

# Pricing american option under stochastic volatility and stochastic interest rate: implementation in PREMIA

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### Abstract

Using the analytical approach provided in Medvedev and Scaillet (2009), We price the American option in a three factor model with stochastic volatility and stochastic interest rate.

## 1 Introduction

We consider the pricing problem of American option in a three factor model with stochastic volatility and stochastic interest rate. The implementation of this problem is based on the approach given by Medvedev and Scaillet. By introducing an explicit and intuitive proxy for the exercise rule to the American option, they derived an asymptotic expansion for the price of American option price with short maturity.

The rest of this file is as follows: in Section 2 we introduce the model and the modified problem, then we present the solution of the modified problem in Section 3, the program manual of the implementation in PREMIA is given in Section 4.

## 2 Model description and its PDE

The risk-neutral dynamic of the three factor model is given as follows:

$$\begin{aligned}dS_t &= (r_t - \delta)S_t dt + \sigma_t S_t dW_t^{(1)} \\d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)} \\dr_t &= \alpha(r_t, t)dt + \beta(r_t)dW_t^{(3)},\end{aligned}$$

where  $dW_t^{(i)}dW_t^{(j)} = \rho_{ij}dt, i, j = 1, 2, 3$ . From now on we assume that  $\rho_{23} = 0$ .

Then the put option price  $\mathbf{P}(S, \sigma, r, t)$  satisfies the partial differential equation (PDE):

$$\begin{aligned} 0 = & \partial_t \mathbf{P} + \partial_S \mathbf{P} S(r - \delta) + \partial_\sigma \mathbf{P} a(\sigma) + \partial_r \mathbf{P} \alpha(r, t) + \frac{1}{2} \partial_{SS}^2 \mathbf{P} S^2 \sigma^2 \\ & + \frac{1}{2} \partial_{\sigma\sigma}^2 \mathbf{P} b^2(\sigma) + \frac{1}{2} \partial_{rr}^2 \mathbf{P} \beta^2(r) + \partial_{S\sigma}^2 \mathbf{P} \sigma S b(\sigma) \rho_{12} \\ & + \partial_{Sr}^2 \mathbf{P} \sigma S \beta(\sigma) \rho_{13} + \partial_{\sigma r}^2 \mathbf{P} b(\sigma) \beta(\sigma) \rho_{23} - r \mathbf{P}. \end{aligned} \quad (2.1)$$

with boundary conditions:

$$\begin{aligned} \mathbf{P}(\infty, \sigma, r, t) &= 0, \quad \mathbf{P}(S, \sigma, r, T) = \max\{K - S, 0\}, \\ \mathbf{P}(\bar{S}(T - t), \sigma, r, t) &= \max\{K - \bar{S}(T - t), 0\}, \quad \partial_S \mathbf{P}(\bar{S}(T - t), \sigma, r, t) = -1, \end{aligned} \quad (2.2)$$

where we use notation  $\mathbf{P} := \mathbf{P}(S, \sigma, r, t)$  for convenience in (2.1).

Change variables of  $(S, \sigma, r, t)$  to  $(\theta, \sigma, r, \tau)$  with  $\theta = \frac{\log(K/S)}{\sigma\sqrt{T-t}}$  and  $\tau = T - t$ , then denote the put option price  $\mathbf{P}(S, \sigma, r, t)$  as  $P(\theta, \sigma, r, \tau)$ , make use of the following relationships:

$$\begin{aligned} \partial_t \mathbf{P} &= \frac{1}{2} \frac{\theta}{\tau} \partial_\theta P - \partial_\tau P, & \partial_S \mathbf{P} &= -\frac{\partial_\theta P}{\sigma S \sqrt{\tau}} \\ \partial_\sigma \mathbf{P} &= -\frac{\theta}{\sigma} \partial_\theta P + \partial_\sigma P, & \partial_r \mathbf{P} &= \partial_r P, \\ \partial_{SS}^2 \mathbf{P} &= \frac{\partial_{\theta\theta}^2 \mathbf{P}}{\sigma^2 S^2 \tau} + \frac{\partial_\theta P}{\sigma S^2 \sqrt{\tau}}, \\ \partial_{\sigma\sigma}^2 \mathbf{P} &= \partial_{\sigma\sigma}^2 P - \frac{2\theta}{\sigma} \partial_{\theta\sigma}^2 P + \frac{2\theta}{\sigma^2} \partial_\theta P + \frac{\theta^2}{\sigma^2} \partial_{\theta\theta}^2 P, \\ \partial_{rr}^2 \mathbf{P} &= \partial_{rr}^2 P, & \partial_{S\sigma}^2 \mathbf{P} &= \frac{\theta \partial_{\theta\theta}^2 P}{S \sigma^2 \sqrt{\tau}} - \frac{\partial_{\theta\sigma}^2 P}{S \sigma \sqrt{\tau}} + \frac{\partial_\theta P}{S \sigma^2 \sqrt{\tau}}, \\ \partial_{Sr}^2 \mathbf{P} &= -\frac{\partial_{\theta r}^2 P}{S \sigma \sqrt{\tau}}, & \partial_{\sigma r}^2 \mathbf{P} &= -\frac{\theta}{\sigma} \partial_{\theta r}^2 P + \partial_{\sigma r}^2 P, \end{aligned}$$

we can transform PDE (2.1) as

$$\begin{aligned} & \frac{\theta}{2} \partial_\theta P + \frac{\partial_{\theta\theta}^2 P}{2} - \tau \partial_\tau P + \sqrt{\tau} \left[ \frac{\partial_\theta P}{\sigma} \left( \frac{\sigma^2}{2} - r + \delta \right) + b \rho_{12} \left( \frac{\theta \partial_{\theta\theta}^2 P}{\sigma} - \partial_{\theta\sigma}^2 P + \frac{\partial_\theta P}{\sigma} \right) - \beta \rho_{13} \partial_{\theta r}^2 P \right] \\ & + \tau \left[ a \left( \partial_\sigma P - \frac{\theta}{\sigma} \partial_\theta P \right) + \frac{b^2}{2} \left( \partial_{\sigma\sigma}^2 P - \frac{2\theta}{\sigma} \partial_{\theta\sigma}^2 P + \frac{2\theta}{\sigma^2} \partial_\theta P + \frac{\theta^2}{\sigma^2} \partial_{\theta\theta}^2 P \right) \right] \\ & + \tau \left[ \frac{\beta^2 \partial_{rr}^2 P}{2} + b \beta \rho_{23} \left( \partial_{\sigma r}^2 P - \frac{\theta}{\sigma} \partial_{\theta r}^2 P \right) - r P + \alpha_0(r) \partial_r P \right] + \sum_{i=1}^{\infty} \tau^{i+1} \alpha_i(r) \partial_r P = 0 \end{aligned} \quad (2.3)$$

with boundary condition

$$P(\infty, \sigma, r, \tau) = 0, \quad (2.4)$$

$$P(y, \sigma, r, \tau) = K \max\{1 - e^{-\sigma y \sqrt{\tau}}, 0\} = K(1 - e^{-\sigma y \sqrt{\tau}}), \quad (2.5)$$

where  $\alpha(r, t) = \alpha(r, T - \tau) = \sum_{i=0}^{\infty} \alpha_i(r) \tau^i$ .

