

Stratified lognormal approximation for Asian options

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Abstract

In this note we describe the stratified lognormal approximation of [2] applied to the pricing and hedging of Asian options. In addition we provide an approximation for hedging strategies.

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1 Introduction

Asian options on the time integral $\Lambda_T := \int_0^T S_t dt$ of geometric Brownian motion

$$S_t = S_0 e^{\sigma B_t + (r - \sigma^2/2)t}, \quad t \in [0, T], \quad (1.1)$$

have been priced in [3], [1] by approximating Λ_T by a lognormal random variable, as

$$e^{-rT} E \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] \simeq e^{-rT} \left(\frac{1}{T} e^{\hat{\mu} + \hat{\sigma}^2/2} \Phi(d_1) - K \Phi(d_2) \right), \quad (1.2)$$

where

$$d_1 = \frac{\log(E[\Lambda_T]/(KT))}{\hat{\sigma}\sqrt{T}} + \hat{\sigma} \frac{\sqrt{T}}{2} = \frac{\hat{\mu}T + \hat{\sigma}^2T - \log(KT)}{\hat{\sigma}\sqrt{T}}$$

and

$$d_2 = d_1 - \hat{\sigma}\sqrt{T} = \frac{\log(E[\Lambda_T]/(KT))}{\hat{\sigma}\sqrt{T}} - \hat{\sigma} \frac{\sqrt{T}}{2},$$

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and $\hat{\mu}$, $\hat{\sigma}$ are estimated as

$$\hat{\sigma}^2 = \frac{1}{T} \log \left(\frac{E[\Lambda_T^2]}{(E[\Lambda_T])^2} \right) \quad (1.3)$$

and

$$\hat{\mu} = \frac{1}{T} \log E[\Lambda_T] - \frac{1}{2} \hat{\sigma}^2, \quad (1.4)$$

based on the first two moments of the lognormal distribution, cf. (3.1) below.

In [2], a more accurate approximation has been proposed by applying *stratified sampling* to the computation of (1.2), via the conditioning

$$E \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] = \int_0^\infty E \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \middle| S_T = z \right] dP(S_T = z).$$

Stratified sampling usually acts as a variance reduction method in Monte Carlo simulations, and in the present setting it also improves numerical precision as seen in the graphs of Figure 1 below.

2 Conditional calculus

In this section we recall and state some facts on the conditional distribution and moments of Λ_T given S_T . Rewriting (1.1) as the solution of

$$dS_t = (1 - p) \frac{\sigma^2}{2} S_t dt + \sigma S_t dB_t,$$

with $p = 1 - 2r/\sigma^2$, and

$$dP(S_T = z \mid S_0 = x) = \frac{1}{\sigma \sqrt{2\pi T}} e^{-(p\sigma^2 T/2 + \log(z/x))^2 / (2\sigma^2 T)} \frac{dz}{z},$$

we can rewrite the conditional law of Λ_T given $S_T = z$ without using the parameter $p \in \mathbb{R}$.

Lemma 2.1. *For all $z, T > 0$ we have*

$$\begin{aligned} \mathbb{P} \left(\Lambda_T \in dx \mid S_T = z, S_0 = 1 \right) &= \sigma \sqrt{\frac{\pi T}{2}} \exp \left(\frac{(\log z)^2}{2\sigma^2 T} - 2 \frac{1+z}{\sigma^2 x} \right) \theta \left(\frac{4\sqrt{z}}{\sigma^2 x}, \frac{\sigma^2 T}{4} \right) \frac{dx}{x}, \\ x &> 0. \end{aligned} \quad (2.1)$$

Next we define the functions

$$a_T(z) := \frac{1}{\sigma^2 p(z)} \left(\Phi \left(\frac{\log z}{\sqrt{\sigma^2 T}} + \frac{1}{2} \sqrt{\sigma^2 T} \right) - \Phi \left(\frac{\log z}{\sqrt{\sigma^2 T}} - \frac{1}{2} \sqrt{\sigma^2 T} \right) \right), \quad (2.2)$$

and

$$b_T(z) = \frac{1}{\sigma^2 q(z)} \left(\Phi \left(\frac{\log z}{\sqrt{\sigma^2 T}} + \sqrt{\sigma^2 T} \right) - \Phi \left(\frac{\log z}{\sqrt{\sigma^2 T}} - \sqrt{\sigma^2 T} \right) \right),$$

where

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-(\sigma^2 T/2 + \log z)^2 / (2\sigma^2 T)}, \quad \text{and} \quad q(z) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-(\sigma^2 T + \log z)^2 / (2\sigma^2 T)},$$

$z > 0$, and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \quad x \in \mathbb{R},$$

is the standard Gaussian cumulative distribution function.

