Iterative Construction of the Optimal Bermudan Stopping Time

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1 Discrete optimal stopping problem

Let us consider a discrete non-negative random process $(H_i)_{0\leq i\leq k}$ adapted to a filtration $(\mathcal{F}_i)_{0\leq i\leq k}$. We suppose that this process is a function of another underlying process X_i: $H_i = h_i(X_i)$ for some function h_i.

We are interested in the problem of finding a stopping time that maximizes the $\mathcal{F}_i$-conditional expectation $\mathbb{E}_i[H_\tau]$ over stopping time τ taking value in $\{i, ..., k\}$

$$\sup_{\tau\in\{i,\dots,k\}} \mathbb{E}_i[H_\tau] \tag{1}$$

We call an *optimal stopping time* for this problem an $\mathcal{F}_i$-stopping time τ_i^* such that

$$\mathbb{E}_i[H_{\tau_i^*}] = \sup_{\tau\in\{i,\dots,k\}} \mathbb{E}_i[H_\tau]$$

We define the process $Q_i^* := \mathbb{E}_i[H_{\tau_i^*}]$. The process $(Q_i^*)_{0\leq i\leq k}$ is called the *Snell envelope* process of $(H_i)_{0\leq i\leq k}$. It can be constructed using the well known *backward dynamic programming* algorithm. In fact, by definition we have $Q_k^* = H_k$

$$\left\{\begin{array}{lcl} Q_k^* & = & H_k \\ Q_i^* & = & \max\Big(H_i,\ \mathbb{E}_i\left[Q_{i+1}^*\right]\Big)\ \text{, for } 0\leq i\leq k-1 \end{array}\right. \tag{2}$$

Then τ_i^* can be represented by

$$\begin{array}{rcl} \tau_i^* & = & \inf\left\{j, i\leq j\leq k : H_j \geq \mathbb{E}_j[Q_{j+1}^*]\right\} \\ & = & \inf\left\{j, i\leq j\leq k : H_j \geq \mathbb{E}_j[H_{\tau_{j+1}^*}]\right\} \end{array}$$

or

$$\left\{\begin{array}{lcl} \tau_k^* & = & k \\ \tau_i^* & = & i\mathbb{I}_A + \tau_{i+1}^*\mathbb{I}_{A^c} \end{array}\right. \tag{3}$$

where $A = \{H_i \geq \mathbb{E}_{i+1}[H_{\tau_{i+1}^*}]\}$

2 Examples of sub-optimal family of stopping time

The optimal stopping time family $(\tau_i^*)_i$ (or equivalently the price process (Q_i^*)) can be estimated via plain Monte Carlo simulation of the backward dynamic program (3). But this raises the problem of how to estimates the conditional expectation $\mathbb{E}_{i+1}[H_{\tau_{i+1}^*}]$. Using Monte Carlo method to estimate this latter implies nested simulations, which makes the algorithm potentially very slow, as illustrated in the figure (1).

In fact, to estimate $Q_i^* = \max\Big(H_i,\ \mathbb{E}_i\left[Q_{i+1}^*\right]\Big)$, we need the conditional expectation $\mathbb{E}_i\left[Q_{i+1}^*\right]$. We estimate this latter by sampling conditional on the state at time i, hence the nested simulations.

One possible way to circumvent this problem is the well known Longstaff-Schwartz algorithm. We note the price given by this algorithm by Q_i^{LS}.

The idea is to estimate the conditional expectations $\mathbb{E}_i\left[Q_{i+1}^{LS}\right]$ using some regression methods, without any nested simulation. In fact, in the case of a Markovian setting, the conditional expectation is a function of the underlying process X_i

$$\mathbb{E}_i\left[Q_{i+1}^{LS}\right] = \phi_i(X_i)$$

The function ϕ_i is then approximated by an orthogonal projection $\langle\alpha_i, g_i\rangle$. We then choose α_i as the one that minimize the second order moment

$$\left\{\begin{array}{l} \min\ \mathbb{E}\left[\Big(Q_{i+1}^{LS} - \langle\alpha_i, g_i\rangle(X_i)\Big)^2\right] \\ \alpha_i \in \mathbb{R}^d \end{array}\right.$$

