Pricing European option under Heston-CIR model with Fourier Cosine Method: implementation in PREMIA

Xiao Wei

Projet Mathfi, INRIA, 78153 Le Chesnay, France
China Institute for Acturial Science, CUFE, 100081 Beijing, China

Abstract

In a Heston hybrid model with stochastic interest rate, we implement European option pricing by Fourier Cosine method. Baced on [3], we apply two different approximations of the hybrid model to obtain a closed form solution of the characteristic function for Fourier cosine method.

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1 Introduction

We consider the pricing problem of European option in hybrid model and stochastic interest rate. The implementation of this problem is based on the Fourier Cosine expansion approach given by Grzelak and Oosterlee (2010).

The rest of this file is as follows: we introduce the model in Section 2, then in section 3 we present the pricing method of Fourier Cosine expansion incoperated with a closed form solution of characteristic function of the model, the program manual of the implementation in PREMIA is given in Section 4.

2 Model description

The model we considered is a Heston model assciated with Cox-Ingersoll-Ross stochastic interest rate process, it is given as follows:

$$
\begin{aligned}
dS_t &= r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^x, \quad S_0 > 0, \\
d\sigma_t &= k(\overline{\sigma} - \sigma_t)dt + \gamma\sqrt{\sigma_t} dW_t^\sigma, \quad \sigma_0 > 0, \\
dr_t &= \lambda(\theta_t - r_t)dt + \eta\sqrt{r_t} dW_t^r, \quad r_0 > 0,
\end{aligned}
\tag{2.1}
$$

where $k > 0$ determines the speed of adjustment of the volatility towards its theoretical mean, $\overline{\sigma} > 0, \gamma > 0$ is the second-order volatility, i.e. the variance of the volatility, $\lambda > 0$ determines the speed of mean reversion for the interest rate process, θ_t is the interest rate term-structure and η controls thevolatility of the interest rate, the correlations are given by $dW_t^x dW_t^\sigma = \rho_{x,\sigma}dt, dW_t^x dW_t^r = \rho_{x,r}dt, dW_t^\sigma dW_t^r = \rho_{\sigma,r}dt$. Note that we assume independence between the instantaneous short rate, r_t, and the volatility process σ_t, i.e. $\rho_{\sigma,r} = 0$.

To compute the price of an European option in such a model, [3] proposed to reformulat the HCIR model in the following way:

$$
\begin{cases}
dS_t = r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^x + \Omega_t \sqrt{r_t} S_t dW_t^r + \Delta\sqrt{\sigma_t} S_t dW_t^\sigma, \quad S_0 > 0, \\
d\sigma_t = k(\overline{\sigma} - \sigma_t)dt + \gamma\sqrt{\sigma_t} dW_t^\sigma, \quad \sigma_0 > 0, \\
dr_t = \lambda(\theta_t - r_t)dt + \eta\sqrt{r_t} dW_t^r, \quad r_0 > 0,
\end{cases}
\tag{2.2}
$$

with $dW_t^x dW_t^\sigma = \hat{\rho}_{x,\sigma}dt, dW_t^x dW_t^r = 0, dW_t^\sigma dW_t^r = 0$.

Let $x_t = \log(S_t)$ we have

$$
\begin{aligned}
dx_t &= \left[r_t - \frac{1}{2}\left(\Omega_t^2 r_t^{2p} + \sigma_t(1 + \Delta^2 + 2\hat{\rho}_{x,\sigma}\Delta)\right)\right] dt + \Omega_t\sqrt{r_t} S_t dW_t^r + \Delta\sqrt{\sigma_t} S_t dW_t^\sigma \\
&= \left(r_t - \frac{1}{2}\sigma_t \right) + \Omega_t\sqrt{r_t} S_t dW_t^r + \Delta\sqrt{\sigma_t} S_t dW_t^\sigma.
\end{aligned}
$$

