COMPUTATION OF VALUE AT RISK AND EXPECTED SHORTFALL IN A CREDIT PORTFOLIO UNDER THE ONE-FACTOR MERTON MODEL

JOSEP J. MASDEMONT AND LUIS ORTIZ-GRACIA

Abstract. We propose a new methodology to compute *Value at Risk* (VaR) and *Expected Shortfall* (ES) for quantifying losses in credit portfolios. We approximate the cumulative distribution of the loss function by a finite combination of Haar wavelet basis functions and calculate the coefficients of the approximation by inverting its Fourier transform. The Wavelet Approximation (WA) method is particularly suitable for non-smooth distributions, often arising in small or concentrated portfolios, when the hypothesis of the Basel II formulas are violated. To test the methodology we consider the one-factor Merton model as our model framework. WA is an accurate, robust and fast method, allowing to estimate VaR and ES much more quickly than with a Monte Carlo (MC) method at the same level of accuracy and reliability. This work is based on [Mas11, Ort12].

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1. Introduction

Financial companies need to evaluate and to manage risks originated from their business activities. In particular, the credit risk underlying the credit portfolio is often the largest risk in a bank and its measure is used to assign capital in order to absorb potential losses arising from the credit portfolio.

The Merton model is the basis of the Basel II IRB approach. It is a Gaussian one factor model such that default events are driven by a latent common factor that is assumed to follow a Gaussian distribution. Under this model, loss only occurs when an obligor defaults in a fixed time horizon. If we assume certain homogeneity conditions, this one factor model leads to a simple analytical asymptotic approximation for the loss distribution and *Value at Risk* (VaR), also called the *Asymptotic Single Risk Factor* (ASRF) model. This approximation works well for a large number of small exposures but can underestimate risks in the presence of exposure concentrations (see [Gie06]).

Concentration risks in credit portfolios arise from an unequal distribution of loans to single borrowers (*name concentration*) or different industry or regional sectors (*sector* or *country concentration*). Moreover, certain dependencies like, for example, direct business links between different borrowers, can increase the credit risk in a portfolio since the default of one borrower can cause the default of a dependent second borrower. This effect is called *default contagion* and is linked to both name and sector concentration.

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In credit risk management one is particularly interested in the portfolio loss distribution. Since the portfolio loss is usually modeled as a sum of random variables, the main task is to evaluate the probability density function (PDF) of such a sum. The PDF of a sum of random variables is equal to the convolution of the respective PDFs of the individual asset loss distributions. The analytical evaluation of this convolution is a difficult problem and even computationally is very intensive. In full generality it is impractical for realistic size portfolios.

Monte Carlo simulation is a standard method for measuring the risk of a credit portfolio. However this method is very time-consuming when the size of the portfolio increases. Computations can become unworkable in many situations, taking also into account that financial companies have to re-balance their credit portfolios frequently.

For all these reasons, several methods have been developed during the last years. The saddle point approximation due to [Mar01] gives an analytical approximation of the Laplace inversion of the moment generating function (MGF). This method has been improved by [Mar06] based on conditional independence models. [Gla07] applies the methodology developed by [Aba00] to the one-factor Merton model. First, the Bromwich integral is approximated by an infinite series using the Trapezoidal Rule and second, the convergence of the infinite series is accelerated by a method called Euler summation. They have shown that the cumulative distribution function (CDF) is comparatively accurate in the regions associated with small losses but it worsens in the tail region, i.e. for big losses. This is due to the fact that the infinite series obtained by the Euler summation is an alternating series where each term is very big in absolute value.

Another approach to numerically invert the Laplace transform has been studied by [Hoo82] and [Ahn03]. Following [Aba00], it consists in applying the Poisson algorithm to approximate the Bromwich integral by an infinite series, and then to use the quotient-difference (QD) algorithm to accelerate its slow convergence. We refer to this approach as *the Hoog algorithm*. Also [Tak08] applies this methodology to the multi-factor Merton model. The numerical examples presented in these papers show that, in contrast with the Euler summation technique, Hoog algorithm is quite efficient in measuring tail probabilities.

