

Pricing Asian options via Fourier and Laplace transforms

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Abstract

By means of Fourier and Laplace transform, we obtain a simple expression for the double transform (with respect the logarithm of the strike and time to maturity) of the price of continuously monitored Asian options. The double transform is expressed in terms of Gamma functions only. The computation of the price requires a multivariate numerical inversion. We show that the numerical inversion can be performed with great accuracy and low computational cost.

Keywords: Black-Scholes model, Asian options, Laplace, Fourier and Mellin Transform, Numerical Inversion of multidimensional transform.

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1 Introduction

In this note we show how to price Asian options using Fourier and Laplace transform. These options are very popular: the payoff depends on the arithmetic average of the underlying asset price over a defined time period. Therefore, Asian options reduce the possibility of market manipulation near the expiry date and offer a better hedge to firms with a stream of positions.

Under the standard Black-Scholes framework, the arithmetic average of prices is a sum of correlated lognormal distributions. Since the distribution of this sum does not admit a simple analytical expression, several approaches have been proposed to price Asian options and detailed references can be found in Kat [19]. Among them, we recall: a) the approximation for the density of the average by fitting the integer moments, as in Turnbull and Wakeman [29], Levy [22], Milevsky and Posner [24], the inverse moments as in Dufresne [10] or the logarithmic moments as in Fusai and Tagliani [13]; b) the numerical solution of either a two-dimensional degenerate parabolic PDE, as in Barraquand and Pudet [3], Zhang [31] or a rescaled one-dimensional equation as in Rogers and Shi [27] and in Vecer [30]; c) Monte Carlo simulations with variance reduction techniques, Kemna and Vorst [21] and Clelow and Carverhill [5]; d) binomial trees, as in Hull and White [16]; e) the lower and upper bounds for the price, as in Curran [6], Rogers and Shi [27] and more recently Thompson [28]; f) the approximation of the characteristic function of the average rate, as in Ju [17]; g) the Laplace transform approach in Geman and Yor [14] and in Fu et al. [12]; h) the spectral expansion of the infinitesimal generator, as in Linetsky [23].

In this paper we price Asian options by computing a Laplace transform with respect to time to maturity and a Fourier transform with respect to the *logarithm of the strike*. Such double transform is related to the characteristic function of a normal random variable. In fact, it is not surprisingly that, as a consequence, the double transform can be expressed easily in terms of Gamma functions. We present this result in Section 2, where we show how the proposed method is easily extended to the computation of the Greeks, like

delta and gamma. In order, to numerically invert our double transform and obtain the option price, we use a multivariate version of the Fourier-Euler algorithm introduced in Abate and Whitt [1]. It results that the numerical inversion is highly accurate also in correspondence of low volatility levels, i.e. when other numerical methods, such as finite difference solution of the pricing PDE, usually fail.

We remark that our approach differs from Geman and Yor [14]. They obtain the Laplace transform with respect to time to maturity of the option price. They exploit the relationship between the Geometric Brownian motion and the Bessel process with a stochastic time change and the additivity property of the Bessel process. Instead, we obtain our double transform thanks to a simpler procedure detailed in Appendix A. Moreover, the numerical inversion of the Laplace transform given in Geman and Yor [14] cannot be performed in correspondence of low volatility levels, for the limited computer precision, see Fu et al. [12], Craddock et al. [8]. Instead, the numerical inversion of our double transform can be computed with great accuracy in correspondence of low volatility values as well.

Also, our approach differ from Fu et al. [12] who investigated a double transform of the option price but with respect to time and *strike*. They obtain a rather complicated expression in terms of non-standard functions, since their result is related to the Laplace transform of a lognormal variable, which does not admit an analytical expression. Moreover, their double transform proves hard to invert numerically.

In Section 2 we give the main result of the paper. In Section 3 we present the numerical scheme and the numerical results. In Section 4 we conclude.

2 The double transform for the Asian option

We start with the standard assumption that the risk-neutral process for the underlying asset is given by a Geometric Brownian process:

$$dS = rSdt + \sigma SdW,$$

where W_t is a Brownian process, r is the interest rate σ is the volatility and t is time.