Proposition 2.2. *We have*

$$E[\Lambda_T \mid S_T = z, S_0 = 1] = \frac{1}{\sigma^2 p(z)} \left(\Phi \left(\frac{\log z}{\sqrt{\sigma^2 T}} + \frac{1}{2} \sqrt{\sigma^2 T} \right) - \Phi \left(\frac{\log z}{\sqrt{\sigma^2 T}} - \frac{1}{2} \sqrt{\sigma^2 T} \right) \right),$$

and

$$E[(\Lambda_T)^2 \mid S_T = z, S_0 = 1] = \frac{2}{\sigma^2} (b_T(z) - (1+z)a_T(z)), \quad z > 0. \quad (2.3)$$

3 Stratified lognormal Asian option pricing

The lognormal distribution with mean $-p\sigma^2 T/2$ and variance $\sigma^2 T$ has the probability density function

$$g(x) = \frac{1}{\sigma x \sqrt{2\pi T}} e^{-(p\sigma^2 T/2 + \log x)^2 / (2\sigma^2 T)},$$

where $x > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and moments

$$E[X] = e^{(1-p)\sigma^2 T/2} \quad \text{and} \quad E[X^2] = e^{(2-p)\sigma^2 T}, \quad (3.1)$$

i.e.

$$p = 1 - \frac{2}{\sigma^2 T} \log E[X] \quad \text{and} \quad \sigma^2 = \frac{1}{T} \log \left(\frac{E[X^2]}{(E[X])^2} \right). \quad (3.2)$$

In the next proposition, as a consequence of (3.2) and Proposition 2.2 we fit the conditional distribution of Λ_T given $S_T = z$ and $S_0 = 1$ to a lognormal distribution using its first two moments.

Proposition 3.1. *Given $z > 0$, letting*

$$\sigma^2(z) = \frac{1}{T} \log \left(\frac{2}{\sigma^2 a_T(z)} \left(\frac{b_T(z)}{a_T(z)} - 1 - z \right) \right) \quad \text{and} \quad p(z) := 1 - \frac{2}{T \sigma^2(z)} \log a_T(z),$$

the lognormal random variable with parameter $(-p(z)\sigma^2(z)T/2, \sigma^2(z)T)$ has same first and second moments as Λ_T given $S_T = z$ and $S_0 = 1$.

Based on Proposition 3.1 we will approximate the law of Λ_T given $S_T = z$ and $S_0 = 1$ as

$$d\mathbb{P} \left(\Lambda_T = x \mid S_T = z, S_0 = 1 \right) \simeq \frac{1}{\sigma(z)\sqrt{2\pi T}} e^{-(p(z)\sigma^2(z)T/2 + \log x)^2/(2\sigma^2(z)T)} \frac{dx}{x}, \quad (3.3)$$

$x > 0$. As a consequence of this approximation we have

$$\begin{aligned} e^{-rT} E \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] &= \frac{1}{T} e^{-rT} E \left[\left(\int_0^T S_t dt - KT \right)^+ \right] \\ &\simeq \frac{e^{-rT}}{T} \int_0^\infty \left(e^{-p(z/x)\sigma^2(z/x)T/2 + \sigma^2(z/x)T/2} \Phi(d_1(K, z, x)) - KT \Phi(d_2(K, z, x)) \right) dP(S_T = z, S_0 = x), \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} d_1(K, z, x) &= \frac{\log(E[\Lambda_T \mid S_T = z, S_0 = x]/(KT))}{\sigma(z/x)\sqrt{T}} + \sigma(z/x) \frac{\sqrt{T}}{2} \\ &= \frac{1}{2\sigma(z/x)\sqrt{T}} \log \left(\frac{2x(b_T(z/x) - (1 + z/x)a_T(z/x))}{\sigma^2 K^2 T^2} \right) + \sigma(z/x) \frac{\sqrt{T}}{2}, \end{aligned}$$

and

$$\begin{aligned} d_2(K, z, x) &= d_1(K, z, x) - \sigma(z/x)\sqrt{T} \\ &= \frac{1}{2\sigma(z/x)\sqrt{T}} \log \left(\frac{2x(b_T(z/x) - (1 + z/x)a_T(z/x))}{\sigma^2 K^2 T^2} \right) - \sigma(z/x)\sqrt{T}. \end{aligned}$$

Figure 1 compares the Asian option prices obtained from (3.4) (stratified lognormal approximation), with the standard lognormal approximation (1.2) of [1] with the Monte Carlo method. Significant discrepancies in the approximations can be observed for large values of time to maturity, and the stratified approximations appear to perform better than the standard lognormal approximation.