Denote the solution of (2.3) under the boundary conditions (2.4) and (2.5) by  $P(\theta, \sigma, r, \tau; y)$ , then the American put price  $P^*(\theta, \sigma, r, \tau)$  can be approximated from below by:

$$P^*(\theta, \sigma, r, \tau) \simeq \max_{y \geq \theta} P(\theta, \sigma, r, \tau; y) = P(\theta, \sigma, r, \tau; \bar{y}(\theta, \sigma, r, \tau)). \quad (2.6)$$

### 3 Asymptotic expansion of American option price

Now we present the asymptotic expansion of American option price given by Medvedev and Scaillet (2009).

Assume that  $P(\theta, \sigma, r, \tau)$  as in PDE (2.3) has regular asymptotic expansion near maturity of the form:

$$P(\theta, \sigma, r, \tau) = \sum_{n=1}^{\infty} P_n(\theta, \sigma, r) \tau^{n/2}, \quad (3.1)$$

where  $P_n(\theta, \sigma, r)$ ,  $n = 1, 2, \dots$  are the coefficients of the short-maturity asymptotic expansion in  $\tau$ . Next proposition from Medvedev and Scaillet (2009) gives the characteristic of the coefficients in (3.1).

**Proposition 1** Consider the PDE (2.3) with boundary condition (2.4) and (2.5) with the regular asymptotic expansion (3.1) with  $(\theta, \tau)$  in the vicinity of  $(0, 0)$ . For any solution to this problem there exists functions  $C_1(\sigma, r)$ ,  $C_2(\sigma, r)$ , ... such that for each  $n$ :

$$P_n(\theta, \sigma, r) = C_n(\sigma, r)[p_n^0(\theta, \sigma, r)\Phi(\theta) + q_n^0(\theta, \sigma, r)\phi(\theta)] + p_n^1(\theta, \sigma, r)\Phi(\theta) + q_n^1(\theta, \sigma, r)\phi(\theta),$$

where  $p_n^0(\theta, \sigma, r) \in \Pi^0(n, \theta, \sigma, r)$ ,  $p_n^1(\theta, \sigma, r) \in \Pi^1(n-2, \theta, \sigma, r)$ ,  $q_n^0(\theta, \sigma, r) \in \Pi^0(n-1, \theta, \sigma, r)$ ,  $q_n^1(\theta, \sigma, r) \in \Pi^1(n-3, \theta, \sigma, r)$  with the coefficients depending on model parameters and  $C_1(\sigma, r)$ ,  $C_2(\sigma, r)$ ,  $\dots$ ,  $C_{n-1}(\sigma, r)$ ,  $\Phi(\theta)$  and  $\phi(\theta)$  is the cumulative distribution function and probability distribution function of normal distribution respectively.

$p_n^0(\theta, \sigma, r)$ ,  $q_n^0(\theta, \sigma, r)$ ,  $p_n^1(\theta, \sigma, r)$ ,  $q_n^1(\theta, \sigma, r)$  can be solved by substituting the expansion (3.1) into PDE (2.3). To determine functions  $C_n(\sigma, r)$ ,  $n = 1, 2, \dots$ , we impose an explicit early exercise rule by requiring that the put option is exercised as soon as it hits the barrier  $\theta = y$ . This condition allows us to uniquely identify the coefficients  $C_n(\sigma, r)$ ,  $n = 1, 2, \dots$ . Note that the put option payoff is  $P(y, \sigma, r, \tau) = K[1 - \exp(-\sigma y \sqrt{\tau})]$ , by Tylor's expansion we have

$$P(y, \sigma, r, \tau) = K[1 - \exp(-\sigma y \sqrt{\tau})] = \sigma y K \sqrt{\tau} - \frac{\sigma^2 y^2 K}{2} \tau + \dots \quad (3.2)$$

By equating  $P_1(\theta, \sigma, r)$  in (3.1) evaluated at  $\theta = y$  to  $\sigma y K$  in (3.2), we determine  $C_1(\sigma, r)$ . Then we find  $p_2^0$ ,  $p_2^1$ ,  $q_2^0$ ,  $q_2^1$ , and equating  $P_1(\theta, \sigma, r)$  to  $\frac{\sigma^2 y^2 K}{2}$ , we obtain  $C_2(\sigma, r)$ . All the other functions can be determined recursively by the same way.

## 4 Program Manual

We implement the American option pricing of the affine model

$$\begin{aligned}dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\dv_t &= k_v(\bar{v} - v_t)dt + \sigma_v \sqrt{v_t} dW_t^{(2)}, \\dr_t &= k_r(\bar{r} - r_t)dt + \sigma_r \sqrt{r_t} dW_t^{(3)}.\end{aligned}$$

For the affine model above, the coefficients and functions of the 4th order expansion for the American option price is given in the appendix.

### Included files:

The program directory contains 3 files:  
this documentation file "docu.pdf"  
"ame\_put.cpp"  
"ame\_put.out"

### Compile and run program:

Compilation commande under Linux:  
g++ ame\_put.cpp -o ame\_put.out

Commande to execute the algorithm:  
./ame\_put.out

### Parameters of the product:

spot: stock price at the initial time  
strike: strike of the American option  
maturity: maturity of the American option, the expansion asymptotic works well for small maturity.  
volatility: stock volatility at the initial time  
interest: interest rate at the initial time  
dividend: dividend of the underlying stock

### Model Parameters:

kv:  $k_v$  in model (4.1)  
vbar:  $\bar{v}$  in model (4.1)  
sigmav:  $\sigma_v$  in model (4.1)  
kr:  $k_r$  in model (4.1)  
rbar:  $\bar{r}$  in model (4.1)  
sigmar:  $\sigma_r$  in model (4.1)  
rho12:  $\rho_{12}$  in model (4.1)  
rho13:  $\rho_{13}$  in model (4.1)  
rho23:  $\rho_{23}$  in model (4.1), note that our method works well only when  $\rho_{23} = 0$ .

