mathcal{F}_t^\xi$. Assume that conditional on the path of the chain, defaults of the $N$ names will be independent, the survival probability of the $i^{th}$ reference entity being given by

$$q_t^i = \mathbb{P}(\tau^i \geq t | \mathcal{F}_t^\xi) = exp(-C_t^i), \tag{1}$$

where $C_t^i$ is some additive functional of the chain of the form

$$C_t^i = \int_0^t \lambda^i(\xi_u)du + \sum_{j\neq k} w_{jk}^i J_{jk}(t).$$

Here, $\tau^i$ is the default time of the $i^{th}$ name in the portfolio, $\lambda^i$ is a deterministic function of the chain, $J_{jk}(t)$ denotes the number of jumps by time $t$ from state $j$ to state $k$, and the $w_{jk}^i$ are non-negative weights.

In order to gain some intuition, one could think of the chain as representing the state of health of the economy. If the chain jumps from a state of economic growth to a state of recession, this may cause the conditional default intensity of some of the reference entities to go up, increasing the chances of observing a larger number of defaults in the portfolio. Note that the information about how the various credits in the portfolio are **correlated** is contained in the $\lambda^i$, the $w_{jk}^i$, and $Q$. (See section 3 of [1] for the derived expression of the default correlation). In the following, we assume that the money market account takes the following form

$$B_t = exp\Big(\int_0^t r(\xi_u)du\Big),$$

where $r$ is a deterministic function of the chain.

**Remark 1.** • *Note that the vectors $\lambda^i(.), r$, the matrix $w$ and the infinitesimal generator $Q$ are seen as parameters of the problem and are calibrated to market data.*
• *One of the nice features of the model is that the number of parameters can be ajusted, by modifying the number of the chain states to best reflect the availability of market data. As CDO markets become more liquid, a higher number of quotes are likely to become available. By increasing the number of parameters we are more likely to capture the extra information available in the market. In our code we just consider 4 states of the Markov chain.*

In order to price derivatives on a portfolio of $N$ defaultable securities, we need to be able to find the distributions of some non trivial random variable. If $l_i = A_i(1 - R_i)$ denotes the loss on the $i^{th}$ name, in terms of the notional $A_i$ and the (possibly random) recovery rate $R_i$ , then the portfolio cumulative loss process is given by

$$L_t = \sum_{i=1}^N l_i \mathbf{1}_{\{\tau_i \leq t\}}.$$

By conditioning firstly on the path of the chain, it is easy to see that the (discounted) Laplace transform of $L_t$ is given by:

$$\mathbb{E}exp\Big(-\int_0^t r(\xi_s)ds - \alpha L_t\Big) = \mathbb{E}\Big[exp\Big(-\int_0^t r(\xi_s)ds\prod_{i=1}^N\big((1-q_t^i)\zeta_i(\alpha)+q_t^i\big)\Big],$$

where $\zeta_i(\alpha) = \mathbb{E}\exp(-\alpha l_i)$.

# 2 Poisson Computational approximation

The expression (1) for the survival probability of name $i$ can be understood in terms of a standard Poisson process $\nu$ independent of the chain $\xi$. If the jump times of $\nu$ are denoted $S_1 < S_2 < \dots$, then we may set

$$\tau^i = \inf\{t : C_t^i > S_1\},$$

and then the relation (1) holds. The Poisson approximation we propose here is to allow name i to default more than once, at times

$$\tau_m^i = \inf\{t : C_t^i > S_m\}, \quad m = 0, 1, \dots.$$

By doing this, we arrive at an expression $L_t$ for the portfolio cumulative loss which overestimates $L_t$ , because it includes (non-existent) second and subsequent losses of each of the names. The error we are committing by this is of the same order as the default probabilities themselves; typically this would be of the order of a few percent, which would be comparable to the error we could expect from a Monte Carlo approach. However, there is some simple trick we can employ to improve the approximation. The expected (discounted) number of losses for name $i$ by time t using the Poisson method is given by $\mathbb{E}[B_t^{-1}C_t^i]$ compared with a true value of $\mathbb{E}[B_t^{-1}(1-\exp(-C_t^i))]$. So if we define

$$\beta_t^i = \frac{\mathbb{E}[B_t^{-1}(1-\exp(-C_t^i))]}{\mathbb{E}[B_t^{-1}C_t^i]} \tag{2}$$

we can get a fairly good approximation for the Laplace transform of the cumulative loss by letting

$$\mathbb{E}exp\Big(-\int_0^t r(\xi_s)ds - \alpha L_t\Big) = \mathbb{E}\Big[exp\Big(-\int_0^t r(\xi_s)ds + \sum_{i=1}^N \beta_t^i(\zeta_i(\alpha)-1)C_t^i\Big)\Big], \tag{3}$$

This is the key relation linking this modelling approach to the kinds of calculation needed to price credit derivatives of various sort.

# 3 Synthetic CDOs

In the following, we assume that the credit portfolio is homogenous. Let $B$ and $A$ be the upper and lower attachment points of the tranche respectively. At each payment date, investors receive a coupon which is proportional to the notional of the tranche, net of the losses suffered by the credit portfolio up to that point.

