

Numerical approximations for Heston-Hull-White type models

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Abstract

We study a hybrid tree-finite difference method which permits to obtain efficient and accurate European and American option prices in the Heston Hull-White and Heston Hull-White2d models. Moreover, as a by-product, we provide a new simulation scheme to be used for Monte Carlo evaluations. Numerical results show the reliability and the efficiency of the proposed methods.

Keywords: stochastic volatility; stochastic interest rate; tree methods; finite difference; Monte Carlo; European and American options

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1 Introduction

In this paper we consider the Heston-Hull-White model, which is a joint evolution for the equity value with a Heston-like stochastic volatility and a generalized Hull-White stochastic interest rate model which is consistent with the term structure of the interest rates. We consider a further situation where the dividend rate is stochastic, a case which is called here the “Heston Hull-White2d model”. We concern the problem of pricing European and American options written on these models.

At the present time, the literature on this subject is quite poor and includes Fourier-Cosine methods, semi-closed approximations and finite difference methods to price vanilla options. In [8], Grzelak and Oosterlee introduce two approximations of the non-affine models. The Fourier-Cosine method is then used on this approximated affine model. The authors remark that for accurate modeling of hybrid derivatives it is necessary to be able to describe a non-zero correlation between the processes driving the equity and the interest rate. This is possible in the approximations presented in their paper but only using approximated affine models. Haentjens and in’t Hout propose in [6] a finite difference Alternating Direction Implicit (ADI) scheme for pricing European options solving the original three-dimensional Heston-Hull-White partial differential equation. The Heston Hull-White2d model is treated using semi-closed approximations in the FX model [9].

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In this paper, we generalize the hybrid tree-finite difference approach that has been introduced for the Heston model in the paper [3]. In practice, this means to write down an algorithm to price European and American options by means of a backward induction that works following a finite difference PDE method in the direction of the share process and following a tree method in the direction of the other random sources (volatility, interest rate and possibly dividend rate). This means to suitably approximate the original dynamics and the description of the approximating processes suggests moreover a way to simulate paths from the Heston-Hull-White models. Therefore, we propose here also a new Monte Carlo algorithm for pricing options which seems to be a real alternative to the Monte Carlo method that makes use of the efficient simulations provided by Alfonsi [1]. Our approaches both allow one to price options in the original Heston-Hull-White processes with non-zero correlations. Here, we consider the case of a non null correlation between the equity and the interest rate process, as well as between the equity and the stochastic volatility. Moreover, in the Heston-Hull-White2d model, we allow the dividend rate to be stochastic and correlated to the equity process. But let us note that other sets of correlations can surely be selected.

The paper is organized as follows. In Section 2 we introduce the Heston-Hull-White model. Then in Section 3 we construct a tree approximation for the pair given by the volatility and the interest rate process. Section 4 refers to the approximation of functions of the underlying asset price process by means of PDE arguments. In Section 5 we describe the hybrid tree-finite difference scheme and we apply it for the computation of American options. In Section 6 we see how to generalize the previous procedure in order to handle the Heston-Hull-White2d process. In Section 7 we show that our arguments can be used also to set-up simulations, to be applied to construct Monte Carlo algorithms. Finally, numerical results and comparisons are given in Section 8.

2 The Heston-Hull-White model

The Heston Hull-White model concerns with cases where the volatility V and the interest rate r are assumed to be stochastic. The dynamics under the risk neutral measure of the share price S and the volatility process V are governed by the stochastic differential equation system

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t - \eta)dt + \sqrt{V_t} dZ_t, \\ dV_t &= \kappa_V(\theta_V - V_t)dt + \sigma_V\sqrt{V_t}dW_t^1, \\ dr_t &= \kappa_r(\theta_r(t) - r_t)dt + \sigma_r dW_t^2,\end{aligned}$$

with initial data $S_0 > 0$, $V_0 > 0$ and $r_0 > 0$, where Z , W^1 and W^2 are suitable and possibly correlated Brownian motions. Recall that V_t is a Cox-Ingersoll-Ross (hereafter CIR) process whereas r_t is a generalized Ornstein Uhlenbeck (hereafter OU) process: here θ_r is not constant but it is a deterministic function which is completely determined by the market values of the zero-coupon bonds (see [4]).