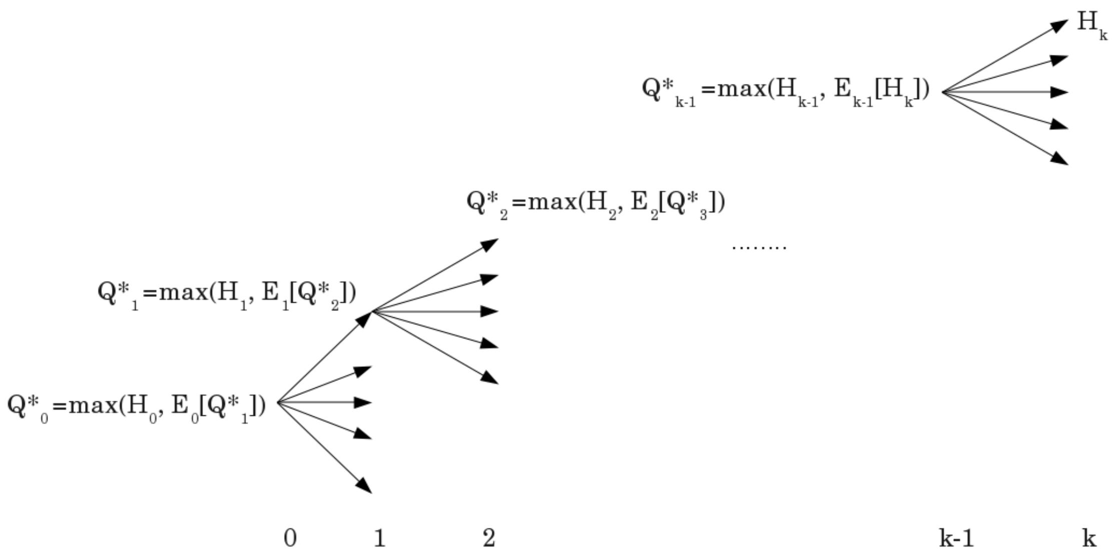

Figure 1: Backward dynamic programming using nested Monte Carlo simulations

Using Monte Carlo method, this minimization problem can be rewritten, for a set of M simulated paths $\{w_1, w_2, ..., w_M\}$:

$$\left\{ \begin{array}{l} \min \sum_m \left[Q_{i+1}^{LS}(w_m) - \langle \alpha_i, g_i \rangle (X_i(w_m)) \right]^2 \\ \alpha_i \in \mathbb{R}^d \end{array} \right.$$

Then Q_i^{LS} is estimated by $Q_i^{LS}(w_m) = \max\left(h_i(X_i(w_m)),\ \langle \alpha_i, g_i \rangle (X_i(w_m))\right)$.
The stopping time given by this method is

$$\tau_i^{LS} = \inf \left\{ j\ ,\ i \leq j \leq k :\ h_j(X_j) \geq \langle \alpha_j, g_j \rangle (X_j) \right\}$$

Of course, this exercise strategy is generally sub-optimal, in the sense that we have $Q_i^{LS} \leq Q_i^*$.

Another possibility is to suppose a parametric form for the stopping time. For example, in the context of bermudan swaption, Andersen proposes in [Andersen 2000] a method that parametrizes the exercise policy and then optimizes these parameters over a set of simulated paths to determine an approximation of the optimal exercise strategy.

In fact, this method considers the following exercise strategy

$$\tau_i^{A_1} = \inf \{ j \ , \ i \leq j \leq k : \ H_j \geq \alpha_j \}$$

where $(\alpha_j)_{0 \leq j \leq k}$ is a vector of parameters that will be estimated by a standard optimization routine that maximizes the price of corresponding bermudan prices.

It's also possible to consider a more refined strategy

$$\tau_i^{A_2} = \inf \left\{ j \ , \ i \leq j \leq k : \ H_j \geq \max(\alpha_j \ , \ \max_{j \leq p \leq k} \mathbb{E}_j[H_p]) \right\}$$

Contrary to the first strategy, we don't exercise if the payoff when exercising at j is less than the price of a European option starting at j with maturity less than k.

See [Andersen 2000] for more details.

3 Iterative improvement upon an initial family of stopping times

We suppose that we have at hand a family of stopping times τ_i that verify the following properties

$$\begin{array}{c} i \leq \tau_i \leq k \ , \ \tau_k = k \\ \tau_i > i \Rightarrow \tau_i = \tau_{i+1} \ , \ 0 \leq i \leq k-1 \end{array} \tag{4}$$

We note $Q_i = \mathbb{E}_i \left[H_{\tau_i} \right]$ the price given by the exercise policy τ_i. We introduce the intermediate process

$$\widetilde{Q}_i = \max_{j \leq p \leq \min(j+\kappa, \ k)} \mathbb{E}_j[H_{\tau_p}]$$

where κ is a fixed window parameter such that $1 \leq \kappa \leq k$.

[Kolodko, Schoenmakers 2009] shows that it's possible to construct a new family of stopping times that improves the original one, in the sense that we get a higher price for the bermudan option. They propose a new exercise strategy $\widehat{\tau}_i$

$$\begin{aligned} \widehat{\tau}_i &= \inf \left\{ j \in [i,k] \ : \ H_j \geq \widetilde{Q}_i \right\} \\ &= \inf \left\{ j \in [i,k] \ : \ H_j \geq \max_{j \leq p \leq \min(j+\kappa,k)} \mathbb{E}_j[H_{\tau_p}] \right\} \end{aligned} \tag{5}$$

This new stopping time verify the conditions (4).

