Computing XVA with Finite Elements combined with characteristic method

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Premia 22

1 Problem formulation

We consider the following model for the stock price:

$$\begin{cases} dS_t &= (r-q)S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t &= \alpha(\beta - \nu_t)dt + \omega\sqrt{\nu_t} dW_t^\nu, \end{cases}$$

with α, β and $\omega \in \mathbb{R}_+$, where W_t^S and W_t^ν are Gaussian processes, where r is the interest rate, q is a foreign interest or dividend, and with the correlation between the two implied Gaussian processes given by

$$\langle dW_t^S, dW_t^\nu \rangle = \rho dt$$

We also consider two counter-parties, a seller B and a buyer C, with zero recovery bond price, such that

$$\begin{cases} dP_{B_t} &= r_{P_B}(t)P_{B_t}dt - P_{B_t}dJ_t^B \\ dP_{C_t} &= r_{P_C}(t)P_{C_t}dt - P_{C_t}dJ_t^C \end{cases}$$

where J_t^B and J_t^C are two independent jump processes that may jump from 0 to 1 on default of B or C, respectively.

Using a martingale approach for an European or an American option (call or put), we can prove that the price is given by the solution of the following partial differential equation

The value of the defaultable derivative, $\hat{V}(t, S_t, \nu_t, J_t^B, J_t^C)$, includes the various adjustments, whereas the value without default risk, $V(t, S_t, \nu_t)$, does not include any counterparty adjustment and equals the well-known Heston PDE derivative value.

Under these assumptions, we solve the following PDE

$$\begin{cases} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} &= s_F M^+ \\ -\lambda_B(M^+ + R_B M^-) - \lambda_C(M^- + R_C M^+) & \\ \hat{V}(T, S, \nu) &= H(S) \end{cases}$$

where s_F represents the funding costs of the entity, $\lambda_B = r_{P_B} - r$, $\lambda_C = r_{P_C} - r$ and M the mark-to-market value, the differential operator $\mathcal{A}$ is given by

$$\mathcal{A}V = \frac{1}{2}S^2\nu\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\omega^2\nu\frac{\partial^2 V}{\partial \nu^2} + \rho\omega S\nu\frac{\partial^2 V}{\partial \nu \partial S} + (r-q)S\frac{\partial V}{\partial S} + \alpha(\beta-\nu)\frac{\partial V}{\partial \nu}$$

and $H(S)$ denotes the payoff function.

According to the two common scenarios for the choice of the mark-to-market value at default, M, two different PDEs problems are obtained. When $M = \hat{V}$, a nonlinear problem is posed

$$\left\{ \begin{array}{rcl} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - r\hat{V} & = & s_F\hat{V}^+ + \lambda_B(1-R_B)\hat{V}^- + \lambda_C(1-R_C)\hat{V}^+ \\ \hat{V}(T,S,\nu) & = & H(S). \end{array} \right.$$

With $M = V$, a linear problem is deduced

$$\left\{ \begin{array}{rcl} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r+\lambda_B+\lambda_C)\hat{V} & = & s_F\hat{V}^+ \\ -(\lambda_B + R_C\lambda_C)\hat{V}^- - (\lambda_C + R_B\lambda_B)\hat{V}^+ & & \\ \hat{V}(T,S,\nu) & = & H(S). \end{array} \right.$$

Note that the function V is the solution of the classical Heston PE model given by

$$\left\{ \begin{array}{rcl} \dfrac{\partial V}{\partial t} + \mathcal{A}V - rV & = & 0 \\ V(T,S,\nu) & = & H(S). \end{array} \right.$$

Considering $\hat{V} = V + U$, we obtain the problem which models the XVA under the following forms.

With $M = \hat{V}$

$$\left\{ \begin{array}{rcl} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU & = & s_F(V+U)^+ + \lambda_B(1-R_B)(V+U)^- + \lambda_C(1-R_C)(V+U)^+ \\ U(T,S,\nu) & = & 0. \end{array} \right.$$

With $M = V$

$$\left\{ \begin{array}{rcl} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r+\lambda_B+\lambda_C)U & = & s_F V^+ \\ +(\lambda_B - R_C\lambda_C)V^- + (\lambda_C - R_B\lambda_B)V^+ & & \\ U(T,S,\nu) & = & 0. \end{array} \right.$$

Following [2], we define the exposure of an option at a future time $t < T$ by

$$E(t) := \max(U(S_t, V_t, t), 0)$$

where $U(S_t, \nu_t, t)$ is the (mark-to-market) value of financial derivatives contract at time t. The present Expected Exposure at a future time $t < T$ is defined by

$$EE(t) := \mathbb{E}\left[E(t)|\mathcal{F}_0\right]$$

where $\mathcal{F}_0$ is the filtration at time $t = 0$ and the expectation is computed under the risk-neutral measure $\mathbb{Q}$.

Finally the XVA is given by

$$XVA(0,T) := (1 - R_C) \int_0^T EE^*(t) dPD_C(t) + (1 - R_B) \int_0^T EE^*(t) dPD_B(t) + FS \int_0^T EE^*(t) dt$$

where R_B and R_C are the recovery rates form B and C respectively, $D(0,t)$ is the risk-free discount factor and $PD_B(t)$ and $PD_C(t)$ denote the default probability of the counter-party, B and C respectively, at time t.

By default, the initial values are $S_0 = 100$ and $V_0 = 0.01$, the maturity T is one year and the strike value is 100, such that $H(S, \nu, 0) = (b(S - K))^+$ where $b = 1$ for the call and $b = -1$ for the put. In the case of the American options, we should add the possibility to exercice the option before the maturity which is easily implemented in the partial differential equation by taking the maximum compared to the pay-off at any time.

2 Numerical solution methods

In order to solve the previously defined models numerically, various numerical techniques are proposed. We focus on the nonlinear problem, as the linear version can be addressed in a very similar way. First of all, we need to apply a localization procedure to define a suitable finite domain and define appropriate boundary conditions. Moreover the time discretization is based on the Lagrangian method, which is combined with a piecewise linear finite element spatial discretization. Comparing with the literature, finite difference methods, in particular ADI-type methods [1], have been successfully applied to solve the classical Heston PDE. Non-uniform meshes in both spatial directions, S and ν are then used. In particular, a mesh with clustered grid points at $(S, \nu) = (K, 0)$ is built. Here, we will work with a finite element method to solve the PDE, and employ a uniform mesh for simplicity. This set of numerical methods is proposed to solve the nonlinear problem, the solution of which will be the discrete adjustment value, including CVA, DVA and FVA. For the linear problem the same numerical techniques can be applied. Note that the fixed point iterative scheme is not needed in this case.

3 Implementation

The main program fixes the variables and compute the grid in space and variance variables. Next it creates the matrix representing the operator $\mathcal{A}$.

The main core of the program is a loop over the time where we successively

- Build the second member of the problem (using the method of caracteristics)
- Compute the solution without boundary condition of a linear system by calling a LU procedure
- Impose the conditions of the problem
- Compute the solution without boundary condition of a linear system by calling a LU procedure
- Make the same four steps on the XVA vector

References

[1] Karel. J. in 't Hout and S. Foulon. ADI finite difference schemes for option pricing in the Heston model with correlation. International Journal of Numerical Analysis and Modeling, 7:1-27 (2014). 3

[2] Cornelis S. L. de Graaf, Drona Kandhai and Peter M.A. Sloot. Efficient Estimation of Sensitivities for Counterparty Credit Risk with the Finite Difference Monte-Carlo Method. 2