Our contribution is a novel methodology for computing VaR and ES via numerically inverting the Fourier transform of the CDF of the loss function, once we have approximated it by a finite sum of Haar wavelets basis functions. Up to certain extent, the idea is similar to the one in [Aba96], which uses Laguerre polynomials instead of wavelets. In the financial context, [Hav09] also performs a Laplace transform inversion for option pricing purposes using a series expansion in terms of the Franklin hat wavelets. The authors numerically compute the coefficients of the approximation by minimizing the average of squared errors between the true option prices and estimated prices. The technique to get the coefficients in our method is quite different in the sense that, our analytical treatment provides an expression for the wavelet coefficients by means of the Cauchy's integral theorem. Then one can compute them using an ordinary trapezoidal rule avoiding this way the infinite series of [Gla07] and [Tak08]. The power of the WA method mostly resides in the good balance between computational time and accuracy both for small and high loss levels, and also for a wide range of portfolios, independent of concentration types and sizes. The saddle point approach, as an asymptotic method, tends in general to work better for high VaR confidence levels when the size of the portfolio increases. Moreover, if the loss distribution is not smooth due to exposure concentration, a straightforward implementation may be insufficient. Finally, it is important to remark that Haar wavelets are naturally capable to reproduce the step-like form distribution derived from the Vasicek model, even when dealing with extremely small or concentrated portfolios.

2. Portfolio Loss and Value at Risk

To represent the uncertainty about future events, we specify a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with sample space Ω, σ-algebra $\mathcal{F}$, probability measure $\mathbb{P}$ and with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We fix a time horizon $T > 0$. Usually T equals one year.

Consider a credit portfolio consisting of N obligors. Any obligor n can be characterized by three parameters: the *exposure at default* E_n, the *loss given default* which without loss of generality we assume to be 100% and the *probability of default* P_n, assuming that each of them can be estimated from empirical default data. The exposure at default of an obligor denotes the portion of the exposure of the obligor that is lost in case of default. Let D_n be the default indicator of obligor n taking the following values

$$D_n = \begin{cases} 1, & \text{if obligor } n \text{ is in default,} \\ 0, & \text{if obligor } n \text{ is not in default,} \end{cases}$$

Let L be the portfolio loss given by:

$$L = \sum_{n=1}^{N} L_n,$$

where $L_n = E_n \cdot D_n$.

To test our methodology we consider the one-factor Merton model as our model framework. The Merton model is a one period default model, i.e., loss only occurs when an obligor defaults in a fixed time horizon. Based on the firm-value model, to describe the obligor's default and its correlation structure, we assign to each obligor a random variable called firm-value. The firm-value of obligor n at time T, $V_n(T)$, is represented by a common, standard normally distributed factor Y component (the state of the world or business cycle, usually called systematic factor) and an idiosyncratic noise component ϵ_n:

$$V_n(T) = \sqrt{\rho_n} Y + \sqrt{1-\rho_n}\epsilon_n,$$

where Y and ϵ_n, $\forall n \leq N$ are i.i.d. standard normally distributed.

In case that $\rho_n = \rho$ for all n, the parameter ρ is called the common asset correlation and it corresponds to the correlation between obligors. The important point is that conditional to the realization of the systematic factor Y, the firm's values and defaults are independent. From now on, we assume ρ_n to be constant.

Let us explain in detail the meaning of systematic and idiosyncratic risk. The first one can be viewed as the macro-economic conditions and affect the credit-worthiness of all obligors simultaneously. The second one represents conditions inherent to each obligor and this is why they are assumed to be independent of each other.

In the Merton model, obligor n defaults when its firm-value falls below the threshold level T_n, defined by $T_n \equiv \Phi^{-1}(P_n)$, where $\Phi^{-1}(x)$ denotes the inverse of the standard normal cumulative distribution function. The probability of default of obligor n conditional to a realization $Y = y$ is then given by,

$$p_n(y) \equiv \mathbb{P}(V_n < T_n \mid Y = y) = \Phi\left(\frac{T_n - \sqrt{\rho}y}{\sqrt{1-\rho}}\right).$$

Consequently, the conditional probability of default depends on the systematic factor, reflecting the fact that the business cycle affects the possibility of an obligor's default.

Let us consider a portfolio with N obligors, let f_L be the density function of L and F_L the cumulative distribution function of L.