The pricing method is derived based on the SDE system $\mathbf{X}_t^*$, in fact the SDE system $\mathbf{X}_t^* := [r_t, \sigma_t, x_t]^T$ is coodinated with that of $\mathbf{X}_t := [S_t, \sigma_t, r_t]^T$ in the sense that

$$
\Omega_t = \rho_{x,r} r_t^{-1/2}\sqrt{\sigma_t}, \hat{\rho}_{x,\sigma}^2 = \rho_{x,\sigma}^2 + \rho_{x,r}^2, \Delta = \rho_{x,\sigma} - \hat{\rho}_{x,\sigma}. \tag{2.3}
$$

3 Sketch of the Pricing Method

The idea of the pricing method proposed by [3] is by fitting the model into a class of affine diffusion processes to have a closed form solution of the characteristic function of the discount logarithm of equity process and then use Fourier-Cosine expansion to derive the option price.

3.1 Closed form Solutions of Discount Characteristic Function by two approximations

From [1], if a process is a affine diffusion process(AD), a closed form solution of the characteristic function exists. The AD processes is given by a system of SDEs

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t \tag{3.1}$$

satisfies:

$$\begin{aligned} \mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t, \quad \text{for any}(a_0, a_1) \in \mathbf{R}^n \times \mathbb{R}^{n\times n}, \\ \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, \quad \text{for abitrary}(c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \\ r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \quad \text{for}(r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n, \end{aligned} \tag{3.2}$$

for any $i, j = 1, \cdots, n$ with $r(\mathbf{X}_t)$ being an interest rate component.

Then, by [1], the discounted characteristic function (ChF) of (3.2) is of the following form:

$$\phi(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r_s ds + i\mathbf{u}^T\mathbf{X}_T\right)|\mathcal{F}+t\right) = e^{\mathbf{A}(\mathbf{u},\tau)+\mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}_t},$$

where the expectation is taken under the risk-neutral measure, $\mathbb{Q}$. For a time lag, $\tau := T - t$, the coefficients $\mathbf{A}(\mathbf{u}, \tau)$ and $\mathbf{B}^T(\mathbf{u}, \tau)$ have to satisfy the following complex-valued ordinary differential equation (ODEs):

$$\begin{cases} \frac{d}{d\tau}\mathbf{B}(\mathbf{u}, \tau) = -r_1 + a_1^T\mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1\mathbf{B}, \\ \frac{d}{d\tau}\mathbf{A}(\mathbf{u}, \tau) = -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0\mathbf{B}, \end{cases} \tag{3.3}$$

with $a_i, c_i, r_i, i = 0, 1$ as in (3.2).

For the HCIR model, the sysmetric instantaneous covariance matrix in (3.2) for the SDE system $\mathbf{X}_t^*$ is given by:

$$\Sigma := \sigma(\mathbf{X}_\mathbf{t}^*)\sigma(\mathbf{X}_\mathbf{t}^*)^T = \begin{bmatrix} \eta^2 r_t & 0 & \eta\Omega_t r_t \\ * & \gamma^2\sigma_t & \gamma\hat{\rho}_{x,\sigma}\sigma_t + \gamma\Delta\sigma_t \\ * & * & \sigma_t + \Omega_t^2 r_t + \Delta^2\sigma_t + 2\hat{\rho}_{x,\sigma}\Delta\sigma_t \end{bmatrix}. \tag{3.4}$$

To make the HCIR model affine we need to approximate the non-affine terms $\sqrt{\sigma_t}\sqrt{r_t}$ in $\Sigma_{(1,3)} = \eta\Omega_t r_t = \eta\rho_{x,r}\sqrt{\sigma_t}\sqrt{r_t}$ of the instantaneous covariance matrix. $\Sigma_{(3,3)}$ does not seems to be of the affine form, but in fact by (2.3), it equals $\Sigma_{(3,3)} = \sigma_t$.

[3] proposed two ways to approximate the non-affine terms, one is deterministic approximation, which approximates $\sqrt{\sigma_t}$ and $\sqrt{r_t}$ by their expectations and the other is a stochastic approximation by a normal distributed random variable. Here we just provide the approximations of the two methods, for the detail of derivation, we refers the reader to [3].