Under this condition, in order to price continuously monitored Asian options we need the probability density of the random variable:

$$A_t = \int_0^t e^{((r-\sigma^2/2)s + \sigma W_s)} ds. \quad (1)$$

Indeed, the payoff of a fixed strike Asian option is given by:

$$\left(\frac{S_0 A_t}{t} - K \right)^+.$$

The case of floating strike Asian options, characterized by a payoff $\left(\frac{S_0 A_t}{t} - S_t \right)^+$ can be dealt with by using the parity result in Henderson and Wojakowski [15]. The presence of a continuous dividend yield q can be taken into account thanks to the replacement of r by $r - q$ and of the spot price by $S_0 e^{-qt}$. Instead, it does not seem to be easy to cope with non-constant interest rate or volatility even if they are deterministic.

Therefore the price of the Asian option is obtained computing the following discounted expected value:

$$e^{-rt} E_0 \left(\frac{S_0 A_t}{t} - K \right)^+ = e^{-rt} \frac{S_0}{t} E_0 (A_t - \bar{K})^+$$

where E_0 is the expected value under the risk-neutral probability measure and $\bar{K} \equiv (K/S_0)t$. In order to compute this expectation, we first use the scaling property of the Brownian motion, see Karatzas and Shreve [18] Lemma 9.4 page 104, to express A_t as

$$A_t = \frac{4}{\sigma^2} D_h^{(v)}, \quad (2)$$

where:

$$D_h^{(v)} \equiv \int_0^h e^{2(W_s + vs)} ds \quad (3)$$

and $\nu = 2r/\sigma^2 - 1$. Thus we obtain:

$$\begin{aligned} E_0 (A_t - \bar{K})^+ &= E_0 \left(\frac{4}{\sigma^2} D_h^{(v)} - \bar{K} \right)^+ \\ &= \frac{4}{\sigma^2} E_0 (D_h^{(v)} - \tilde{K})^+ \\ &= \frac{4}{\sigma^2} \int_{\tilde{K}}^{\infty} (x - \tilde{K}) f_D(x, h) dx \end{aligned} \quad (4)$$

where f_D is the density function of the r.v. $D_h^{(v)}$; $\tilde{K} \equiv \bar{K}\sigma^2/4$; and $h \equiv \sigma^2 t/4$. After a final change of variable, $w = \ln x$, we are interested in the function:

$$c(k, h) \equiv \frac{4}{\sigma^2} \int_k^{\infty} (e^w - e^k) f_{\ln D}(w, h) dw \quad (5)$$

where $k = \ln \tilde{K}$. Note that we have used the fact that the density law of the logarithm of a r.v. is related to the density of the same r.v. by the relationship:

$$f_{\ln D}(w, h) = f_D(e^w, h) e^w, \quad -\infty < w < \infty$$

It is our aim is to compute the analytical expression of the double transform (Fourier wrt k and Laplace wrt h) of $c(k, h)$. Following Carr and Madan [4], we multiply the option price $c(k, h)$ by an exponentially decaying term so that it is square integrable in k over the negative axis. Therefore, we replace the function $c(k, h)$ by $c(k, h; a_f) \equiv c(k, h) e^{-a_f k}$, $a_f > 0$, and we compute the double transform of $c(k, h; a_f)$:

$$\mathcal{L}(\mathcal{F}(c(k, h; a_f); k \rightarrow \gamma); h \rightarrow \lambda) \equiv \int_0^{+\infty} e^{-\lambda h} \int_{-\infty}^{+\infty} e^{i\gamma k} c(k, h; a_f) dk dh \quad (6)$$

It turns out that:

Theorem 1: *The double transform of $c(k, h; a_f)$, for $\lambda > 2\gamma(\gamma + v)$, reads*

$$\mathcal{L}(\mathcal{F}(c(k, h; a_f); k \rightarrow \gamma); h \rightarrow \lambda) = C(\gamma + ia_f, \lambda)$$

where:

$$C(\gamma, \lambda) = \frac{4}{\sigma^2} \frac{1}{\lambda} \frac{1}{2^{1+i\gamma}} \frac{\Gamma(i\gamma) \Gamma\left(\frac{\mu+v}{2} + 1\right) \Gamma\left(\frac{\mu-v}{2} - 1 - i\gamma\right)}{\Gamma\left(\frac{\mu+v}{2} + 2 + i\gamma\right) \Gamma\left(\frac{\mu-v}{2}\right)}, \quad (7)$$

and $\Gamma(\cdot)$ is the gamma function of complex argument (see Press et al. [25], page 213) and $\mu = \sqrt{2\lambda + v^2}$.