Let $\alpha \in (0,1)$ be a given confidence level (usually α of interest are very close to 1). The α-quantile of the loss distribution of L in this context is called *Value at Risk* (VaR):

$$\mathrm{VaR}_\alpha = \inf\{l \in \mathbb{R} : \mathbb{P}(L \leq l) \geq \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}.$$

This is the measure chosen in the Basel II Accord for the computation of capital requirement, meaning that a bank that manages its risks according to Basel II, must reserve capital by an amount of VaR_α to cover potential extreme losses.

By definition, the Expected Shortfall at confidence level α is given by,

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_{VaR_\alpha}^{+\infty} x f_L(x)dx. \tag{1}$$

3. The Haar Basis Wavelets System

Consider the space $L^2(\mathbb{R}) = \{f : \int_{-\infty}^{+\infty} |f(x)|^2 \, dx < \infty\}$. For simplicity we can view this set as the set of functions $f(x)$ which get small in magnitude fast enough as x goes to plus and minus infinity.

A general structure for wavelets in $L^2(\mathbb{R})$ is called a *Multi-resolution Analysis* (MRA). We start with a family of closed nested subspaces,

$$... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$$

in $L^2(\mathbb{R})$ where,

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}),$$

and

$$f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}.$$

If these conditions are met, then there exists a function $\phi \in V_0$ such that $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j, where,

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k).$$

In other words, the function ϕ, called the *father function*, generates an orthonormal basis for each V_j subspace.

Let us define W_j in such a way that $V_{j+1} = V_j \oplus W_j$. This is, W_j is the space of functions in V_{j+1} but not in V_j, and so, $L^2(\mathbb{R}) = \sum_j \oplus W_j$. Then (see [Dau92]) there exists a function $\psi \in W_0$ such that defining,

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),$$

$\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j and $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a wavelet basis of $L^2(\mathbb{R})$. The ψ function is called *mother function* and the $\psi_{j,k}$ functions are known as *wavelet functions*.

For any $f \in L^2(\mathbb{R})$ a projection map of $L^2(\mathbb{R})$ onto V_m,

$$\mathcal{P}_m : L^2(\mathbb{R}) \to V_m,$$

is defined by means of

$$\mathcal{P}_m f(x) = \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{k=+\infty} d_{j,k}\psi_{j,k}(x) = \sum_{k\in\mathbb{Z}} c_{m,k}\phi_{m,k}(x), \tag{2}$$

where $d_{j,k} = \int_{-\infty}^{+\infty} f(x)\psi_{j,k}(x)dx$ are the wavelet coefficients and the $c_{m,k} = \int_{-\infty}^{+\infty} f(x)\phi_{m,k}(x)dx$ are the scaling coefficients. Note that the first part in (2) is a truncated wavelet series. If j were allowed to go to infinity, we would have the full wavelet summation. The second part of (2) gives an equivalent sum in terms of the scaling functions $\phi_{m,k}$. Considering higher m values (i.e. when more terms are used), the truncated series representation of the function f improves. There exists also an interesting relation between the wavelet coefficients and the scaling coefficients at different scales:

$$c_{j,k} = \frac{c_{j+1,2k} + c_{j+1,2k+1}}{\sqrt{2}}, \quad d_{j,k} = \frac{c_{j+1,2k} - c_{j+1,2k+1}}{\sqrt{2}}. \tag{3}$$

To develop our work we consider Haar wavelets (see [Dau92]). Using these wavelets, V_j is the set of $L^2(\mathbb{R})$ functions which are constant on each interval of the form $[\frac{k}{2^j}, \frac{k+1}{2^j})$ for all integers k. In this case the father and mother functions are given by,

$$\phi(x) = \begin{cases} 1, & \text{if } 0 \le x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

As opposed to Fourier series, a key fact about using wavelets is that wavelets can be moved (choosing the k value), stretched or compressed (choosing the j value) to accurately represent the local properties of a function. Moreover, $\phi_{j,k}$ is nonzero only inside the interval $[\frac{k}{2^j}, \frac{k+1}{2^j})$. In what follows we take worth of this fact to compute VaR without the need of knowing the whole distribution of the loss function.