Lemma 3.1. *The expectation, $\mathbb{E}(\sqrt{\sigma_t})$, with stochastic process given by equation (2.1) can be approximated by*

$$\mathbb{E}(\sigma_t) \approx a + be^{-ct}, \tag{3.5}$$

where

$$a = \sqrt{\overline{\sigma} - \frac{\gamma^2}{8k}}, b = \sqrt{\sigma_0} - a, c = -\log(b^{-1}(\Lambda(1) - a)),$$

with

$$\Lambda(t) := \sqrt{c(t)(\lambda(t) - t) + c(t)d + \frac{c(t)d}{2(d + \lambda(t))}},$$

$$c(t) = \frac{1}{4k}\gamma^2(1 - e^{-kt}), d = \frac{4k\overline{\sigma}}{\gamma^2}, \lambda(t) = \frac{4k\sigma_0 e^{-kt}}{\gamma^2(1 - e^{-kt})},$$

$k, \overline{\sigma}, \gamma$ and σ_0 are the parameters given in (2.1).

Remarks 3.2. *Note that the interest rate process r_t is given by an equation with the similar structure of σ_t, then the expectation of its square root can be approximated by the same way.*

By the approximation given above, the HCIR model can be fitted into the AD class, thus we have a closed form solution of the discount ChF.

Theorem 3.3. *The discount ChF of X_t^* is given by*

$$\phi(u, \mathbf{X}_t, \tau) = \exp(A(u, \tau) + B_x(u, \tau)x_t + B_\sigma(u, \tau)\sigma_t + B_r(u, \tau)r_t), \tag{3.6}$$

where

$$\begin{aligned} B_x(u, \tau) &= iu, \\ B_r(u, \tau) &= \frac{1 - e^{-D_1\tau}}{\eta^2(1 - G_1 e^{-D_1\tau})}(\lambda - D_1), \\ B_\sigma(u, \tau) &= \frac{1 - e^{-D_2\tau}}{\gamma^2(1 - G_2 e^{-D_2\tau})}(k - \gamma\zeta iu - D_2), \end{aligned}$$

and

$$A(u, \tau) = \int_0^\tau (k\overline{\sigma}B_\sigma(u, s) + \lambda\theta B_r(u, s) + \rho_{x,r}\eta iu\mathbb{E}(\sqrt{\sigma}_{T-s})\mathbb{E}(\sqrt{r}_{T-s})B_r(u, s))ds,$$

with $\zeta = \hat{\rho}_{x,\sigma} + \Delta$, $D_1 = \sqrt{\lambda^2 + 2\eta^2(1 - iu)}$, $D_2 = \sqrt{(\gamma\zeta iu - k)^2 - (iu - 1)iu\gamma^2}$, $D_1 = \frac{\lambda - D_1}{\lambda + D_1}$, and $G_2 = \frac{k - \gamma\zeta iu - D_2}{k - \gamma\zeta iu + D_2}$.

For the stochastic approximation we consider the SDE system of $X_t := [x_t, \sigma_t m r_t, v_t, R_t, z_t]^T$ with $R_t = \sqrt{(r_t)}, v_t = \sqrt{\sigma_t}, z_t = v_t R_t$. Then for X_t the discount ChF is given by the following theorem.

Theorem 3.4. *The discount ChF of X_t is given by*

$$\phi(u,\mathbf{X}_t,\tau)=\exp(A(u,\tau)+B_x(u,\tau)x_t+B_\sigma(u,\tau)\sigma_t)+B_r(u,\tau)r_t+B_v(u,\tau)v_t+B_R(u,\tau)R_t+B_z(u,\tau)z_t, \tag{3.7}$$

where $B_x(u,\tau):=B_x, B_\sigma(u,\tau):=B_\sigma, B_r(u,\tau):=B_r, B_v(u,\tau):=B_v, B_R(u,\tau):=B_R, B_z(u,\tau):=B_z$ can be solved from the following ODEs:

$$\begin{aligned}
\frac{dB_x}{d\tau} &= 0,\\
\frac{dB_r}{d\tau} &= -1+B_x-\lambda B_r+\frac{1}{2}\eta^2B_r^2+\frac{1}{2}(\psi_t^v)^2B_z^2,\\
\frac{dB_R}{d\tau} &= \mu_t^vB_z+\psi_t^R\eta B_rB_R+(\psi_t^v)^2B_vB_z,\\
\frac{dB_z}{d\tau} &= \eta\rho_{x,r}B_xB_r+\zeta\psi_t^vB_xB_z+\gamma\psi_t^vB_\sigma B_z+\eta\psi_t^RB_rB_z,\\
\frac{dA}{d\tau} &= k\overline{\sigma}B_\sigma+\lambda\theta B_r+\mu_t^vB_v+\mu_t^RB_R+\frac{1}{2}(\psi_t^v)^2B_v^2+\frac{1}{2}(\psi_t^R)^2B_R^2,\\
\frac{dB_v}{\tau} &= \mu_t^RB_z+\psi_t^v\zeta B_xB_v+\gamma\psi_t^vB_\sigma B_v+\rho_{x,r}\psi_t^RB_xB_R+(\psi_t^R)^2B_rB_z,
\end{aligned}$$

and

$$\frac{dB_\sigma}{d\tau}=\frac{1}{2}B_x(B_x-1)-kB_\sigma+\gamma\zeta B_xB_\sigma+\frac{1}{2}\gamma^2B_\sigma^2+\rho_{x,r}\psi_t^RB_xB_z+\frac{1}{2}(\psi_t^R)^2B_x^2, \tag{3.8}$$

with the boundaray conditions: $B_x(u,0)=iu, B_r(u,0)=0, B_R(u,0)=0, B_z(u,0)=0, B_\sigma(u,0)=0, B_v(u,0)=0$, and $A(u,0)=0$, where $\mu_t^v,\mu_t^R,\psi_t^v,\psi_t^R$ are specified in , and $\zeta=\hat{\rho}_{x,\sigma}+\Delta$.

In the implementation of the option pricing, we use Lunge-Kutta algorithm to solve the coefficient functions $B_x, B_\sigma, B_r, B_v, B_R, B_z$.

3.2 Pricing Option by Fourier-Cosine expansion

With the closed form of characteristic function provided above, the European price can be derived by Fourier-Cosine expansion. Denote by $v(x,t_0)$ the present value of option price at time t_0 with initial option value x, it can be expressed as risk neutral valuation formula:

$$v(x,t_0)=\mathbb{E}^{\mathbb{Q}}[v(y,T)|x]=\int_{\mathbb{R}}v(y,T)f(y|x)dy,$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is the expectation operator under risk-neutral measure $\mathbb{Q}$, x and y are states variables at time t_0 and T, respectively, $f(y|x)$ is the probability density of y given x, and r is the risk-neutral interest rate.

Firstly, we truncate the infinite integration range without loosing significant accuracy to $[amb]\in\mathbb{R}$, and we obtain approximation v_1 :

$$v_1(x,t_0)=\int_a^b v(y,T)f(y|x)dy, \tag{3.9}$$

where

$$[a,b] := \left[c_1 - L\sqrt{c_2+\sqrt{c_4}}, c_1 + L\sqrt{c_2+\sqrt{c_4}}\right], \tag{3.10}$$

with $L = 10$ and c_n denotes the n-th cumulant of $\ln(S_T/K)$.

Secondly, we replace teh density by its cosine expansion in y,

$$f(y|x) = \sum_{k=0}^{+\infty} A_k(x) cos\left(k\pi \frac{y-a}{b-a}\right), \tag{3.11}$$

where the summation Σ here with the first term is weightedd by one-half and so thus for the following summation and

$$A_k(x) := \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right) \equiv \frac{2}{b-a} Re\left\{\phi\left(\frac{k\pi}{b-a}\right)\exp\left(-i\frac{ka\pi}{b-a}\right)\right\}, \tag{3.12}$$

where the second equation is obtained by comparing the cosine coefficient A_k of $f(y|x)$ with the definition of discount characteristic function of the state variables.

