Lévy model free volatility index and volatility swap

Oleg Kudryavtsev and Alexander Grechko

Premia 22

1. Lévy processes: basic facts

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1})\nu(dy), \tag{1}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2}$$

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = S_0 e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\Im\xi \in (-1, 0)$ and continuous continuation into the closed strip $\Im\xi \in [-1, 0]$.

Example 1. If Lévy measure of a jump diffusion process is given by normal distribution:

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}}\exp\left(-\frac{(x-\gamma)^2}{2\delta^2}\right)dx,$$

then we obtain Merton model. The parameter λ characterizes the intensity of jumps. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\left(1 - \exp\left(-\frac{\delta^2\xi^2}{2} + i\gamma\xi\right)\right), \tag{3}$$

where $\sigma, \delta, \lambda \ge 0$, $\mu, \gamma \in \mathbf{R}$.

Example 2. The characteristic exponent of a pure jump KoBoL process (a.k.a. CGMY model) of order $\nu \in (0,2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{4}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Note that Boyarchenko and Levendorskiĭ (2000, 2002) consider a more general version with $c_\pm$ instead of c, as well as the case $\nu = 1$ and cases of different exponents $\nu_\pm$. If $\nu \geq 1$ or $\mu = 0$, then the order of the KoBoL process equals to the order of the infinitesimal generator as PDO, but if $\nu < 1$ and $\mu \neq 0$, then the order of the process is ν, and the order of the PDO $-L = \psi(D)$ is 1.

Example 3. If Lévy density is given by exponential functions on negative and positive axis:

$$F(dy) = \mathbf{1}_{(-\infty;0)}(y)c_+\lambda_+e^{\lambda_+ y}dy + \mathbf{1}_{(0;+\infty)}(y)c_-(-\lambda_-)e^{\lambda_- y},$$

where $c_\pm \geq 0$ and $\lambda_- < -1 < 0 < \lambda_+$, then we obtain Kou model (see Kou (2002)). The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{5}$$

2. Model free volatility index and volatility swap

Let S_t denote the value of a stock or stock index at time t. We assume that the volatility product starts at time zero and ends at time T. Assume that S_t is modeled by some stochastic process, then the annualized expected quadratic variation of log-returns over the time interval $[0;T]$ is determined by

$$Q_T = \frac{1}{T}\mathbf{E}\Big[[\ln S, \ln S]_T\Big],$$

where $[\ln S, \ln S]$ denotes the quadratic variation of $\ln S$.

Now consider swaps written on the volatility $Q_T^{1/2}$. A volatility swap is an instrument which allows investors to trade future realized (or historical) volatility against current implied volatility. The quantity

$$IV = Q_T^{1/2} \cdot 100$$

is called "fair strike of a volatility swap in annual volatility points". If the underlying asset is a stock index, IV can be used as a volatility index of the local financial market.

Volatility swaps on volatility are derivatives written on $Q_T^{1/2}$:

Volatility swap with fixed strike K pays the holder

$$VolS(K,T) = Q_T^{1/2} - K.$$

In the most well-known model-free approaches such as the CBOE method, the price of variance derivative may be approximated by some portfolio (the so called replicating portfolio) of some amount of underlying and derivatives on it. One may find the construction of replicating portfolio e.g. in Bossu et al. (2005) and Buehler

(2009). The alternative approach developed in Fukasawa et. al. (2011) is based on the implied volatility integration. Both methods begin with the assumption that S_t follows some diffusion process of the type:

$$\frac{dS_t}{S_t} = \mu(t, S_t, ...)dt + \sigma(t, S_t, ...)dW_t$$

where W_t is a Wiener process, and the drift μ and the volatility σ are unknown coefficients (either deterministic or stochastic). If S_t is a continuous semimartingale, the following auxiliary formula can be applied:

$$\mathbf{E}\Big[[\ln S, \ln S]_T\Big] = -Q_S \mathbf{E}\left[\ln\left(\frac{S_T}{\mathbf{E}[S_T]}\right)\right], \tag{6}$$

where $Q_S = 2$ and $\mathbf{E}[]$ is the risk-neutral expectation. Notice that in the case of models admitting jumps (e.g. Lévy models) the multiplier Q_S in (6) may be different from 2.