## 5 Appendix

$$\begin{aligned}
C_1(v, r) &= \frac{Ky\sqrt{v_t}}{\Phi(y)y + \phi(y)}, \\
C_2(v, r) &= \frac{-\Phi(y)C_1\sqrt{v_t}^2 + 2\Phi(y)C_1r_t - \phi(y)yb_1\rho_{12}C_1 - Ky^2v_t^3}{2v_t[\phi(y)(y^2 + y) + \Phi(y)]}, \\
C_3(v, r) &= \frac{1}{24v_t^2[\Phi(y)(y^3 + 3y) + \phi(y)(y^2 + 2)]} \\
&\times \left[ -3C_1v_t^4 + \phi(y) + 48\Phi(y)yb_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^2 + 48\phi(y)b_1\frac{\partial C_2}{\partial v}v_t^2 + 24\phi(y)r_tC_1v_t^2 \right. \\
&- 4\phi(y)b_1^2y^2C_1 - 24\Phi(y)yC_2v_t^3 + 24\Phi(y)yr_tC_1v_t^2 - 3\phi(y)b_1^2\rho_{12}^2C_1y^4 \\
&+ 6\phi(y)b_1y^2\rho_{12}C_1v_t^2 - 24\phi(y)C_2v_t^3 - 96\Phi(y)yb_1\rho_{12}C_2v_t \\
&- 72\phi(y)b_1\rho_{12}C_2v_t - 12\phi(y)a_vv_tC_1 - 12\phi(y)b_1y^2\rho_{12}C_1r_t + 48\Phi(y)yC_2v_tr_t \\
&+ 12\phi(y)b_1\rho_{12}C_1r_t + 48\Phi(y)yb_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} + 48\phi(y)b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} - 12\phi(y)C_1r_t^2 \\
&- 12\phi(y)b_r\rho_{13}v_tC_1 + 12\phi(y)C_1v_t^2r_t + 48\phi(y)C_2v_tr_t - 2\phi(y)b_1^2C_1 \\
&\left. + 6\phi(y)b_1\rho_{12}C_1v_t^2 - \phi(y)b_1^2\rho_{12}^2C_1 + 10\phi(y)b_1^2y^2\rho_{12}^2C_1 + 4Ky^3v_t^5 \right],
\end{aligned}$$

$$\begin{aligned}
C_4(v, r) = & -\frac{1}{48(v_t^3(3\Phi(y) + \Phi(y)y^4 + 6\Phi(y)y^2 + 5\phi(y)y + \phi(y)y^3))} \\
& \times [6\phi(y)y^5b_1^2\rho_{12}^2C_1r_t + 12\Phi(y)C_2v_t^5 + 2Ky^4v_t^7 + 12\phi(y)yb_1\rho_{12}C_1v_t^2r_t \\
& - 36\phi(y)yb_1\rho_{12}C_1r_t^2 + 24\phi(y)yb_r\rho_{13}v_tC_1r_t + 96\Phi(y)b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t - 144\Phi(y)y^2v_t^2C_3r_t \\
& + 42\phi(y)yb_1^2\rho_{12}^2C_1r_t - 10\phi(y)yb_1^2C_1v_t^2 - 13\phi(y)y^5b_1^3\rho_{12}^3C_1 - 12\phi(y)yC_1v_t^2r_t^2 \\
& + 6\phi(y)yC_1v_t^4r_t - 44\phi(y)y^3b_1^2\rho_{12}^2C_1r_t + 24\phi(y)y a_v v_t C_1 r_t + 48\Phi(y)a_r v_t^3\frac{\partial C_2}{\partial r} \\
& - 24\Phi(y)a_r v_t^2 C_1 + 8\phi(y)yC_1r_t^3 + 72\Phi(y)C_3v_t^4 - 28\phi(y)yb_1\rho_{12}b_r\rho_{13}v_tC_1 \\
& + 24\Phi(y)b_r\rho_{13}v_t^3C_1 + 12\phi(y)y^3b_1b_r\rho_{13}v_tC_1 - 12\phi(y)C_1 - 12\phi(y)yb_r\rho_{13}v_t^3C_1 \\
& + 48\Phi(y)b_r\rho_{13}v_t^2C_2 + 48\Phi(y)b_r\rho_{13}^2v_t^3b_{r1}\frac{\partial C_2}{\partial r} - 48\phi(y)yb_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} \\
& - 48\Phi(y)b_r\rho_{13}v_t^4r - 144\Phi(y)b_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} - 144\Phi(y)b_r\rho_{13}v_t^3\frac{\partial C_3}{\partial r} \\
& - 144\Phi(y)y^2v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r} - 144\phi(y)yb_r\rho_{13}v_t^3\frac{\partial r}{\partial} + 8\phi(y)yb_r\rho_{13}^2v_t^2b_{r1}C_1 \\
& - 144\Phi(y)b_1\rho_{12}\frac{\partial C_3}{\partial v}v_t^3 - 144\Phi(y)y^2v_t^3b_1\rho_{12}\frac{\partial C_3}{\partial v} - 144\phi(y)yb_1\rho_{12}\frac{\partial C_3}{\partial v}v_t^3 \\
& + 48\Phi(y)C_2v_t^2r_t - 48\Phi(y)C_2v_t^3r_t - 144\Phi(y)C_3v_t^2r_t + 2\phi(y)yb_1^3\rho_{12}C_1 \\
& - 18\phi(y)y^3b_1^3\rho_{12}C_1 - 12\phi(y)y^3b_1\rho_{12}C_1v_t^2r_t + 48\Phi(y)r_tC_1v_t^2r_t - 12\phi(y)yb_1^2C_1r_t \\
& + 8\phi(y)yb_1\rho_{12}v_t^2a_{v1}C_1 + 432\Phi(y)y^2v_t^2b_1\rho_{12}C_3 + 288\Phi(y)b_1\rho_{12}C_3v_t^2 \\
& + 432\phi(y)yb_1\rho_{12}C_3v_t^2 + 48\phi(y)y a_r v_t^3\frac{\partial C_2}{\partial r} + 48\Phi(y)\phi(y)y a_r v_t^3\frac{\partial C_2}{\partial r} + 48\Phi(y)y^2v_t^3a_r\frac{\partial C_2}{\partial r} \\
& + 12\phi(y)y^3b_1\rho_{12}C_1r_t^2 + 96\Phi(y)b_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^2r_t + 8\phi(y)y^3b_1^2C_1r_t - 24\Phi(y)r_tC_1v_t^4 \\
& + 12\Phi(y)b_1^2C_1v_t^2 - \phi(y)yC_1v_t^6 - 48\Phi(y)C_2v_t^3r_t + 72\Phi(y)b_1^2C_2v_t \\
& - 48\Phi(y)a_v v_t^2 C_2 + 24\Phi(y)a_v v_t^3 C_1 + 48\Phi(y)a_v v_t^3\frac{\partial C_2}{\partial v} + 24\phi(y)yb_1^2\frac{\partial^2 C_2}{\partial v^2}v_t^3 \\
& - 96\phi(y)yb_1^2\frac{\partial C_2}{\partial v}v_t^2 + 33\phi(y)y^3b_1^3\rho_{12}^3C_1 + 10\phi(y)y^3b_1^2\rho_{12}^2C_1v_t^2 + 48\Phi(y)b_1^2\rho_{12}^2v_t^3\frac{\partial^2 C_2}{\partial v^2} \\
& - 96\phi(y)y a_v v_t^2 C_2 - 12\phi(y)y a_v v_t^3 C_1 - 12\phi(y)yb_1\rho_{12}a_v v_t C_1 + 48\phi(y)y a_v v_t^3\frac{\partial C_2}{\partial v} \\
& + 12\phi(y)y^3b_1\rho_{12}a_v v_t C_1 + 48\Phi(y)y^2v_t^3a_v\frac{\partial C_2}{\partial v} - 96\Phi(y)y^2v_t^2a_v C_2 - 12\phi(y)y^3b_1^2\rho_{12}^2C_2v_t \\
& - 48\phi(y)yC_2v_t^3r_t - 9\phi(y)yb_1^3\rho_{12}^3C_1 - \phi(y)yb_1^2\rho_{12}^2C_1v_t^2 + 3\phi(y)yb_1\rho_{12}C_1v_t^4 \\
& - 24\phi(y)yb_1\rho_{12}v_t^2r_tC_1 - 192\Phi(y)b_1\rho_{12}C_2v_t r_t - 48\phi(y)yb_1\rho_{12}C_2v_t r_t - 144\phi(y)yC_3v_t^2r_t \\
& - 144\Phi(y)b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2 + 24\Phi(y)b_1^2\frac{\partial^2 C_2}{\partial v^2}v_t^3 - 48\Phi(y)b_1^2\frac{\partial C_2}{\partial v}v_t^2 - 3\phi(y)y^5b_1^2\rho_{12}^2C_1v_t^2 \\
& + \phi(y)y^7b_1^3\rho_{12}^3C_1 - 4\phi(y)y^3b_1^2C_1v_t^2 + 4\phi(y)y^5b_1^3\rho_{12}C_1 + 3\phi(y)y^3b_1\rho_{12}C_1v_t^4 \\
& + 24\Phi(y)y^2v_t^3b_1^2\frac{\partial^2 C_2}{\partial v^2} - 96\Phi(y)y^2v_t^2b_1^2\frac{\partial C_2}{\partial v} - 48\Phi(y)b_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^4 - 48\phi(y)yb_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2 \\
& + 128\phi(y)yb_1^2C_2v_t - 48\Phi(y)y^2v_t^3C_2r_t + 144\Phi(y)y^2v_t b_1^2C_2 + 24\phi(y)yb_1\rho_{12}C_2v_t^3 \\
& + 100\phi(y)yb_1^2\rho_{12}^2C_2v_t + 72\phi(y)yC_3v_t^4 + 48\Phi(y)b_1\rho_{12}C_2v_t^3 + 192\Phi(y)b_1^2\rho_{12}^2C_2v_t \\
& + 72\Phi(y)y^2v_t^4C_3] ,
\end{aligned}$$