4. Fourier Transform Inversion in a Fixed Interval $[a, b]$: the WA$^{[a,b]}$ Method

Let us consider a function $f \in L^2(\mathbb{R})$ and its Fourier transform, whenever it exists:

$$\widehat{f}(w) = \int_{-\infty}^{+\infty} e^{-iwx} f(x)dx. \tag{4}$$

Since $f \in L^2(\mathbb{R})$ we can expect that f decays to zero, so it can be well approximated in a finite interval $[a,b]$ by,

$$f^c(x) = \begin{cases} f(x), & \text{if } x \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

Following the theory of MRA (in a bounded interval), we can approximate $f^c(x) \simeq f_m^c(x)$ for all $x \in [a,b]$, where,

$$f_m^c(x) = \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}\left(\frac{x-a}{b-a}\right),$$

with convergence in L^2-norm.

The main idea behind the Wavelet Approximation method is to approximate $\widehat{f}$ by $\widehat{f_m^c}$ and then to compute the coefficients $c_{m,k}$ by inverting the Fourier Transform. Proceeding this way, we have,

$$\begin{aligned}\widehat{f}(w) &= \int_{-\infty}^{+\infty} e^{-iwx} f(x)dx \simeq \int_{-\infty}^{+\infty} e^{-iwx} f_m^c(x)dx \\ &= \sum_{k=0}^{2^m-1} c_{m,k}\left(\int_{-\infty}^{+\infty} e^{-iwx}\phi_{m,k}\left(\frac{x-a}{b-a}\right)dx\right).\end{aligned}$$

Introducing a change of variables, $y = \frac{x-a}{b-a}$, gives us,

$$\begin{aligned}\widehat{f}(w) &\simeq (b-a)\cdot e^{-iaw}\sum_{k=0}^{2^m-1} c_{m,k}\int_{-\infty}^{+\infty} e^{-iw(b-a)y}\phi_{m,k}(y)dy \\ &= (b-a)\cdot e^{-iaw}\sum_{k=0}^{2^m-1} c_{m,k}\widehat{\phi}_{m,k}\left((b-a)\cdot w\right).\end{aligned}$$

Finally, taking into account that $\widehat{\phi}_{m,k}(\xi) = 2^{-\frac{m}{2}}\widehat{\phi}(\frac{\xi}{2^m})e^{-i\frac{k}{2^m}\xi}$ and performing a change of variables, $z = e^{-i\frac{b-a}{2^m}w}$, we find,

$$\widehat{f}\left(\frac{2^m}{b-a}i\cdot\log(z)\right) \simeq 2^{-\frac{m}{2}}(b-a)\cdot z^{\frac{2^m a}{b-a}}\widehat{\phi}\left(i\cdot\log(z)\right)\sum_{k=0}^{2^m-1} c_{m,k}z^k. \tag{5}$$

If we define,

$$P_m(z) := \sum_{k=0}^{2^m-1} c_{m,k}z^k \quad \text{and} \quad Q_m(z) := \frac{2^{\frac{m}{2}} z^{-\frac{2^m a}{b-a}}\widehat{f}\left(\frac{2^m}{b-a}i\cdot\log(z)\right)}{(b-a)\widehat{\phi}\left(i\cdot\log(z)\right)},$$

then, according to the previous formula (5), we have,

$$P_m(z) \simeq Q_m(z). \tag{6}$$

Since $P_m(z)$ is a polynomial, it is (in particular) analytic inside a disc of the complex plane $\{z \in \mathbb{C} : |z| < r\}$ for $r > 0$. We can obtain expressions for the coefficients $c_{m,k}$ by means of Cauchy's integral formula. This is,

$$c_{m,k} = \frac{1}{2\pi i}\int_\gamma \frac{P_m(z)}{z^{k+1}}dz, \quad k = 0, ..., 2^m-1,$$

where γ denotes a circle of radius r, $r > 0$, about the origin.

Considering now the change of variables $z = re^{iu}$, $r > 0$, gives us,

$$c_{m,k} = \frac{1}{2\pi r^k} \int_0^{2\pi} \frac{P_m(re^{iu})}{e^{iku}} du, \tag{7}$$

where $k = 0, ..., 2^m - 1$.

