Analytical formulas for local volatility model with stochastic rates

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1 Model specification

This paper presents a new approximation formulae of European options in local volatility model with stochastic interest rates. More precisely, we consider the one factor Hull and white model for interest rates $(r_t)_{t\geq 0}$, the CEV diffusion for the spot process $(S_t)_{t\geq 0}$ and constant correlation ρ. The dynamics of $(S_t, r_t)_{t\geq 0}$ are given by

$$\begin{cases} S_t &= S_0 + \int_0^t r_u S_u du + \int_0^t \nu S_u^\beta dW_u \\ r_t &= f(0,t) - \int_0^t \gamma(u,t)\Gamma(u,t)du + \int_0^t \gamma(u,t)dB_u, \end{cases} \tag{1}$$

where $\gamma(t,T) = \xi e^{-\kappa(T-t)}$ and $\Gamma(t,T) = \frac{\xi}{\kappa}\left(e^{-\kappa(T-t)} - 1\right)$.

2 Second order approximation formula for pricing European call

In the following, we introduce the log discounted process $(X_t)_{t\geq 0}$ given by $X_t = \log(S_t) - \int_0^t r_s ds$. We use the smart expansion introduced by Benhamou et al [1] to give an analytical accurate approximation of a call european price, written as the expected value under the risk neutral probability measure of the payoff function $h(x) = (e^x - K)_+$ evaluated at the maturity time T:

$$Call\ Price = \mathbb{E}\left[e^{-\int_0^T r_s ds} h\left(\int_0^T r_s ds + X_T\right)\right].$$

More precisely, we use the second order approximation price formula, given by Theorem 3.6 of [1], to write

$$Call\ Price = B(0,T)\left\{\mathbb{E}_T\left[h\left(\int_0^T r_s ds + X_T^B\right)\right] + \sum_{i=1}^3 \alpha_{i,T} Greek_i^h\left(\int_0^T r_s ds + X_T^B\right) + Resid_2\right\} \tag{2}$$

where $\mathbb{E}_T$ is the expectation under the forward neutral probability $\mathbb{Q}^T$, $B(0,T)$ is the zero coupon bond paying 1 Euro at time T. The process X_T^B is given by

$$X_T^B = \log(S_0) + \int_0^T \sigma_t dW_t - \frac{1}{2}\int_0^T \sigma_t^2 dt, \quad \sigma_t \equiv \nu S_0^\beta.$$

The leading order in this approximation is given by the quantity $A = B(0,T)\mathbb{E}_T\left[h\left(\int_0^T r_s ds + X_T^B\right)\right]$ which is given by the Black formula

$$A = S_0\mathcal{N}\left(\frac{1}{\sigma^{\text{Black}}\sqrt{T}}\log\left(\frac{S_0}{B(0,T)K}\right) + \frac{1}{2}\sigma^{\text{Black}}\sqrt{T}\right) - KB(0,T)\mathcal{N}\left(\frac{1}{\sigma^{\text{Black}}\sqrt{T}}\log\left(\frac{S_0}{B(0,T)K}\right) - \frac{1}{2}\sigma^{\text{Black}}\sqrt{T}\right).$$

The quantity $Greek_i^h\left(\int_0^T r_s ds + X_T^B\right)$ is the ith derivative of the leading term w.r.t. the initial value $x_0 = \log(S_0)$ and the error term $Resid_2$ is well analyzed in [1]. In the particular case of model (1) the coefficients $\alpha_{i,T}$ are explicit.

$$
\begin{aligned}
\alpha_{1,T} &= \frac{e^{-2\kappa T}(\beta-1)\nu^2 S_0^{2(\beta-1)}}{4\kappa^4}[2\rho^2\xi^2 + 2e^{\kappa T}\rho(\kappa\nu S_0^{\beta-1}(2\kappa T+1) + 2\rho(\kappa T-1)\xi)\xi + \\
&\quad e^{2\kappa T}(\nu^2 S_0^{2(\beta-1)}T^2\kappa^4 + \rho\nu S_0^{\beta-1}(\kappa T(3\kappa T-2)-2)\xi\kappa + 2\rho^2(\kappa T-1)^2\xi^2)], \\
\alpha_{2,T} &= -\alpha_{1,T} - \alpha_{3,T}, \\
\alpha_{3,T} &= \frac{e^{-2\kappa T}(\beta-1)\nu^2 S_0^{2(\beta-1)}}{2\kappa^4}[\rho\xi + e^{\kappa T}(\nu S_0^{\beta-1}T\kappa^2 + \rho T\xi\kappa - \rho\xi)]^2.
\end{aligned}
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Finally, the Call price, given by the approximation formula (2), is computed by the Premia code source function.

References

[1] Benhamou Eric Gobet Emmanuel and Miri Mohammed. Analytical formulas for local volatility model with stochastic rates. *Quantitative Finance*, 12(2), 2012. 1, 2