$$\begin{aligned}
C_5(v, r) = & \frac{1}{5760v_t^4[\Phi(y)(15y + y^5 + 10y^3) + \phi(y)(y^4 + 9y^2 + 8)]} \\
& \times \left[ \Phi(y) \left( -5760y^3v_t^4\rho_{23}b_rb_1\frac{\partial^2 C_3}{\partial v\partial r} + 86400yb_1\rho_{12}v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r} \right. \right. \\
& - 11520yb_r\rho_{13}v_t^4\frac{\partial C_2}{\partial r}r_t + 5760y\rho_{23}b_rb_1v_t^3\frac{\partial C_2}{\partial v} + 23040yb_r\rho_{13}v_t^2b_1^2\frac{\partial C_2}{\partial r} \\
& + 23040y^3v_t^4b_r\rho_{13}\frac{\partial C_4}{\partial r} + 69120yb_r\rho_{13}v_t^4\frac{\partial C_4}{\partial r} - 17280yv_t^2C_3r_t^2 \\
& + 5760yv_t^3b_1^2\frac{\partial^2 C_2}{\partial r^2}r_t - 23040yv_t^2a_vC_2r_t - 34560yv_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r}r_t \\
& - 23040yb_1\rho_{12}v_t^3a_r\frac{\partial C_2}{\partial r} - 11520yv_t^3C_2r_t^2 - 17280yb_r\rho_{13}v_t^3C_3 \\
& - 207360yb_1\rho_{12}C_4v_t^3 - 5760y^3v_t^4a_v\frac{\partial C_3}{\partial v} - 92160y^3v_t^3b_1\rho_{12}C_4 \\
& - 23040yv_t^2b_1^2\frac{\partial C_2}{\partial v}r_t + 69120yC_4v_t^3r_t + 11520yb_1\rho_{12}v_t^4a_v\frac{\partial^2 C_2}{\partial r^2} - 2880y^3v_t^4b_1^2\frac{\partial^2 C_3}{\partial r^2} \\
& - 17280yb_r\rho_{13}v_t^3a_v\frac{\partial C_2}{\partial r} - 17280y\rho_{23}b_rb_1\frac{\partial C_3^2}{\partial v\partial r}v_t^4 + 5760yb_r^2v_t^3\frac{\partial C_2}{\partial r} \\
& - 17280yb_r^2\rho_{13}v_t^4\frac{\partial^2 C_3}{\partial r^2} + 5760y^3v_t^4C_3r_t + 11520yv_t^3a_r\frac{\partial C_2}{\partial r}r_t \\
& + 5760yb_r\rho_{13}v_t^4a_{r1}\frac{\partial C_2}{\partial r} - 2880y\rho_{23}b_rb_1v_t^4\frac{\partial C_2}{\partial r} + 11520y\rho_{23}b_rb_1^2v_t^2\rho_{12}\frac{\partial C_2}{\partial r} \\
& - 2880y^3v_t^4b_r^2\frac{\partial C_3^2}{\partial^2 r} + 34560yb_1^2\frac{\partial C_3}{\partial v}v_t^3 - 34560y^3v_t^2b_1^2C_3 - 4320yv_t^6C_3 \\
& - 34560yv_t^3b_1\rho_{12}\frac{\partial C_3}{\partial v}r_t + 34560yv_t b_1^2C_2r_t - 28800yb_1^3\rho_{12}v_t^3\frac{\partial^2 C_2}{\partial r^2} \\
& + 23040y^3v_t^3C_4r_t + 17280yv_t^4C_3r_t - 155520yb_1^2\rho_{12}v_t^2C_3 - 8640yb_1^2\frac{\partial^2 C_3}{\partial r^2}v_t^4 \\
& + 11520yv_t^3a_v\frac{\partial C_2}{\partial v}r_t - 34560yv_t^4b_1\rho_{12}C_3 - 60480yb_1^2v_t^2C_3 \\
& - 5760yv_t^5a_r\frac{\partial C_2}{\partial r} - 5760yv_t^5a_v\frac{\partial C_2}{\partial v} - 17280yb_1^2\rho_{12}v_t^4\frac{\partial^2 C_3}{\partial r^2} - 17280ya_vv_t^4\frac{\partial C_3}{\partial v} \\
& + 34560ya_vv_t^3C_3 + 5760yv_t^4b_1^2\frac{\partial C_2}{\partial v} + 5760yv_t^5C_2r_t - 5760y\rho_{23}b_rb_1v_t^2\frac{\partial C_2}{\partial r}r_t \\
& + 86400yb_1^2\rho_{12}v_t^3\frac{\partial C_3}{\partial v} - 2880yv_t^5b_1^2\frac{\partial^2 C_2}{\partial r^2} + 103680yv_t^2b_1\rho_{12}C_3r_t + 17280yv_t^4C_3r_t \\
& + 23040y^3v_t^4b_1\rho_{12}\frac{\partial C_4}{\partial v} + 5760yb_1\rho_{12}v_t^4a_{v1}\frac{\partial C_2}{\partial v} + 69120yb_1\rho_{12}\frac{\partial C_4}{\partial v}v_t^4 \\
& - 103680yb_1^3\rho_{12}v_tC_2 + 23040yb_1\rho_{12}v_t^3C_2r_t - 11520yb_1\rho_{12}v_t^4\frac{\partial C_2}{\partial v}r_t + 80640yb_1^3\rho_{12}v_t^2\frac{\partial C_2}{\partial v} \\
& + 5760ya_rv_t^3C_2 + 2880yb_r^2v_t^4b_{r2}\rho_{13}\frac{\partial C_2}{\partial r} - 5760yb_r\rho_{13}v_t^4C_2 - 17280yb_r\rho_{13}^2v_t^4b_{r1}\frac{\partial C_3}{\partial r} \\
& - 8640yb_r^2v_t^4\frac{\partial^2 C_3}{\partial r^2} - 5760y\rho_{23}b_rb_1v_t^2C_2 - 40320yb_1\rho_{12}v_t^3a_v\frac{\partial C_2}{\partial v}
\end{aligned}$$