Then, we can further expand expression (7) by,

$$c_{m,0} = \frac{1}{\pi} \int_0^{\pi} \Re(P_m(re^{iu}))du, \tag{8}$$

and,

$$c_{m,k} = \frac{2}{\pi r^k} \int_0^{\pi} \Re(P_m(re^{iu}))\cos(ku)du, \quad k = 1, \dots, 2^m - 1. \tag{9}$$

On the other side, since $\widehat{\phi}\,(i \cdot \log(z)) = \frac{z-1}{\log(z)}$, we have,

$$Q_m(z) = \frac{2^{\frac{m}{2}} z^{-\frac{2^m a}{b-a}} \widehat{f}\left(\frac{2^m}{b-a} i \cdot \log(z)\right) \log(z)}{(b-a)(z-1)},$$

and it has a pole at $z = 1$. Finally, making use of (6) and taking into account the former observation, we can exchange P_m by Q_m in (8) and (9) to obtain, respectively,

$$c_{m,0} \simeq \frac{1}{\pi} \int_0^{\pi} \Re(Q_m(re^{iu}))du, \tag{10}$$

and,

$$c_{m,k} \simeq \frac{2}{\pi r^k} \int_0^{\pi} \Re(Q_m(re^{iu}))\cos(ku)du, \quad k = 1, ..., 2^m - 1, \tag{11}$$

where $r \neq 1$ is a positive real number.

In practice, both integrals in (10) and (11) are computed by means of the Trapezoidal Rule, and we can define,

$$I(k) = \int_0^{\pi} \Re(Q_{m,j}(re^{iu}))\cos(ku)du, \tag{12}$$

and,

$$I(k;h) = \frac{h}{2}\left(Q_m(r) + (-1)^k Q_m(-r) + 2\sum_{s=1}^{M-1} \Re(Q_m(re^{ih_s}))\cos(kh_s)\right), \tag{13}$$

where $h = \frac{\pi}{M}$ and $h_s = sh$ for all $s = 0, \dots, M$. Proceeding this way we find,

$$\begin{aligned} c_{m,k} &\simeq \frac{2}{\pi r^k} I(k) \simeq \frac{2}{\pi r^k} I(k;h) \\ &= \frac{1}{M r^k}\left(Q_m(r) + (-1)^k Q_m(-r) + 2\sum_{s=1}^{M-1} \Re(Q_m(re^{ih_s}))\cos(kh_s)\right), \end{aligned} \tag{14}$$

where $k = 1, ..., 2^m - 1$.

5. Haar Wavelets Approximation

Here, we show that a semi-analytic formula, based on the Haar wavelets can be derived based on the characteristic function of the portfolio loss, which is defined as,

$$\varphi_{\text{loss}}(w) = \mathbb{E}\left[e^{-iwL}\right]. \tag{15}$$

The starting point for the derivation of the characteristic function is the tower property to calculate the conditional expectation, conditional on variable Y,

$$\varphi_{\text{loss}}(w) := \mathbb{E}\left[\mathbb{E}\left[e^{-iwL}\middle| Y\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-iw\sum_{n=1}^{N} E_n \cdot D_n\right)|Y\right]\right].$$

We recall that in a one-factor model framework, if the systematic factor Y is fixed, default occurs independently since the only remaining uncertainty is the idiosyncratic risk. Then,

$$\varphi_{\text{loss}}(w) = \mathbb{E}\left[\prod_{n=1}^{N} \mathbb{E}\left[e^{-iwE_n\cdot D_n}|Y\right]\right] = \mathbb{E}\left[\prod_{n=1}^{N}\vartheta_n(w;y)\right] = \int_{\mathbb{R}} f_Y(y)\cdot\prod_{n=1}^{N}\vartheta_n(w;y)dy, \tag{16}$$

where f_Y is the Gaussian density and,

$$\vartheta_n(w;y) := e^{-iwE_n}p_n(y) + 1 - p_n(y).$$

The conditional characteristic function for an individual obligor ϑ_n can be obtained analytically, and we only need to evaluate (16), the integral over y, numerically to find $\varphi_{\text{loss}}(w)$.

According to (15), the characteristic function φ_{loss} is the Fourier transform of the loss density f_L associated to the random variable L. Then,

$$\varphi_{\text{loss}}(w) = \int_{\mathbb{R}} e^{-iwl} f_L(l)dl = \int_{\mathbb{R}} e^{-iwl} F_L'(l)dl, \tag{17}$$

where F_L' is the derivative of distribution function F_L associated to the random variable L.

Without loss of generality, we can assume that $\sum_{n=1}^{N} E_n = 1$, and therefore, we can consider,

$$F_L(l) = \begin{cases} \overline{F}_L(l), & \text{if} \quad 0\le l\le 1,\\ 1, & \text{if} \quad l>1,\end{cases}$$

for certain $\overline{F}_L$ defined in $[0,1]$.