Also, we can obtain the delta and gamma of the Asian option. Indeed, after some algebra it turns out that:

$$\begin{aligned} \Delta(S_0, K, t, r, \sigma) &\equiv e^{-rt} \frac{\partial E_0(A_t - \bar{K})^+}{\partial S_0} = \frac{e^{-rt}}{t} \left(c(k, h) - \frac{\partial c(k, h)}{\partial k} \right) \Big|_{k=\ln\left(\frac{K}{S_0} \frac{\sigma^2 t}{4}\right), h=\frac{\sigma^2 t}{4}}, \\ \Gamma(S_0, K, t, r, \sigma) &\equiv e^{-rt} \frac{\partial^2 E_0(A_t - \bar{K})^+}{\partial S_0^2} = \frac{e^{-rt}}{S_0 t} \left(\frac{\partial c(k, h)}{\partial k} - \frac{\partial^2 c(k, h)}{\partial k^2} \right) \Big|_{k=\ln\left(\frac{K}{S_0} \frac{\sigma^2 t}{4}\right), h=\frac{\sigma^2 t}{4}} \end{aligned}$$

The above quantities can be again recovered by numerically inverting their double transforms¹:

$$\begin{aligned} \int_0^{+\infty} e^{-\lambda h} \int_{-\infty}^{+-\infty} e^{i\gamma k} e^{-a_f k} \left(c(k, h) - \frac{\partial c(k, h)}{\partial k} \right) dk dh &= D(\gamma + ia_f, \lambda), \\ \int_0^{+\infty} e^{-\lambda h} \int_{-\infty}^{+-\infty} e^{i\gamma k} e^{-a_f k} \left(\frac{\partial c(k, h)}{\partial k} - \frac{\partial^2 c(k, h)}{\partial k^2} \right) dk dh &= G(\gamma + ia_f, \lambda). \end{aligned} \quad (8)$$

where $D(\gamma, \lambda)$ and $G(\gamma, \lambda)$ are a simple rescaling of the function $C(\gamma, \lambda)$:

$$\begin{aligned} D(\gamma, \lambda) &= (1 + i\gamma) C(\gamma, \lambda), \\ G(\gamma, \lambda) &= i\gamma (1 + i\gamma) C(\gamma, \lambda). \end{aligned}$$

3 Numerical inversion and numerical results

In this Section we discuss how to obtain the original function $c(k, h)$ by double numerical inversion. This given, the price of the Asian option reads:

$$e^{-rt} E_0 \left(\frac{S_0 A_t}{t} - K \right)^+ = e^{-rt} \frac{S_0}{t} e^{a_f k} c(k, h; a_f) \Big|_{k=\ln\left(\frac{K}{S_0} \frac{\sigma^2 t}{4}\right), h=\frac{\sigma^2 t}{4}}. \quad (9)$$

The numerical inversion of the double transform in (7) can be performed by resorting to the multivariate version of the Fourier-Euler algorithm presented in Choudhury et al. [7]. This algorithm consists in the iterated one-dimensional numerical inversion formula proposed in Abate and Whitt [1] that improves the Fourier-series method originally proposed in Dubner and Abate [9]. Given the double transform $C(\gamma, \lambda)$ we first compute

¹In order to obtain (8) we have used the property that relates the Fourier transform of a derivative of a function to the transform of the function itself, $\mathcal{F}(\partial c(k, h; a_f) / \partial k) = i\gamma \mathcal{F}(c(k, h; a_f))$. This requires the functions $c(k, h; a_f)$ and $\partial c(k, h; a_f) / \partial k$ to be integrable and $\partial c(k, h; a_f) / \partial k$ to be continuous, see Priestley [26], Theorem 33.7, pag. 267.

numerically the Fourier inverse wrt γ . Then, we numerically invert the Laplace transform wrt λ by using again the numerical univariate inversion formula.