$$\begin{aligned}
& +2880y\rho_{23}b_rb_1v_t^3C_1 - 8640yv_t^3b_1^2C_2 - 34560yb_1\rho_{12}v_t^4b_r\rho_{13}\frac{\partial^2C_3}{\partial v\partial r} + 5760yv_t^4a_vC_2 \\
& +17280yv_t^5b_r\rho_{13}\frac{\partial C_3}{\partial r} - 2880yr_t^2v_t^4C_1 + 17280y^3v_t^3a_vC_3 \\
& -11520yb_1\rho_{12}v_t^3a_{v_1}C_2 + 57600yb_1\rho_{12}v_t^2a_vC_2 - 34560yv_t^5C_4 - 11520y^3v_t^5C_4 \\
& -5760y^3v_t^4a_r\frac{\partial C_3}{\partial r} - 17280ya_rv_t^4\frac{\partial C_3}{\partial r} + 2880ya_rv_t^4C_1 \\
& +17280y^3v_t^3b_1^2\frac{\partial C_3}{\partial v} + 17280yv_t^5b_1\rho_{12}\frac{\partial C_3}{\partial v} + 5760ya_rv_t^4b_{r_1}\rho_{13}\frac{\partial C_2}{\partial r} \Big) \\
& +\phi(y) \left( -17280b_r\rho_{13}v_t^3C_3 - 360y^2v_t^5b_r\rho_{13}C_1 + 16320b_r\rho_{13}v_t^2b_1^2\frac{\partial C_2}{\partial r} \right. \\
& -2880v^5b_r\rho_{13}^2b_{r_1}\frac{\partial C_2}{\partial r} - 17280b_r^2\rho_{13}^2v_t^4\frac{\partial^2C_3}{\partial r^2} - 960b_r\rho_{13}v_t^3a_{r_1}C_1 \\
& +1440y^2b_r\rho_{13}v_t^2b_1^2C_1 - 13440y^2b_1^2\rho_{12}^2b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} + 720y^2b_1^2\rho_{12}\rho_{23}b_rv_tC_1 \\
& -11520b_r\rho_{13}v_t^4\frac{\partial C_2}{\partial r}r_t + 1920y^2b_r\rho_{13}v_t^2b_1^2\frac{\partial C_2}{\partial r} - 240y^2b_r^2\rho_{13}^3v_t^3b_{r_2}C_1 + 1920b_r^2\rho_{13}^3v_t^4b_{r_2}\frac{\partial C_2}{\partial r} \\
& -1080v_t^5b_r\rho_{13}C_1 + 1440b_1\rho_{12}a_rv_t^2C_1 - 120v_t^2b_1^2\rho_{12}^2C_1r_t + 240b_1^3\rho_{12}C_1r_t \\
& +2160y^2b_1^2C_1r_t^2 + 5760b_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^2r_t^2 - 120b_1^4\rho_{12}^2y^8C_1 - 15y^2v_t^8C_1 - 840b_1^3y^4v_t^2\rho_{12}C_1 \\
& +390b_1^2y^4v_t^4\rho_{12}^2C_1 - 3771y^2b_1^4\rho_{12}^4C_1 - 1440y^2v_t^4b_1\rho_{12}r_tC_1 + 2400b_1^3y^4v_t^2\rho_{12}^3C_1 \\
& -177b_1^4\rho_{12}^4C_1 - 6720y^2b_1^3\rho_{12}C_1r_t - 11760b_1^3y^4\rho_{12}^3C_1r_t \\
& +12960y^2b_1^3\rho_{12}^3C_1r_t + 240C_1r_t^4 + 150v_t^4b_1^2\rho_{12}^2C_1 + 60b_1^3\rho_{12}^3y^8v_t^2C_1 \\
& +17280b_1^2\frac{\partial C_3}{\partial v}v_t^3 - 480b_1y^4\rho_{12}C_1r_t^3 + 5760v_t^3b_1^2\frac{\partial^2C_2}{\partial r^2}r_t - 34560v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r}r_t \\
& -960a_vv_t^3a_{v_1}C_1 + 2880y^2b_1\rho_{12}v_t^2a_vC_2 - 720v_t^5b_1\rho_{12}C_2 - 11520a_vv_t^4\frac{\partial C_3}{\partial v} \\
& -5760v_t^5a_v\frac{\partial C_2}{\partial v} + 2880v_t^4a_vC_2 - 1080v_t^5a_vC_1 + 17280a_vv_t^3C_3 - 240y^2C_1r_t^4 + 1920C_2v_tr_t^3 \\
& +17280v_t^4C_3r_t - 17280v_t^2C_3r_t^2 + 2880y^2b_1\rho_{12}C_1r_t^3 + 360v_t^4b_1\rho_{12}C_1r_t \\
& +360b_1^2y^6v_t^2\rho_{12}^2C_1r_t - 240y^2b_1^2C_1v_t^2r_t + 1440v_t^3b_r\rho_{13}C_1r_t - 1440y^2a_vv_tC_1r_t^2 \\
& -1440b_1y^4\rho_{12}a_vv_tC_1r_t - 23040b_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t + 5760y^2b_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t \\
& +7680y^2b_1\rho_{12}b_r\rho_{13}v_tC_1r_t + 24000v_tb_1^2C_2r_t - 120b_1^3\rho_{12}^3y^8C_1r_t - 7200v_t^3b_1^2C_2 \\
& -14400b_1\rho_{12}C_2v_tr_t^2 - 3360b_1\rho_{12}b_r\rho_{13}v_tC_1r_t - 960y^2v_t^3b_1\rho_{12}b_r\rho_{13}C_1 \\
& -3530b_1^4y^6\rho_{12}^4C_1 - 840b_1^3y^6v_t^2\rho_{12}^3C_1 + 435b_1^4\rho_{12}^4y^8C_1 + 960b_1\rho_{12}v_t^2a_{v_1}C_1r_t \\
& -11520\rho_{23}b_rb_1\frac{\partial^2C_3}{\partial v\partial r}v_t^4 + 5760b_1\rho_{12}b_r\rho_{13}v_t^3\frac{\partial C_2}{\partial v} + 1900b_1^4y^6\rho_{12}^2C_1 \\
& -960y^2b_1\rho_{12}v_t^2a_{v_1}C_1r_t + 120y^2v_t^6C_1r_t - 360y^2v_t^4C_1r_t^2 - 960a_rv_t^3b_{r_1}\rho_{13}C_1 \\
& +720b_1y^4v_t^3\rho_{12}b_r\rho_{13}C_1 + 2880v_t^3b_r\rho_{13}C_1r_t - 480b_1^2y^4b_r\rho_{13}v_tC_1 + 480y^2v_t^4b_r\rho_{13}^2b_{r_1}C_1
\end{aligned}$$