## 3.1 Premium leg

The premium leg is equal to

$$pl = \sum_{j=0}^M \Delta_j \mathbb{E}\Big[B_{T_j}^{-1}\Phi(L_{T_j})\Big],$$

where

$$\Phi(x) = \frac{1}{B-A}\Big[(B-x)_+ - (A-x)^+\Big]$$

and $M$ is the number of total payments occurring at dates $T_1, \dots, T_M$. In order to evaluate the $pl$, we need to calculate the price of a portfolio of put options on the portfolio cumulative losses at each payment date $T_j$. In particular, $pl$ is the difference of two put options of the form

$$pl = \sum_{j=0}^M \Delta_j(P_{T_j}(B) - P_{T_j}(A)),$$

where $P_t(x) = \mathbb{E}\Big[B_t^{-1}(K - L_t)_+\Big]$. Standard computations shows that the laplace transform $\hat{P}_t(.)$ of $P_t(.)$ is given by

$$\hat{P}_t(\alpha) = \frac{1}{\alpha^2}\mathbb{E}exp\Big(-\int_0^t r(\xi_s)ds - \alpha L_t\Big) \tag{4}$$

All that remains to do is to compute $P_{T_i}(B)$ and $P_{T_i}(A)$ for $1 \leq j \leq M$ by inverting the corresponding Laplace transforms $\hat{P}_{T_j}$. By expressions (3), (4) and using standard formulas for Markov chain processes (see [2]) we prove that

$$\hat{P}_t(\alpha) =< V_t(\alpha), \Pi >, \quad \text{where} \quad <.,.> \quad \text{denotes the inner product}$$

with $V_t(\alpha) = \exp(\tilde{Q}t).\mathbf{1}$, ($\mathbf{1} = (1,1,1,1)$) and $\Pi$ is the initial distribution of the Markov chain. Here the matrix $\tilde{Q}$ is obtained from the infintesimal generator $Q$ of the Markov chain $\xi$ using the following transformation:

$$\tilde{Q}_{ii} = Q_{ii} - \nu_i, \quad \tilde{Q}_{ij} = \exp(-\tilde{w}_{ij})Q_{ij}, \quad \text{for } i \neq j,$$

where $\nu_j = N\beta_t(1-\zeta(\alpha))\lambda$, $\tilde{w}_{ij} = Nw_{ij}\beta_t(1-\zeta(\alpha))$.
The function computing $\hat{P}_t(\alpha)$ is called `double Laplace_transform`. Concequently, in order to compute $P_t(\alpha)$ we have just to compute the inverse laplace transform of the computed quantity $\hat{P}_t(\alpha)$. The function computing $P_t(\alpha)$ is called `double InverseTransform`. Then, it's easy to compute the premium leg using that

$$pl = \sum_{j=0}^{M} \Delta_j (P_{T_j}(B) - P_{T_j}(A)).$$

## 3.2 Default leg

Concerning the default leg, it can be easily shown that

$$dl = 1 - \mathbb{E}\Big[\exp(B_T^{-1})\Phi(L_T)\Big] - \mathbb{E}\Big[\int_0^T rB_u^{-1}\Phi(L_u)du\Big].$$

Again all the quantities in the above expression can be calculated explicitly. Note that the basic elements needed to calculate the default leg are the same as the ones we derived when calculating the premium leg, with some minor modi¿cation to account for the term appearing in the second expectation of the above quantity. The time integral appearing in the last term of the above expression can be approximated by standard quadrature methods. The tranche spread is recovered as usual by dividing the default leg by the premium leg.

## 3.3 Calibration

The model is calibrated to tranches on the CDX (series 7) index (mid levels) for 4 consecutive business days from November 1st to November 6th 2006. We implemented in Premia only the data model for the first November and here are the used data:

$$Q = \begin{pmatrix} -0.0069 & 0.0000 & 0.0004 & 0.0065 \\ 0.0179 & -0.0180 & 0.0001 & 0.0000 \\ 0.0000 & 0.0000 & -0.4291 & 0.4291 \\ 0.0000 & 1.2835 & 0.0014 & -1.2849 \end{pmatrix},$$

$$w = \begin{pmatrix} 0.0000 & 9.3981 & 0.12770 & 14.8746 \\ 0.0000 & 0.0000 & 10.0362 & 19.6856 \\ 9.1688 & 7.5897 & 0.0000 & 0.00000 \\ 6.4959 & 0.0009 & 0.74070 & 0.0000 \end{pmatrix},$$

$$\lambda = (0.0545, 0.0134, 0.0000, 0.0007), \ \Pi = (0.0019, 0.0, 0.9981, 0.0)$$

and the recovery $R = 04701$.

# References


[1] DiGraziano, G. & Rogers, C. (2006), A dynamic approach to the modelling of correlation credit derivatives using Markov chains, working paper, Statistical Laboratory, University of Cambridge. 1

[2] L.C.G. Rogers and D. Williams: Diffusions, Markov Processes and Martingales: Volume 2 , Itô Calculus, Cambridge University Press, Cambridge, 2000. 3