If we denote respectively with \mathcal{L}^{-1} and with \mathcal{F}^{-1} the formal Laplace and the Fourier inverse, the function $c(k, h)$ is given by:

$$\begin{aligned} c(k, h) &= e^{a_f k} \mathcal{L}^{-1}(\mathcal{F}^{-1}(C(\gamma + ia_f, \lambda); \gamma \rightarrow k); \lambda \rightarrow h) \\ &:= e^{a_f k} \mathcal{L}^{-1}(\mathcal{F}^{-1}(C(\gamma + ia_f, \lambda))) \end{aligned}$$

and using the Fourier inversion formula, we obtain:

$$c(k, h) = e^{a_f k} \mathcal{L}^{-1} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\gamma k} C(\gamma + ia_f, \lambda) d\gamma \right).$$

Given that $|C(a_f + i\gamma, \lambda)|$ is integrable, in this case the trapezoidal rule is exact, Abate and Whitt [1], p. 21 eq. 4.8. Then, if we discretize the inversion integral by a step size Δ_f , we obtain:

$$c(k, h) = e^{a_f k} \mathcal{L}^{-1} \left(\frac{1}{2\pi} \Delta_f \sum_{s=-\infty}^{+\infty} e^{-i\Delta_f s k} C(\Delta_f s + ia_f, \lambda) \right).$$

If we set $\Delta_f = \pi/k$ and $a_f = A_f/(2k)$, we have:

$$c(k, h) = e^{\frac{A_f}{2}} \mathcal{L}^{-1} \left(\frac{1}{2k} \sum_{s=-\infty}^{+\infty} (-1)^s C\left(\frac{\pi}{k}s + i\frac{A_f}{2k}, \lambda\right) \right).$$

Then, by means of the Bromwich contour for the inversion of the Laplace transform, we have:

$$c(k, h) = \frac{e^{A_f/2}}{2k} \frac{1}{2\pi i} \int_{a_l - i\infty}^{a_l + i\infty} e^{\lambda h} \left(\sum_{s=-\infty}^{+\infty} (-1)^s C\left(\frac{\pi}{k}s + i\frac{A_f}{2k}, \lambda\right) \right) d\lambda$$

where a_l is at the right of the largest singularity of the function $C(\gamma, \lambda)$. By substituting $\lambda = a_l + i\omega$, we have:

$$c(k, h) = \frac{e^{A_f/2}}{2k} \frac{e^{a_l h}}{2\pi} \int_{-\infty}^{+\infty} e^{\lambda h} \left(\sum_{s=-\infty}^{+\infty} (-1)^s C\left(\frac{\pi}{k}s + i\frac{A_f}{2k}, a_l + i\omega\right) \right) d\omega$$

We can approximate again the above integral by using the trapezoidal rule with step size $\Delta_l = \pi/h$ and by setting $a_l = A_l/(2h)$, with A_l such that a_l is greater than the right-most singularities. Finally we have:

$$c(k, h) \approx \frac{e^{A_f/2 + A_l/2}}{2k} \frac{1}{2h} \sum_{m=-\infty}^{+\infty} (-1)^m \left(\sum_{s=-\infty}^{+\infty} (-1)^s C\left(\frac{\pi}{k}s + i\frac{A_f}{2k}, a_l + is\frac{\pi}{h}\right) \right). \quad (10)$$

In fact, once we have written each sum as two sums over the nonnegative integers, the inversion formula is simply (10). Choudhury et al. [7] discuss the sources of error in the inversion algorithm above and how to control it. In particular, the parameters A_f and A_l control the discretization error. By numerical experiments a good choice for these two parameters is $A_f = A_l = 18.4$.

Note that in the inversion formula we have sums of the form $\sum_{s=1}^{\infty} (-1)^s a_s$ where a_s is complex. In the summation of alternating series like these, it is recommendable to use the Euler transformation since it gives a much faster convergence of the infinite sums, see the discussion in Abate and Whitt [1] and in Choudhury et al. [7]. Specifically, the Euler sum provides an estimate $E(m, n)$ of the series $\sum_{s=1}^{\infty} (-1)^s a_s$, with:

$$E(m, n) = \sum_{k=0}^{m-1} \binom{m}{k} 2^{-m} S_{n+k}$$

and:

$$S_j = \sum_{k=0}^{m-1} (-1)^k a_k$$

Therefore, the use of the Euler algorithm requires $n+m$ evaluation of the complex function a_k . In particular, the numerical inversion requires the application of the Euler algorithm twice, once for the Fourier inversion and once for the Laplace inversion, for a total of $(n_f + m_f)(n_l + m_l)$ evaluations of the double transform. Then, the computational cost of the inversion is directly related to this product. In order to avoid numerical difficulties in the computation of the binomial coefficient in the Euler algorithm, we set $m_f = n_f + 15$ and $m_l = n_l + 15$, where the choice of n_f and n_l has to be tuned according to the volatility level.