$$\begin{aligned}
& +240b_r\rho_{13}^3v_t^3b_{r_1}^2C_1 + 1920b_r\rho_{13}^2v_t^3b_{r_1}C_2 - 360b_1^2y^6\rho_{12}^2b_r\rho_{13}v_tC_1 + 240b_r^2\rho_{13}^3v_t^3b_{r_2}C_1 \\
& -5760y^2v_t^4\rho_{23}b_rb_1\frac{\partial C_3^2}{\partial v\partial r} - 360b_1y^4v_t^4\rho_{12}C_1r_t + 5760y^2b_1\rho_{12}a_vv_tC_1r_t - 2880a_rv_t^2C_1r_t \\
& -1440b_1y^4\rho_{12}b_r\rho_{13}v_tC_1r_t + 15v_t^8C_1 - 5760v_t^4b_r\rho_{13}\frac{\partial C_2}{\partial r}r_t - 1440y^2b_r\rho_{13}v_tC_1r_t^2 \\
& +5760y^2v_t^4C_3r_t + 720v_t^2b_1\rho_{12}C_1r_t^2 - 17280v_t^2a_vC_2r_t - 1440b_1\rho_{12}a_vv_tC_1r_t \\
& +11520v_t^3a_v\frac{\partial C_2}{\partial v}r_t + 1440y^2a_vv_t^3C_1r_t - 240v_t^7C_2 + 480a_vv_tb_1^2C_1 - 360b_1^2y^6\rho_{12}^2a_vv_tC_1 \\
& +2880a_rv_t^3r_tC_1 + 720v_t^3b_1\rho_{12}a_vC_1 + 480b_1^2y^4v_t^2C_1r_t - 480b_1^3y^6\rho_{12}C_1r_t - 120v_t^6C_1r_t \\
& -23040v_t^5C_4 - 4320v_t^6C_3 + 60b_1y^4v_t^6\rho_{12}C_1 + 240b_1^3y^6v_t^2\rho_{12}C_1 - 90b_1^2y^6v_t^4\rho_{12}^2C_1 \\
& -15b_1^4\rho_{12}^4C_1y^10 - 2880r_t^2v_t^4C_1 + 720v_t^6r_tC_1 + 1440b_r\rho_{13}v_tC_1r_t^2 - 5760\rho_{23}b_rb_1v_t^2\frac{\partial C_2}{\partial r}r_t \\
& +5760b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t^2 + 720b_1y^4\rho_{12}C_1v_t^2r_t^2 + 23040y^2v_t^3C_4r_t - 11520v_t^3C_2r_t^2 \\
& -16320y^2b_1^2\rho_{12}^2C_2v_tr_t - 720b_1^2C_1r_t^2 + 1440C_2v_t^5r_t - 2880v_t^3C_2r_t^2 + 360v_t^4C_1r_t^2 \\
& -480C_1v_t^2r_t^3 + 46080C_4v_t^3r_t - 480b_1^2y^4C_1r_t^2 + 2520b_1^2\rho_{12}^2C_1r_t^2 \\
& +240b_1^2\rho_{12}^2v_t^3a_{v_2}C_1 - 4320y^2b_1^2\rho_{12}^2v_t^2C_3 - 720b_1^2\rho_{12}\rho_{23}b_rb_vC_1 - 2880\rho_{23}b_rb_1v_t^4\frac{\partial C_2}{\partial r} \\
& +5760\rho_{23}b_rb_1v_t^3\frac{\partial C_2}{\partial v} + 2880y^2b_1\rho_{12}b_r\rho_{13}v_t^2C_2 - 12480b_1\rho_{12}b_r\rho_{13}v_t^2C_2 \\
& +2880y^2b_1\rho_{12}b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r} + 1320b_1^2\rho_{12}^2b_r\rho_{13}v_tC_1 + 48Ky^5v_t^9 - 120b_1^2y^4v_t^4C_1 \\
& +17280v_t^5b_1\rho_{12}\frac{\partial C_3}{\partial v} + 4440b_1^2y^4\rho_{12}^2C_1r_t^2 - 1080b_1^3\rho_{12}^3C_1r_t - 8640b_1^3\rho_{12}^3v_t^3\frac{\partial^2 C_2}{\partial r^2} \\
& +5760y^2b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2r_t + 86400v_t^2b_1\rho_{12}C_3r_t + 1440y^2v_t^3b_r\rho_{13}C_1r_t \\
& -2400y^2v_t^2b_1^2\rho_{12}^2C_1r_t - 720y^2a_v^2v_t^2C_1 - 120b_1^2\rho_{12}^2a_vv_tC_1 + 720y^2v_t^5b_1\rho_{12}C_2 \\
& +240b_1^3y^6\rho_{12}^3C_2v_t + 23040y^2v_t^4b_1\rho_{12}\frac{\partial C_4}{\partial v} + 720y^2v_t^4b_1\rho_{12}C_1r_t + 2880y^2v_t^2b_1\rho_{12}C_1r_t^2 \\
& -2880y^2v_t^2b_1\rho_{12}C_1r_t^2 + 3720y^2v_t^2b_1^2\rho_{12}^2C_1r_t - 420y^2v_t^4b_1^2C_1 + 2880y^2b_1^3\rho_{12}^3v_t^3\frac{\partial^2 C_2}{\partial r^2} \\
& -360b_1^2y^6\rho_{12}^2C_1r_t^2 + 1440b_1^2y^4\rho_{12}^2C_2v_tr_t + 8640v_t^3b_1\rho_{12}C_2r_t - 25920v_t^4b_1\rho_{12}C_3 \\
& +1440v_t^6b_1\rho_{12}\frac{\partial C_2}{\partial v} - 2880y^2v_t^3b_1\rho_{12}C_2r_t + 5760b_r\rho_{13}v_t^2C_2r_t + 1920y^2v_tb_1^2C_2r_t \\
& +5760a_rv_t^3C_2 + 480b_1^2v_t^2r_tC_1 - 900v_t^4b_1^2C_1 - 13440y^2b_1^3\rho_{12}^3\frac{\partial C_2}{\partial v}v_t^2 \\
& +5760b_1^2\rho_{12}^2v_t^3\frac{\partial^2 C_2}{\partial r^2}r_t - 2880v_t^4r_tC_1r_t - 180v_t^6b_1\rho_{12}C_1 - 840y^2v_t^2b_1^3\rho_{12}^3C_1 \\
& +960y^2a_vv_tb_1^2C_1 - 480b_1^2y^4a_vv_tC_1 + 11520b_1\rho_{12}v_t^4a_v\frac{\partial^2 C_2}{\partial r^2} - 11520b_1\rho_{12}v_t^4\frac{\partial C_2}{\partial v}r_t \\
& -17280b_1^2\rho_{12}^2v_t^4\frac{\partial^2 C_3}{\partial r^2} - 92160y^2v_t^3b_1\rho_{12}C_4 - 115200b_1\rho_{12}C_4v_t^3 + 2880C_1v_t^2r_tr_t^2
\end{aligned}$$