The analysis of power variation of log-returns for RTS index in Grechko and Kudryavtsev (2016,2017) under the RFBR project No.15-32-01390 “Mathematical Methods of Analysis and Risk Management on Russian Stock Market”shows that indices based on the (diffusion) model free volatility formula (see e.g. M. Fukasawa et. al. (2011))

$$\sigma^2 = \frac{1}{T}\mathbf{E}\Big[[\ln S, \ln S]_T\Big] = -\frac{Q_S}{T}\mathbf{E}\left[\ln\left(\frac{S_T}{E[S_T]}\right)\right], Q_S = 2, \tag{7}$$

give a bad estimation for a realized variation in the case of Russian financial market.

The method CBOE applied by Moscow Stock Exchange to estimate RVI is applicable for diffusion processes or jump-diffusion processes with rare jumps only. On the other hand, the research in Grechko and Kudryavtsev (2016,2017) shows that the most adequate models for the RTS index are Lévy processes with infinite activity jumps and without a diffusion component. There are many empirical studies on American and European financial markets (see e.g. Cont and Tankov (2004) and the bibliograpy therein) supporting the fact that pure non-Gaussian Lévy models are more adequate than diffusion models. In the case of Lévy models the multiplier Q_S in formula (7) is differ from 2 and can be found by the following formula in Carr et al. (2012):

$$Q_S = \frac{\mathbf{E}\Big[[\ln S, \ln S]_T\Big]}{\ln E[S_T] - E[\ln S_T]}. \tag{8}$$

The new index formula implemented into Premia is based on the variation representation via market option prices. Denote by F the forward price of S_0 to be paid at time T, K the strike price, $P(K)$ and $C(K)$, respectively, the market put and call prices at strike level K. Then a new model free volatility index for Levy processes can be expressed in terms of "out-of-the-money" (OTM) market option prices:

$$\mathbf{E}\Big[[\ln S, \ln S]_T\Big] = Var(\ln(S_T)) = \mathbf{E}[\ln^2(S_T)] - \mathbf{E}^2[\ln(S_T)], \tag{9}$$

$$\begin{aligned}\mathbf{E}[\ln^2(S_T)] &= \ln^2 K_0 + 2\ln K_0\left(\frac{F}{K_0} - 1\right) + \\ &+ 2\left[\int_{K_0}^{\infty} C(K)\frac{1-\ln K}{K^2}dK + \int_0^{K_0} P(K)\frac{1-\ln K}{K^2}dK\right].\end{aligned} \tag{10}$$

If one needs to construct the replicating portfolio, then Q_S in (8) should be expressed via market option prices. The denominator of Q_S we approximate as follows.

$$\begin{aligned}\ln E[S_T] - E[\ln S_T] &= -\mathbf{E}\left[\ln\left(\frac{S_T}{F}\right)\right] \\ &= \int_0^{K_0}\frac{P(K)}{K}dK + 2\int_{K_0}^{\infty}\frac{C(K)}{K}dK + 2\int_{K_0}^{F}\frac{K-F}{K^2}dK.\end{aligned}$$

Notice that a similar formula is used to approximate $\mathbf{E}^2[\ln(S_T)]$ in (9).

The algorithm implemented into the program platform Premia consist of the following steps.

- input a set of European put and European call prices of the same maturity at different strike levels and a strike volatility swap;
- define "at-the-money" strike price K_0, which corresponds to the strike with a minimal difference between the correspondent put and call prices;
- define the forward price by using the call-put parity formula
- select a set of valid out-of-the-money (OTM) calls and puts to be used;
- approximate the integral in (10) by using Simpson's rule;
- output "fair strike of a volatility swap" IV (see (9)-(10)) and "volatility swap price" $VolS$ in annual volatility points.

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