The code for the inversion has been implemented in C by means of Microsoft Visual C++ Version 5.0. All the calculations have been performed on a Compaq Presario with Pentium 133 processor. Note that the code requires the use of operations between complex numbers. For this purpose, we have used the complex utility (Complex.c and Complex.h) for standard complex arithmetic operations available in Press et al. [25], pp. 948-950. Also, we have used the complex natural logarithm function and the complex Gamma function. This function has been computed by using the Lanczos approximation given in Press et al. [25], pp. 213. No additional subroutines have been used. All calculation have been done by means of standard precision calculation. We have coded the numerical inversion in Mathematica as well. Indeed, the Euler algorithm is already available in the Mathematica package NumericalMath`NLimit². Unfortunately, in some cases, mainly owing to low volatility, Mathematica had some numerical problems in the computation not encountered when using C. Therefore, all the following Tables have produced by using C.

In Tables 1a, 1b and 2 we give some numerical results where we set the following parameters: $S_0 = 100$, $r = 0.09$, $t = 1$ and we let the volatility vary between 5% and 50% and the strike price between 90 and 110.

In Table 1a and 1b we investigate how to chose n_f and n_l in the Euler algorithm in order to achieve a given accuracy. For example, in Table 1a for low volatility levels ($\sigma\sqrt{t} = 0.05$), the estimate of the option price at a five decimal digits accuracy requires a high number of terms. Larger part of the computational cost is due to perform the Laplace inversion (indeed we should set $n_l = 315$ and $m_l = n_l + 15$, whilst $n_f = 55$ and $m_f = n_f + 15$). As the volatility increases, the optimal values of n_f and n_l decrease quickly and consequently the computational time required for estimating the option price. For

²A notebook containing the Mathematica code is available from the Author upon request.

example when $\sigma\sqrt{t} \geq 0.4$, it is sufficient to have $n_f = n_l = 15$ in order to obtain a five digits accuracy. In Table 1b, we investigate how the choice relative to n_f and n_l affects the estimate in the Asian option price.

In Table 2 we compare the transform method with other approximations proposed in the literature: a) the lower and upper bounds for the option price given in Thompson [28] that improves ³ on the bounds given by Rogers and Shi [27], b) the approximations based on the fit to the lognormal density (see Turnbull and Wakeman [29] and Levy [22]) and the Reciprocal Gamma distribution (see Milevsky and Posner [24]) by using the first two integer moments, c) the numerical solution of the reduced form PDE given in Rogers and Shi [27]⁴. We do not consider the Edgeworth series expansion method because it gives completely unreliable results as the volatility parameter increases. Moreover, Ju [17] has proved that the Edgeworth series expansion around a lognormal density is not convergent. In Table 2 we report also the percentage of times that the different approximations give an estimate inside the lower and upper bounds.

³The lower bound is the same as in Rogers and Shi but its computation is easier. The upper bound improves the one in Rogers and Shi.

⁴Antonino Zanette (Università di Udine) has kindly provided the software PREMIA for numerically solving the PDE. The numerical scheme adopted is the Crank-Nicolson one with 3000 spatial and time grids.

From Table 2, we can see that the estimates obtained with the double numerical inversion when $n_l = 315$ and $n_f = 55$ only in one case out of thirty did not stay inside the bounds. This happened when $\sigma\sqrt{t} = 0.05$ and $K = 90$, and the inaccuracy revealed only at the sixth digits (upper bound: 13.378209804, double numerical inversion: 13.378212616)! Other numerical methods lose in accuracy for low volatility levels. For example, the numerical inversion of the Laplace transform given in Geman and Yor [14] cannot be performed if $\sigma\sqrt{t} < 0.08$, for the limited computer precision, see Fu et al. [12], Craddock et al. [8]. A similar problem occurs with the numerical solution of the reduced form PDE given in Rogers and Shi, see column PDE (R-S) in Table 2, and in this case the problem can be solved only thanks to a very fine discretization grid.

Also, from Table 2, we remark that the lognormal approximation, widely used by practitioners, gives price estimates outside the bounds for very low volatility levels too, i.e. when it should perform better. Moreover, this approximation deteriorates quickly as the volatility increases. A similar problem occurs for the Reciprocal Gamma approximation.