$$\begin{aligned}
& +480y^2C_1v_t^2r_t^3 + 5760b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r}r_t - 34560v_t^3b_1\rho_{12}\frac{\partial C_3}{\partial v}r_t - 10440y^2b_1^2\rho_{12}^2C_1r_t^2 \\
& +1440v_t^3a_vC_1r_t - 17280b_1\rho_{12}v_t^3a_r\frac{\partial C_2}{\partial r} - 2880v_t^5b_1^2\rho_{12}^2\frac{\partial^2 C_2}{\partial r^2} - 2880b_1\rho_{12}v_t^2r_tC_1r_t \\
& +2880y^2b_1\rho_{12}C_2v_tr_t^2 + 35040b_1^2\rho_{12}^2C_2v_tr_t - 5760v_t^4b_1\rho_{12}\frac{\partial C_2}{\partial v}r_t - 23040b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2r_t \\
& +1920\rho_{23}b_rb_1v_tC_1r_t + 30y^2v_t^4b_1^2\rho_{12}^2C_1 - 120v_t^2b_1^3\rho_{12}C_1 + 17280y^2v_t^3b_1^2\frac{\partial C_3}{\partial v} \\
& -1440v_t^4b_1\rho_{12}r_tC_1 - 252b_1^4C_1 - 276y^2b_1^4C_1 - 480b_1^2y^4\rho_{12}^2v_t^2a_{v_1}C_1 - 240b_1^2\rho_{12}^2v_t^2a_{v_1}C_1 \\
& +46080b_1\rho_{12}\frac{\partial C_4}{\partial v}v_t^4 + 5760b_r\rho_{13}v_t^4a_{r_1}\frac{\partial C_2}{\partial r} + 46080b_r\rho_{13}v_t^4\frac{\partial C_4}{\partial r} - 80b_1^4y^6C_1 \\
& +5760v_t^5C_2r_t + 240v_t^2b_1^2\rho_{12}^2C_1r_t + 5280y^2v_t^3b_1^2\rho_{12}^2C_2 + 960b_1^3y^4\rho_{12}v_tC_2 \\
& -27600b_1^3\rho_{12}^3C_2v_t + 17280b_1\rho_{12}v_t^3C_2r_t + 22320y^2b_1^3\rho_{12}^3C_2v_t + 4800y^2b_1^3\rho_{12}v_t^2\frac{\partial C_2}{\partial v} \\
& -1440b_1\rho_{12}C_1r_t^3 + 240b_1^2C_1v_t^2r_t - 3000b_1^2y^4v_t^2\rho_{12}^2C_1r_t + 2400b_1^3y^6\rho_{12}^3C_1r_t + 564b_1^4\rho_{12}^2C_1 \\
& +2652y^2b_1^4\rho_{12}^2C_1 + 2880v_t^4b_1^2\frac{\partial C_2}{\partial v} - 5760b_1^2\frac{\partial^2 C_3}{\partial r^2}v_t^4 - 34560y^2v_t^2b_1^2C_3 \\
& +576b_1^4y^4C_1 + 1440b_1^3y^4\rho_{12}^3\frac{\partial C_2}{\partial v}v_t^2 - 60v_t^2b_1^3\rho_{12}^3C_1 + 960y^2b_1^2v_t^2r_tC_1 - 6372b_1^4y^4\rho_{12}^2C_1 \\
& +5760v_t^4b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v} - 2880y^2v_t^4b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v} + 53760b_1^3\rho_{12}v_t^2\frac{\partial C_2}{\partial v} + 22560b_1^3\rho_{12}^3\frac{\partial C_2}{\partial v}v_t^2 \\
& -360y^2v_t^5a_vC_1 + 720b_1y^4v_t^3\rho_{12}a_vC_1 - 5760y^2v_t^4a_r\frac{\partial C_3}{\partial r} - 1440y^2b_1\rho_{12}a_rv_t^2C_1 \\
& -11520a_rv_t^4\frac{\partial C_3}{\partial r} + 4320a_rv_t^4C_1 - 5760v_t^5a_r\frac{\partial C_2}{\partial r} - 7800y^2b_1^2\rho_{12}^2b_r\rho_{13}v_tC_1 \\
& +2880b_r^2v_t^4b_{r_2}\rho_{13}\frac{\partial C_2}{\partial r} - 480b_r^2v_t^3b_{r_2}\rho_{13}C_1 - 480b_1y^4\rho_{12}b_r\rho_{13}v_t^2b_{r_1}C_1 + 3960b_1^2y^4\rho_{12}^2b_r\rho_{13}v_tC_1 \\
& +1440v_t^6b_r\rho_{13}\frac{\partial C_2}{\partial r} + 22560b_1^2\rho_{12}^2b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} + 480v_t^4b_r\rho_{13}^2b_{r_1}C_1 - 240y^2b_r\rho_{13}^3v_t^3b_{r_1}^2C_1 \\
& +2160y^2b_1\rho_{12}b_r\rho_{13}^2v_t^2b_{r_1}C_1 + 1920b_r\rho_{13}^3v_t^4b_{r_1}^2\frac{\partial C_2}{\partial r} - 17280b_r\rho_{13}^2v_t^4b_{r_1}\frac{\partial C_3}{\partial r} \\
& -5760\rho_{23}b_rb_1v_t^2C_2 - 240a_v^2v_t^2C_1 + 8466b_1^4y^4\rho_{12}^4C_1 - 110880b_1^2\rho_{12}^2v_t^2C_3 + 5760a_rv_t^4b_{r_1}\rho_{13}\frac{\partial C_2}{\partial r} \\
& -11520b_r\rho_{13}v_t^3a_v\frac{\partial C_2}{\partial r} + 480b_r\rho_{13}v_t^2a_vC_1 - 720y^2b_r^2\rho_{13}^2v_t^2C_1 + 720b_r^2\rho_{13}^2v_t^2C_1 \\
& -240v_t^3b_1\rho_{12}b_r\rho_{13}C_1 - 34560b_1\rho_{12}v_t^4b_r\rho_{13}\frac{\partial^2 C_3}{\partial v\partial r} - 5760b_r^2v_t^4\frac{\partial^2 C_3}{\partial r^2} + 5760b_r^2v_t^3\frac{\partial C_2}{\partial r} \\
& -960b_r^2v_t^2C_1 - 720y^2b_1^2v_t^2a_{v_1}C_1 + 480v_t^4b_1\rho_{12}a_{v_1}C_1 - 9600b_1\rho_{12}v_t^3a_{v_1}C_2 \\
& +37440b_1\rho_{12}v_t^2a_vC_2 + 17280y^2v_t^3a_vC_3 - 720b_1\rho_{12}b_r\rho_{13}^2v_t^2b_{r_1}C_1 - 2880y^2v_t^4b_r^2\frac{\partial^2 C_3}{\partial r^2} \\
& -8640b_1\rho_{12}b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r} - 1440y^2b_r\rho_{13}v_t^2a_vC_1 + 23040y^2v_t^4b_r\rho_{13}\frac{\partial C_4}{\partial r}
\end{aligned}$$