If we integrate by parts the expression (17), we have,

$$\varphi_{\text{loss}}(w) = e^{-iw} + iw\int_0^1 e^{-iwl}\overline{F}_L(l)dl,$$

and then $(\varphi_{\text{loss}}(w) - e^{-iw})/(iw)$ is the Fourier transform of $\overline{F}_L$.

Since $\overline{F}_L \in L^2([0,1])$, according to the theory of MRA we can approximate $\overline{F}_L$ in $[0,1]$ by a sum of Haar scaling functions,

$$\overline{F}_L(l) \simeq \overline{F}_L^m(l), \quad \overline{F}_L^m(l) = \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x), \tag{18}$$

with convergence in the L^2-norm.

Finally, we can apply the $\mathrm{WA}^{[a,b]}$ method in a bounded interval, as described in Section 4, where $[a,b]=[0,1]$ in this case, to recover the coefficients of the approximation (18).

In summary, with the characteristic function of the portfolio loss determined, we can apply the $\mathrm{WA}^{[a,b]}$ method to perform the inverse Fourier transformation to recover the density or the cumulative probability function of the portfolio loss efficiently. Then, VaR and ES values can be easily extracted from the cumulative probability function.

5.1. **VaR and ES Computation.** It can be easily proved that

$$0 \leq c_{m,k} \leq 2^{-\frac{m}{2}}, \qquad k=0,1,...,2^m-1,$$

and

$$0 \leq c_{m,0} \leq c_{m,1} \leq ... \leq c_{m,2^m-1}.$$

Considering an approximation in a level of resolution m, VaR can now be quickly computed with m coefficients due to the compact support of the basis functions. Observe that due to the approximation (18) we have,

$$\overline{F}_L(\mathrm{VaR}_\alpha) \simeq 2^{\frac{m}{2}} \cdot c_{m,\overline{k}}$$

for a certain $\overline{k} \in \{0,1,...,2^m-1\}$. Thus, we can simply start searching VaR_α by means of the following simple iterative procedure: first we compute $\overline{F}^m_L(\frac{2^{m-1}}{2^m})$. If $\overline{F}^m_L(\frac{2^{m-1}}{2^m}) > \alpha$ then we compute $\overline{F}^m_L(\frac{2^{m-1}-2^{m-2}}{2^m})$, otherwise we compute $\overline{F}^m_L(\frac{2^{m-1}+2^{m-2}}{2^m})$, and so on. This algorithm finishes after m steps storing the $\overline{k}$ value such that $\overline{F}^m_L(\frac{\overline{k}}{2^m})$ is the closest value to α in our m resolution approximation.

In fact, due to the stepped shape of the Haar wavelets approximation, $\overline{F}^m_L(\xi) = \overline{F}^m_L(\frac{\overline{k}}{2^m})$, for all $\xi \in \left[\frac{\overline{k}}{2^m}, \frac{\overline{k}+1}{2^m}\right)$. In what follows let us take, $\mathrm{VaR}^{W(m)}_\alpha = \frac{2\overline{k}+1}{2^{m+1}}$, the middle point of this interval, as the VaR value computed by means of this wavelet algorithm at scale m.

Then, integrating by parts (1) and using the approximation in (18) we have,

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\left(1-\alpha\mathrm{VaR}_\alpha - \int_{VaR_\alpha}^1 \overline{F}_L(x)dx\right) \simeq \mathrm{ES}^{W(m)}_\alpha, \tag{19}$$

where,

$$\mathrm{ES}^{W(m)}_\alpha \equiv \frac{1}{1-\alpha}\left(1-\alpha\mathrm{VaR}^{W(m)}_\alpha - \frac{1}{2^{\frac{m}{2}+1}}c_{m,\overline{k}} - \frac{1}{2^{\frac{m}{2}}}\sum_{k=\overline{k}+1}^{2^m-1} c_{m,k}\right).$$

6. Numerical Experiments Setting

Real situations in financial companies show the existence of strong concentrations in their credit portfolios, while Basel II formulae to calculate VaR are supported under unrealistic hypothesis, such as infinite number of obligors with small exposures. For these reasons, we test our methodology with small and concentrated portfolios.