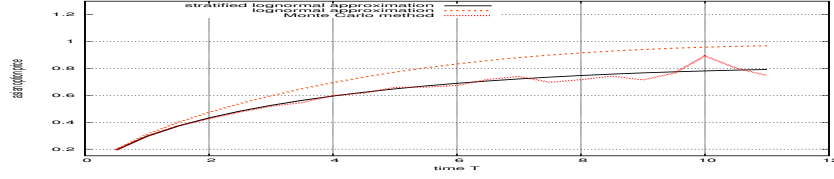


Figure 1: Asian option prices with $\sigma = 1$, $r = 0.05$, $K/S_0 = 1.1$, $S_0 = 1.5$.

Hedging

The Delta of the option with respect to $x = S_0$ can be estimated from the approximation (3.4) as

$$\begin{aligned}
\Delta_t &= e^{-rT} \frac{\partial}{\partial x} E \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \middle| S_0 = x \right] \\
&= e^{-rT} \frac{\partial}{\partial x} \left(x E \left[\left(\frac{1}{T} \int_0^T \frac{S_t}{x} dt - \frac{K}{x} \right)^+ \middle| S_0 = x \right] \right) \\
&= e^{-rT} \frac{\partial}{\partial x} \left(x E \left[\left(\frac{1}{T} \int_0^T S_t dt - \frac{K}{x} \right)^+ \middle| S_0 = 1 \right] \right) \\
&= e^{-rT} E \left[\left(\frac{1}{T} \int_0^T S_t dt - \frac{K}{x} \right)^+ \middle| S_0 = 1 \right] + e^{-rT} x \frac{\partial}{\partial x} E \left[\left(\frac{1}{T} \int_0^T S_t dt - \frac{K}{x} \right)^+ \middle| S_0 = 1 \right] \\
&\simeq \frac{e^{-rT}}{T} \int_0^\infty \left(e^{-p(z)\sigma^2(z)T/2 + \sigma^2(z)T/2} \Phi(d_1(K/x, z, 1)) - \frac{KT}{x} \Phi(d_2(K/x, z, 1)) \right) dP(S_T = z, S_0 = 1) \\
&\quad + \frac{xe^{-rT}}{T} \frac{\partial}{\partial x} \int_0^\infty \left(e^{-p(z)\sigma^2(z)T/2 + \sigma^2(z)T/2} \Phi(d_1(K/x, z, x)) - \frac{KT}{x} \Phi(d_2(K/x, z, x)) \right) dP(S_T = z, S_0 = 1) \\
&= \frac{e^{-rT}}{T} \int_0^\infty \left(e^{-p(z)\sigma^2(z)T/2 + \sigma^2(z)T/2} \Phi(d_1(K/x, z, 1)) - \frac{KT}{x} \Phi(d_2(K/x, z, 1)) \right) dP(S_T = z, S_0 = 1) \\
&\quad + \frac{xe^{-rT}}{T\sigma(z)\sqrt{2\pi T}} \int_0^\infty e^{-p(z)\sigma^2(z)T/2 + \sigma^2(z)T/2} e^{-(d_1(K/x, z, 1))^2/2} dP(S_T = z, S_0 = 1) \\
&\quad - \frac{Ke^{-rT}}{x} \int_0^\infty \left(\Phi(d_2(K/x, z, 1)) + \frac{1}{\sigma(z)\sqrt{2\pi T}} e^{-(d_2(K/x, z, 1))^2/2} \right) dP(S_T = z, S_0 = 1).
\end{aligned}$$

References

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- [3] S. Turnbull and L. Wakeman. A quick algorithm for pricing European average options. *Journal of Financial and Quantitative Analysis*, 26:377–389, 1992. [1](#)