$$\begin{aligned}
& +69120b_1\rho_{12}v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r} + 5760\rho_{13}^2v_t^3\frac{\partial C_2}{\partial r} - 8640b_r\rho_{13}v_t^4C_2 \\
& + 5760v_t^4b_1\rho_{12}b_r\rho_{13}\frac{\partial C_2}{\partial r} + 9600b_1^2v_t^2\rho_{12}\frac{\partial C_2}{\partial r} + 1440b_1^2y^4\rho_{12}^2b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} \\
& + 17280v_t^5b_r\rho_{13}\frac{\partial C_3}{\partial r} + 3840\rho_{23}b_rb_1v_t^3C_1 - 2880y^2v_t^4b_1\rho_{12}b_r\rho_{13}\frac{\partial C_2}{\partial r} \\
& - 67680\rho_{12}v_tC_2 - 6960v_t^3b_1^2\rho_{12}^2C_2 - 10080y^2b_1^3\rho_{12}v_tC_2 - 4560b_1^3y^4C_2v_t \\
& + 5760b_1\rho_{12}v_t^4a_{v_1}\frac{\partial C_2}{\partial v} - 960y^2v_t^3b_1^2C_2 - 2880y^2v_t^4b_1^2\frac{\partial^2 C_3}{\partial r^2} + 960b_r\rho_{13}^2v_t^2r_t \\
& - 960y^2b_r\rho_{13}^2v_t^2b_{r_1}C_1r_t + 1440a_vv_tC_1r_t^2 + 4560b_1^3y^4\rho_{12}C_1r_t + 480y^2v_t^4b_1\rho_{12}a_{v_1}C_1 \\
& - 240b_1^2v_t^2a_{v_1}C_1 + 1680y^2b_1^2\rho_{12}^2v_t^2a_{v_1}C_1 + 11520r_t - 31680b_1^2v_t^2C_3 \\
& - 240y^2b_1^2\rho_{12}^2v_t^3a_{v_2}C_1 + 3000b_1^2y^4\rho_{12}^2a_vv_tC_1 - 5760y^2v_t^4a_v\frac{\partial C_3}{\partial v} \\
& - 28800\rho_{12}v_t^3a_v\frac{\partial C_2}{\partial v} - 3480y^2b_1^2\rho_{12}^2a_vv_tC_1 - 11520y^2v_t^5C_4 - 17280v_t^2b_1^2\frac{\partial C_2}{\partial v}r_t \\
& + 11520v_t^3a_r\frac{\partial C_2}{\partial r}r_t - 20160b_1^3\rho_{12}v_t^3\frac{\partial^2 C_2}{\partial r^2} + 69120b_1^2\rho_{12}^2v_t^3\frac{\partial C_3}{\partial v} + 720b_1^2y^4v_t^2\rho_{12}^2C_1r_t \\
& - 2880v_t^5b_1^2\frac{\partial^2 C_2}{\partial r^2} - 480b_1^2v_t^3a_{v_2}C_1 - 720b_1^2y^4v_t^3\rho_{12}^2C_2 \Big) \Big],
\end{aligned}$$

where

$$\begin{aligned}
a_r &= k_r(\bar{r} - r_t), \\
a_{r_1} &= -k_r \\
a_v &= (k_v(\bar{v} - v_t^2) - b_1^2)/2/v_t \\
a_{v_1} &= -(k_v\bar{v} - b_1^2)/2/v_t^2 - k_v/2 \\
a_{v_2} &= (k_v\bar{v} - b_1^2)/v_t^3 \\
b_1 &= \sigma_v/2 \\
b_r &= \sigma_r\sqrt{r_t} \\
b_{r_1} &= \sigma_r/\sqrt{r_t}/2 \\
b_{r_2} &= -(1/4)\sigma_rr_t^{-3/2}
\end{aligned}$$

## References

- [1] A., Medvedev, O., Scaillet, 2009, Pricing American options under stochastic volatility and stochastic interest rates.