Title : Double Heston Fourier transform

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Premia 21

We consider the following model :

$$\begin{cases} \frac{dS_t}{S_t} & = & (r-\delta)dt + \sum_{j=1}^{2} \sqrt{V_t^j}\left(\rho_j dW_t^j + \sqrt{1-\rho_1^2}dB_t^1\right) \\ dV_t^j & = & b_j(\theta_j - V_t^j)dt + \sigma_j\sqrt{V_t^j}dW_t^j & j=1,2 \end{cases} \tag{1}$$

1 Characteristic function

Let $X_t = \ln(e^{-(r-\delta)t}S_t)$, then we want to compute the following expression (when finite) for any $k \in \mathbb{C}$:

$$\psi(k,t,x,V_0^1,V_0^2) = \mathbb{E}_{(x,v_0,v_1)}\left[e^{ikX_t}\right]. \tag{2}$$

On one hand, thanks to [DFS03], we know that the model is affine, and that the characteristic function has the following form :

$$\psi(k,t,x,V_0^1,V_0^2) = e^{ik_1x + A(k;t) + B_1(k;t)V_0^1 + B_2(k;t)V_0^2} \tag{3}$$

where $A(k;0) = B_1(k;0) = B_2(k;0) = 0$.

On the other hand, we know by Ito's formula that $\partial_t\psi = \mathrm{L}\psi$ where L is the operator associated to the three-dimensionnal affine process (X, V^1, V^2) and that $\psi(k,0,x,V_0^1,V_0^2) = e^{ikx}$.

Proceeding by identification with respect to x, V_0^1, V_0^2, we obtain Ricatti equations on B_1, B_2 and a trivial ODE on A. Fortunately, these equations are solvable. By introducing for $j=1,2$:

$$\begin{array}{rcl} d_j & = & \sqrt{(b_j - i\rho_j\sigma_j k)^2 + \sigma_j^2 k(k+i)} \\ G_j & = & \frac{b_j - i\rho_j\sigma_j k - d_j}{b_j - i\rho_j\sigma_j k + d_j}, \end{array} \tag{4}$$

we get :

$$\begin{array}{rcl} A(k;t) & = & \sum_{j=1}^{2} \frac{b_j\theta_j}{\sigma_j^2}\left[(b_j - i\rho_j\sigma_j k - d_j)\,t - 2\ln\left(\frac{1-G_je^{-d_jt}}{1-G_j}\right)\right] \\ B_j(k;t) & = & \frac{b_j - i\rho_j\sigma_j k - d_j}{\sigma_j^2}\frac{1-e^{-d_jt}}{1-G_je^{-d_jt}} \qquad \text{for } j=1,2 \end{array} \tag{5}$$

where all the complex functions are to be understood as their principal branch. I.e if for some $a, b \in \mathbb{R}$, $z = e^{a+ib}$ then $\sqrt{z} = e^{\frac{a+ib}{2}}$ and if moreover $b \in]-\pi, \pi]$, then $\ln(z) = a + ib$.

2 Fourier pricing

Following [GP10] and [CM99] and [R. 04], for any $\tilde{K} \in \mathbb{R}_+^\star$ we compute the Fourier transform of the following both integrable and square integrable function

$x \in \mathbb{R} \mapsto ((e^x - \tilde{K})^+ - e^x)e^{-\frac{x}{2}} \in \mathbb{R}$. We obtain :

$$((e^x - \tilde{K})^+ - e^x)e^{-\frac{x}{2}} = \int_{\mathbb{R}} e^{i2\pi\xi x} \frac{e^{\left(\frac{1}{2}+i2\pi\xi\right)\ln \tilde{K}}}{-\frac{1}{4} - (2\pi\xi)^2} d\xi. \tag{6}$$

Since we have :

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = e^{-\delta T}\mathbb{E}_{\ln S_0, V_0^1, V_0^2}\left[\left(e^{X_T} - Ke^{-(r-\delta)T}\right)^+\right]. \tag{7}$$

and

$$\mathbb{E}\left[e^{-rT}S_T\right] = S_0 e^{-\delta T} = e^{-\delta T}\mathbb{E}_{\ln S_0, V_0^1, V_0^2}\left[e^{X_T}\right]. \tag{8}$$

Using Fubini's theorem, we get the following pricing formula :

$$\begin{aligned} C =& \mathbb{E}\left[e^{-rT}(S_T - K)^+\right] \\ =& S_0 e^{-\delta T} + e^{-\delta T}\int_{\mathbb{R}} \psi\left(2\pi\xi - \frac{i}{2}, T, \ln S_0, V_0^1, V_0^2\right) \frac{e^{(-(r-\delta)T + \ln K)\left(\frac{1}{2}+i2\pi\xi\right)}}{-\frac{1}{4} - (2\pi\xi)^2} d\xi. \end{aligned}$$

Endly, we use the Gauss Kronrod procedure implemented in the Premia Numerical Library to compute the integral involving in this formula.

References

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