4 Conclusions

In the present paper, by using Fourier and Laplace transform, we have given a simple expression for a double transform of the option price of an Asian option. We discussed the numerical inversion and obtained very accurate results, in particular for the difficult cases of low volatility levels. An open question is whether we can apply the same technique to discrete monitoring of the average and to basket options, which are widely traded. In this direction, Ju [17] gives accurate approximations for the characteristic function of the logarithm of the discrete average.

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Table 1a: Accuracy desired and parameters of the Euler algorithm Parameters setting: $m_f=n_f+15$, $m_l=n_l+15$, $A_f=A_l=22.4$, $S_0=100$, $K=100$, $t=1$, $r=0.09$				
Number of decimal digits	2	3	4	5
Volatility σ	$n_f; n_l$	$n_f; n_l$	$n_f; n_l$	$n_f; n_l$
0,05	35; 175	35; 195	35; 255	55; 315
0,1	15; 115	15; 115	35; 115	35; 135
0,2	15; 15	15; 35	15; 55	15; 55
0,3	15; 15	15; 35	15; 35	15; 35
0,4	15; 15	15; 15	15; 15	15; 15
0,5	15; 15	15; 15	15; 15	15; 15

Figure 1: Table 1

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$\sigma=0.05$	nf; nl	15	35	55	75	95	115	135	155	175	195	215	235	255	275	295	315
15		5.27607	4.38630	4.21618	4.21062	4.23891	4.27513	4.29584	4.30082	4.30179	4.30190	4.30190	4.30190	4.30190	4.30190	4.30190	4.30190
35		5.27607	4.38630	4.21615	4.21029	4.24095	4.27061	4.29037	4.30109	4.30605	4.30771	4.30841	4.30828	4.30822	4.30823	4.30823	4.30823
55		5.27607	4.38630	4.21615	4.21029	4.24095	4.27061	4.29039	4.30111	4.30598	4.30779	4.30829	4.30835	4.30830	4.30827	4.30825	4.30824
75		5.27607	4.38630	4.21615	4.21029	4.24095	4.27061	4.29039	4.30111	4.30598	4.30779	4.30829	4.30835	4.30831	4.30827	4.30825	4.30824
95		5.27607	4.38630	4.21615	4.21029	4.24095	4.27061	4.29039	4.30111	4.30598	4.30779	4.30829	4.30835	4.30831	4.30827	4.30825	4.30824
$\sigma=0.1$	nf; nl	15	35	55	75	95	115	135	155	175	195	215	235	255	275	295	315
15		5.29278	4.91301	4.90399	4.91222	4.91498	4.91534	4.91534	4.91534	4.91534	4.91534	4.91534	4.91534	4.91534	4.91534	4.91534	4.91534
35		5.29278	4.91301	4.90405	4.91255	4.91480	4.91509	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512
55		5.29278	4.91301	4.90405	4.91255	4.91480	4.91509	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512
75		5.29278	4.91301	4.90405	4.91255	4.91480	4.91509	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512
95		5.29278	4.91301	4.90405	4.91255	4.91480	4.91509	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512	4.91512
$\sigma=0.2$	nf; nl	15	35	55	75	95	115	135	155	175	195	215	235	255	275	295	315
15		6.77613	6.77723	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735
35		6.77613	6.77723	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735
55		6.77613	6.77723	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735
75		6.77613	6.77723	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735
95		6.77613	6.77723	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735	6.77735
$\sigma=0.3$	nf; nl	15	35	55	75	95	115	135	155	175	195	215	235	255	275	295	315
15		8.82828	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876
35		8.82828	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876
55		8.82828	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876
75		8.82828	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876
95		8.82828	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876	8.82876
$\sigma=0.4$	nf; nl	15	35	55	75	95	115	135	155	175	195	215	235	255	275	295	315
15		10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377
35		10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377
55		10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377
75		10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377
95		10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377	10.92377
$\sigma=0.5$	nf; nl	15	35	55	75	95	115	135	155	175	195	215	235	255	275	295	315
15		13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816
35		13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816
55		13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816
75		13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816
95		13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816	13.02816