In order to consider concentrated portfolios, we have fixed the exposure taking $E_n = \frac{C}{n}$ (where C is a constant such that $\sum_{n=1}^{N} E_n = 1$). Without loss of generality we assume that $P_n = pd$, where pd is the probability of default associated to each obligor.

For the numerical integration (16) we truncate the domain to $[-5, 5]$ and we apply the rectangle method using 100 points. With this simple and naive method, we can achieve high accurate results at high confidence levels. However, to get more precision further in the tail with probably less computational effort, the authors recommend to use the Gauss-Hermite quadrature. Finally, we consider $M = 2^m$ and $r = 0.9995$ in (14) for the coefficients computation.

As an example, let us consider $N = 100, pd = 0.003, \alpha = 0.999, \rho = 0.15$ and the scale of approximation $m = 10$.

With this setting, the VaR value at 99.9% confidence level with the WA method is $\text{VaR}_{0.999}^{W(10)} = 0.197754$ and the ES value at 99.9% confidence level is $\text{ES}_{0.999}^{W(10)} = 0.217655$.

7. Conclusions

We have presented a numerical approximation to the loss function based on Haar wavelets. First of all we approximate the discontinuous distribution of the loss function by a finite summation of Haar scaling functions, and then we calculate the coefficients of the approximation by inverting its Fourier transform. Due to the compact support property of the Haar system, only a few coefficients are needed for the VaR computation.

A wide range of numerical examples are shown in [Mas11]. The WA method is applicable and very accurate to different sized portfolios needing also of short time computations. Moreover, the Wavelet Approximation is robust since the method is very stable under changes in the parameters of the model. The stepped form of the approximated distribution makes the Haar wavelets natural and very suitable for the approximation.

References

[Aba96] J. Abate, G.L. Choudhury and W. Whitt (2000). On the Laguerre method for numerically inverting Laplace transforms. *Journal on Computing*. Vol 8. No. 4. 2

[Aba00] J. Abate, G.L. Choudhury and W. Whitt (2000). An introduction to numerical transform inversion and its application to probability models. *Computational Probability*, ed. W.K. Grassman, Kluwer, Norwell, MA. pp. 257-323. 2

[Ahn03] J. Ahn, S. Kang and Y. Kwon (2003). A flexible inverse Laplace transform algorithm and its application. *Computing* 71. 115-131. 2

[Dau92] I. Daubechies (1992). *Ten lectures on wavelets.* CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM.

[Gie06] G. Giese (2006). A saddle for complex credit portfolio models. *RISK* (July), 84-89. 4, 5

[Gla07] P. Glasserman and J. Ruiz-Mata (2007). Computing the credit loss distribution in the Gaussian copula model: a comparison of methods. *Journal of Credit Risk* 3. 33-66. 1

[Hav09] E. Haven, X. Liu, C. Ma and L. Shen. Revealing the implied risk-neutral MGF from options: The wavelet method. *Journal of Economics Dynamics & Control* 33. 692-709. 2

[Hoo82] F. de Hoog, J. Knight and A. Stokes (1982). An improved method for numerical inversion of Laplace transforms. *SIAM Journal on Scientific and Statistical Computing* 3. 357-366. 2

[Mar01] R. Martin, K. Thompson and C. Browne (2001). Taking to the saddle. *RISK* (June), 91-94.

[Mar06] R. Martin and R. Ordovás (2006). An indirect view from the saddle. *RISK* (October), 94-99. 2

[Mas11] J. J. Masdemont and L. Ortiz-Gracia (2011). Haar wavelets-based approach for quantifying credit portfolio losses. *Quantitative Finance*, DOI: 10.1080/14697688.2011.595731. 2

[Ort12] L. Ortiz-Gracia and J. J. Masdemont (2012). Peaks and jumps reconstruction with B-splines scaling functions. *Working paper*, available at www.arxiv.org. 2

[Tak08] Y. Takano and J. Hashiba (2008). A novel methodology for credit portfolio analysis: numerical approximation approach. Available at *www.defaultrisk.com*. 1, 10

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Josep J. Masdemont
Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Diagonal 647
08028 Barcelona
Spain
Email address: josep@barquins.upc.edu

Luis Ortiz-Gracia
Centre de Recerca Matemàtica
Campus de Bellaterra, Edifici C
08193 Bellaterra (Barcelona)
Spain
Email address: lortiz@crm.cat