If we note $\widehat{Q}_i = \mathbb{E}_i \left[H_{\widehat{\tau}_i} \right]$ the price process given by the exercise policy $\widehat{\tau}_i$, then

$$Q_i \leq \widetilde{Q}_i \leq \widehat{Q}_i \leq Q_i^* \ , \ 0 \leq i \leq k$$

witch means that the stopping rule $\widehat{\tau}_i$ is closer to the optimal one τ_i^* than the initial strategy τ_i. It's then tempting to iterate this procedure in order to get a good estimate of the stopping strategy τ_i^*.

In fact, starting from an initial stopping strategy and its corresponding price $(\tau_i^{(0)}, Q_i^{(0)})_{0\leq i\leq k}$, we can construct a sequence of pairs $\left\{(\tau_i^{(m)}, Q_i^{(m)})_{0\leq i\leq k}\right\}_{m\geq 0}$ such that

$$\begin{aligned} \tau_i^{(m+1)} &= \inf\left\{ j \in [i,k] \ : \ H_j \geq \max_{j\leq p\leq \min(j+\kappa,k)} \mathbb{E}_j[H_{\tau_p^{(m)}}] \right\} \\ &= \inf\left\{ j \in [i,k] \ : \ H_j \geq \widetilde{Q}_i^{(m)} \right\} \end{aligned} \tag{6}$$

and

$$Q_i^{(m+1)} = \mathbb{E}_i \left[H_{\tau_i^{(m+1)}} \right] \tag{7}$$

This sequence satisfies the important inequalities, for $m \geq 1 \ , \ 0 \leq i \leq k$,

$$\begin{array}{l} Q_i^{(0)} \leq Q_i^{(m)} \leq \widetilde{Q}_i^{(m)} \leq Q_i^{(m+1)} \leq Q_i^* \\ \tau_i^{(m)} \leq \tau_i^{(m+1)} \leq \tau_i^* \end{array}$$

This means that each iteration improves the estimation of the optimal strategy, in a non-decreasing way. Moreover [Kolodko, Schoenmakers 2009] proves that this estimation become exact after a finite number of iterations. In fact, we recall the Proposition 4.3 of [Kolodko, Schoenmakers 2009].

Proposition: For $i \in [0,k]$ the following identity holds

$$\begin{aligned} Q_i^{(m)} &= Q_i^*, \quad \text{for } m \geq k-i \\ \tau_i^{(m)} &= \tau_i^*, \quad \text{for } m \geq k-i \end{aligned}$$

witch means that we need at most k iterations to get the exact price Q_0^*.

4 Monte Carlo implementation of the iterative Method

We recall the procedure proposed in [Kolodko, Schoenmakers 2009] to implement a Monte Carlo version of the iterative method explained above.

We consider random set of dates $\Theta^{(m)}$ where the strategy $\tau^{(m)}$ says "exercise"

$$\Theta_i^{(m)} = \{ j \geq i \ : \ H_j \geq \widetilde{Q}_j^{(m)} \} \ , \ 0 \leq i \leq k \ , \ m \geq 1$$

Since the sequence $(\widetilde{Q}_j^{(m)})_{m\geq 1}$ is non-decreasing, we have

$$\Theta_i^{(m+1)} \subset \Theta_i^{(m)} \subset \Theta_i^{(1)} \ , \ 0 \leq i \leq k \ , \ m \geq 1$$

If we have at hand the set $\Theta_i^{(m)}$, then we can compute the stopping time $\tau_p^{(m)}$, for $i \leq p \leq k$, by

$$\tau_p^{(m)} = \inf\left\{\Theta_i^{(m)} \cap \{p, ..., k\}\right\}$$

Suppose we have at hand a sample path $X = (X_j)_{0 \leq j \leq k}$, we want to construct $\Theta_i^{(m)}$. We present here a procedure to construct this random set $\Theta_i^{(m)}$.

We note $\mathbf{func}(m,i,w)$ a function that take as arguments two integers m, i and a random path w. This function has as output the set of integers $\Theta_i^{(m)} \subset \{i,...,k\}$.

$$\Theta_i^{(m)} = \mathbf{func}(m,i,w)$$

1. **If** $m = 1$
   Using the given initial stopping family $\tau^{(0)}$, we can write the set of exercise dates

   $$\begin{aligned}\Theta_i^{(1)} &= \{j \geq i \ : \ H_j \geq \widetilde{Q}_j^{(1)}\} \\ &= \{j \geq i \ : \ H_j \geq \max_{j \leq p \leq \min(j+\kappa,k)} \mathbb{E}_j[H_{\tau_p^{(0)}}]\}\end{aligned}$$

   The conditional expectation in the definition of $\Theta_i^{(1)}$ can be estimated by Monte Carlo, or by explicit formulas if available. For example, if we start with a simple stopping family $\tau_i^{(0)} = i$, the conditional expectation is just the price of a European option, witch is in general available in closed formulas.