Figure 3: Table 3

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Table 2: A Comparison of Asian Option Pricing Models
Parameters setting: $m_t=n_t+15$, $m_i=n_i+15$, $A_t=A_i=22.4$, $S_0=100$, $t=1$, $r=0.09$

v	K	Lower Bound	Lognormal	Rec. Gamma	Double numerical Inversion				PDE (R-S)	Upper Bound
					$(n_t, m_t)=(15,30)$	$(n_i, m_i)=(15,30)$	$(n_t, m_t)=(55,70)$	$(n_i, m_i)=(55,70)$		
					$(n_t, m_t)=(15,30)$	$(n_i, m_i)=(175,190)$	$(n_t, m_t)=(175,190)$	$(n_i, m_i)=(315,330)$		
0,05	90	13,37821	13,37821	13,37821	12,06455	13,38473	13,38105	13,37821	13,37821	13,37821
0,05	95	8,80884	8,80888	8,80881	8,29387	8,80917	8,80717	8,80885	8,80869	8,80887
0,05	100	4,30823	4,30972	4,30720	5,27607	4,30179	4,30598	4,30824	4,30145	4,30837
0,05	105	0,95833	0,95815	0,95851	2,99514	0,96618	0,96198	0,95839	0,96037	0,95849
0,05	110	0,05210	0,05088	0,05303	1,37988	0,04751	0,05088	0,05214	0,05780	0,05236
0,1	90	13,38519	13,38629	13,38452	12,53421	13,38610	13,38520	13,38520	13,38490	13,38603
0,1	95	8,91183	8,91721	8,90820	8,50979	8,91137	8,91185	8,91185	8,91041	8,91296
0,1	100	4,91508	4,92310	4,90938	5,29278	4,91534	4,91512	4,91512	4,91298	4,91541
0,1	105	2,06993	2,07045	2,06952	2,91129	2,06999	2,07007	2,07007	2,07030	2,07038
0,1	110	0,63006	0,62338	0,63501	1,29656	0,63029	0,63027	0,63027	0,63236	0,63102
0,2	90	13,83122	13,86167	13,81078	13,73719	13,83150	13,83150	13,83150	13,83099	13,83721
0,2	95	9,99536	10,03043	9,97052	9,92790	9,99566	9,99566	9,99566	9,99515	9,99807
0,2	100	6,77700	6,80355	6,75716	6,77613	6,77735	6,77735	6,77735	6,77697	6,77866
0,2	105	4,29594	4,30408	4,28890	4,34518	4,29647	4,29647	4,29647	4,29649	4,29798
0,2	110	2,54546	2,53453	2,55250	2,60153	2,54622	2,54622	2,54622	2,54653	2,54854
0,3	90	14,98279	15,06704	14,92399	14,98317	14,98396	14,98396	14,98396	14,98369	14,99285
0,3	95	11,65475	11,73287	11,59733	11,65484	11,65589	11,65589	11,65589	11,65570	11,66128
0,3	100	8,82755	8,88576	8,78216	8,82828	8,82876	8,82876	8,82876	8,82861	8,83329
0,3	105	6,51635	6,54628	6,49026	6,51794	6,51779	6,51779	6,51779	6,51780	6,52257
0,3	110	4,69491	4,69511	4,69045	4,69712	4,69671	4,69671	4,69671	4,69678	4,70265
0,4	90	16,49702	16,65395	16,38398	16,49997	16,49997	16,49997	16,49997	16,49981	16,51601
0,4	95	13,50789	13,64791	13,40168	13,51071	13,51071	13,51071	13,51071	13,51062	13,52377
0,4	100	10,92090	11,03114	10,83224	10,92377	10,92377	10,92377	10,92377	10,92369	10,93596
0,4	105	8,72680	8,79965	8,66299	8,72994	8,72994	8,72994	8,72994	8,72994	8,74234
0,4	110	6,89990	6,93321	6,86426	6,90349	6,90349	6,90349	6,90349	6,90350	6,91747
0,5	90	18,18295	18,43698	17,99498	18,18885	18,18885	18,18885	18,18885	18,18874	18,22077
0,5	95	15,43707	15,66486	15,25983	15,44272	15,44272	15,44272	15,44272	15,44267	15,47216
0,5	100	13,02253	13,21198	12,86687	13,02816	13,02816	13,02816	13,02816	13,02810	13,05680
0,5	105	10,92375	11,06751	10,79735	10,92963	10,92963	10,92963	10,92963	10,92962	10,95880
0,5	110	9,11795	9,21323	9,02510	9,12432	9,12432	9,12432	9,12432	9,12431	9,15600
% Inside Bound		100,00%	6,67%	0,00%	50,00%	76,67%	83,33%	96,67%	60,00%	100,00%