So that

$$v_1(x,t_0) = \frac{1}{2}(b-a)\sum_{k=0}^{N-1} A_k(x)V_k, \tag{3.13}$$

where

$$V_k := \frac{2}{b-a}\int_a^b v(y,T)\sum_{k=0}^{+\infty} A_k(x)\cos(k\pi\frac{y-a}{b-a})dy. \tag{3.14}$$

Interchange the summation and integration, we have

$$v_1(x,t_0) = \frac{1}{2}(b-a)\sum_{k=0}^{+\infty} A_k(x)V_k \approx \frac{1}{2}(b-a)\sideset{}{'}\sum_{k=0}^{N-1} A_k(x)V_k, \tag{3.15}$$

with $V_k := \frac{2}{b-a}\int_a^b v(y,T)\cos(k\pi\frac{y-a}{b-a})dy$.

Then replacing (3.12) of A_k in (3.15), we have

$$v(x,t_0) \approx \sum_{k=0}^{N-1} Re\left\{\phi\left(\frac{k\pi}{b-a};x\right)e^{-ik\pi\frac{a}{b-a}}V_k\right\}, \tag{3.16}$$

which is the COS formula for general underlying processes.

At last, we just need to determine V_k in the above COS formula, for a call option, V_k is given by

$$V_k = \frac{2}{b-a}K(\xi_l(0,b) - \phi_k(0,b)), \tag{3.17}$$

where

$$
\begin{aligned}
\xi_k(c,d) &:= \frac{1}{1+\left(\frac{k\pi}{b-a}\right)}^2 \left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - cos\left(k\pi\frac{c-a}{b-a}\right)e^c\right. \\
&\quad \left. +\frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right] \\
\phi_k(c,d) &:= \begin{cases} \left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi}, & k \neq 0, \\ (d-c), & k = 0. \end{cases}
\end{aligned}
$$

We refers to [2] for more details.

4 Program Manual

We implement the European option pricing by Fourier Cosine expansion by using two types of approximation to derive the closed form solution of characteristic functions. The program HAS TO work with the pnl library.

Included files:
The program directory contains the files:
this documentation file "docu.pdf"
"HESTON_COSINE.c"
"HESTON_COSINE"
"Makefile"

Compile and run program:
By excute "make" under Linux, the file "HESTON_COSINE.c" is compiled and an excutable file "HESTON_COSINE" will be regenerated.
Model Parameters:
kv: k in model (2.1)
vbar: $\overline{\sigma}$ in model (2.1)
sigmav: γ in model (2.1)
sigma0: the initial value of σ_t
kr: λ in model (2.1)
rbar: θ in model (2.1)
sigmar: η in model (2.1)
r0: the initial value of interest rate r_t
rho12: $\rho_{x,\sigma}$ in model (2.1)
rho13: $\rho_{x,r}$ in model (2.1)
rho23: $\rho_{\sigma,r}$ in model (2.1), note that our method works well only when $\rho_{23} = 0$.

Parameters of the product:
S0: stock price at the initial time
K: strike of the American option

T: maturity of the American option, the expansion asymptotic works well for small maturity.

Flags to choose products and approximation methods :
approximation_flag: approximation method flag with 0 for deterministic approximation, 1 for stochastic approximation
callput_flag:callput flag: 0 for call, 1 for put

Parameters for COSINE method:
N: discrect steps in the integration range N in (3.16)
L: parameter in the truncate bound of $[a, b]$ as given in (3.10).

References

[1] Duffiem D., Pan, J., Singleton, K., 1999, Transform analysis and asset pricing for affine jump-diffusions. 3

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[3] Grzeket, A.L., Oosterlee, C.W., 2010, On the heston model with stochastic interest rates. Preprint. 1, 2, 3