2. **If** $m > 1$
   We construct $\Theta_i^{(m)}$ recursively, by nested Monte Carlo simulations, in the following steps.

   (a) Construct the set $\Theta_i^{(m-1)} = \mathbf{func}(m-1,i,w)$

   (b) $\forall j \in \Theta_i^{(m-1)}$

      i. Simulate N_m paths w_α starting from $X_j(w)$, for $1 \leq \alpha \leq N_m$.

      ii. $\forall w_\alpha \in \{w_1,...,w_{N_m}\}$

         A. Construct the set $\Theta_j^{(m-1)}(w_\alpha) = \mathbf{func}(m-1,j,w_\alpha)$

         B. $\forall p = j,...,\min(j+\kappa,k)$, $\tau_p^{(m-1)}(w_\alpha) = \inf\left\{\Theta_j^{(m-1)}(w_\alpha) \cap \{p,...,k\}\right\}$

      iii. $\forall p = j,...,\min(j+\kappa,k)$, $\mathbb{E}_j[H_{\tau_p^{(m-1)}}] = \dfrac{1}{N_m}\displaystyle\sum_{\alpha=0}^{N_m} H_{\tau_p^{(m-1)}}(w_\alpha)$

      iv. $\widetilde{Q}_j^{(m-1)} = \displaystyle\max_{j \leq p \leq \min(j+\kappa,k)} \mathbb{E}_j[H_{\tau_p^{(m-1)}}]$

      v. If $H_j \geq \widetilde{Q}_j^{(m-1)}$, then $j \in \Theta_i^{(m)}$.

4.1 Practical implementation

Due to the computation time constraints, we will only consider the case of $m = 1$ i.e. we start with an initial exercise strategy $\tau^{(0)}$ then we construct the improved exercise strategy $\tau^{(1)}$. For the initial strategy $\tau^{(0)}$, we chose the two following strategies:

1. The strategy that exercises when the payoff is greater than all the lasting European options:
$$\tau_p^{(0)} = \inf \left\{ i \geq p \ , \ H_i \geq \max_{i \leq l \leq k} \mathbb{E}_i[H_l] \right\}$$
We suppose that we have at hand some exact or approximate formula to evaluate European option price, so that we don't need any nested Monte Carlo simulations to evaluate them.

2. The strategy given by Longstaff-Schwartz algorithm:
$$\tau_p^{(0)} = \inf \left\{ i \geq p \ , \ H_i \geq \langle \alpha_i, g_i \rangle (X_i) \right\}$$
where α_i and g_i are defined in section 2.

Now, to estimate $\tau_0^{(1)}$, we need to evaluate the conditional expectations in $\max_{j \leq p \leq k} \mathbb{E}_j[H_{\tau_p^{(0)}}]$ for $0 \leq j \leq k$.

One possibility would be to use nested Monte Carlo simulation as explained above in the procedure. It's also possible to use the same idea as in Longstaff-Schwartz algorithm and approximate this conditional expectations with regression methods.

In fact, we can look for the coefficients β_j^p such that $\langle \beta_j^p, g_j \rangle (X_j)$ is a good approximate of $\mathbb{E}_j[H_{\tau_p^{(0)}}]$, by solving the least squares problem

$$\begin{cases} \min \displaystyle\sum_m \left[H_{\tau_p^{(0)}}(w_m) - \langle \beta_j^p, g_j \rangle (X_j(w_m)) \right]^2 \\ \beta_j^p \in \mathbb{R}^d \end{cases}$$

Then, the stopping time $\tau_0^{(1)}$ can be approximated by

$$\tau_0^{(1)} = \inf \left\{ j \geq 0 \ , \ H_j \geq \max_{j \leq p \leq \min(j+\kappa, k)} \langle \beta_j^p, g_j \rangle (X_j) \right\}$$

That way, we don't have to simulate nested paths, hence the speed of the algorithm improves.

References

[Andersen 2000] Andersen, L. (2000) A simple Approach to the Pricing of Bermudan Swaptions in the Multi-Factor LIBOR Market Model. *Computational Finance*, 3, 1-32. 3, 4

[Kolodko, Schoenmakers 2009] Kolodko, A., Schoenmakers, J. (2006) Iterative construction of the optimal Bermudan stopping time *Finance and Stochastics*, 10, 27-49. 4, 5