Let us fix the correlations among the Brownian motions. As observed in [8], the important correlations are between the pairs (S, V) and (S, r) . So, we assume that $W = (W^1, W^2)$ is a standard Brownian motion in \mathbb{R}^2 and Z is a Brownian motion in \mathbb{R} which is correlated both with W^1 and W^2 :

$$d\langle Z, W_1 \rangle_t = \rho_1 dt \text{ and } d\langle Z, W_2 \rangle_t = \rho_2 dt.$$

By passing to the logarithm $Y = \ln S$ in the first component and taking into account the above

mentioned correlations, we reduce to the dynamics

$$\begin{aligned} dY_t &= (r_t - \eta - \frac{1}{2}V_t)dt + \sqrt{V_t}(\rho_1 dW_t^1 + \rho_2 dW_t^2 + \rho_3 dW_t^3), \quad Y_0 = \ln S_0 \in \mathbb{R}, \\ dV_t &= \kappa_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dW_t^1, \quad V_0 > 0 \\ dr_t &= \kappa_r(\theta_r(t) - r_t)dt + \sigma_r dW_t^2, \quad r_0 > 0 \end{aligned}$$

where $W = (W^1, W^2, W^3)$ is a standard Brownian motion in \mathbb{R}^3 and the correlation parameter ρ_3 is given by

$$\rho_3 = \sqrt{1 - \rho_1^2 - \rho_2^2}, \quad (\rho_1, \rho_2) \in B_1(0),$$

$B_1(0)$ denoting the open ball in \mathbb{R}^2 centered in 0 and with radius 1.

As already done in [7], the process r can be written in the following way:

$$r_t = \sigma_r X_t + \varphi_t \tag{2.1}$$

where

$$X_t = -\kappa_r \int_0^t X_s ds + W_t^2 \quad \text{and} \quad \varphi_t = r_0 e^{-\kappa_r t} + \kappa_r \int_0^t \theta_r(s) e^{-\kappa_r(t-s)} ds. \tag{2.2}$$

So, we can consider the triple (Y, V, X) , whose dynamics is given by

$$\begin{aligned} dY_t &= \mu_Y(V_t, X_t, t)dt + \sqrt{V_t}(\rho_1 dW_t^1 + \rho_2 dW_t^2 + \rho_3 dW_t^3), \quad Y_0 = \ln S_0 \in \mathbb{R}, \\ dV_t &= \mu_V(V_t)dt + \sigma_V \sqrt{V_t} dW_t^1, \quad V_0 > 0 \\ dX_t &= \mu_X(X_t)dt + dW_t^2, \quad X_0 = 0, \end{aligned} \tag{2.3}$$

where

$$\mu_Y(v, x, t) = \sigma_r x + \varphi_t - \eta - \frac{1}{2}v, \tag{2.4}$$

$$\mu_V(v) = \kappa_V(\theta_V - v) \tag{2.5}$$

$$\mu_X(x) = -\kappa_r x. \tag{2.6}$$

The purpose of this paper is to efficiently approximate the process (Y, V, X) in order to numerically compute the price of options written on the share process S .

3 The binomial tree for the pair X and V

First of all, we consider an approximation for the pair (V, X) on the time-interval $[0, T]$ by means of a 2-dimensional *computationally simple tree*, that is by means of a Markov chain that runs over a 2-dimensional recombining bivariate lattice.

We start by considering a discretization of the time-interval $[0, T]$ in N subintervals $[nh, (n+1)h]$, $n = 0, 1, \dots, N$, with $h = T/N$.

3.1 The tree for X

A binomial tree for the process X is quite standard. We consider the “multiple-jumps” approach by Nelson and Ramaswamy [11].

For $n = 0, 1, \dots, N$, consider the lattice for the process X

$$\mathcal{X}_n^h = \{x_{n,j}\}_{j=0,1,\dots,n} \quad \text{with} \quad x_{n,j} = (2j - n)\sqrt{h} \tag{3.1}$$

(notice that $x_{0,0} = 0 = X_0$). For each fixed $x_{n,j} \in \mathcal{X}_n^h$, we define the “up” and “down” jump by means of $j_u^h(n, j)$ and $j_d^h(n, j)$ defined by

$$j_u^h(n, j) = \min\{j^* : j + 1 \leq j^* \leq n + 1 \text{ and } x_{n,j} + \mu_X(x_{n,j})h \leq x_{n+1,j^*}\}, \quad (3.2)$$

$$j_d^h(n, j) = \max\{j^* : 0 \leq j^* \leq j \text{ and } x_{n,j} + \mu_X(x_{n,j})h \geq x_{n+1,j^*}\}, \quad (3.3)$$

μ_X being the drift of the process X , see (2.6). As usual, one sets $j_u^h(n, j) = n + 1$ if $\{j^* : j + 1 \leq j^* \leq n + 1 \text{ and } x_{n,j} + \mu_X(x_{n,j})h \leq x_{n+1,j^*}\} = \emptyset$ and $j_d^h(n, j) = 0$ if $\{j^* : 0 \leq j^* \leq j \text{ and } x_{n,j} + \mu_X(x_{n,j})h \geq x_{n+1,j^*}\} = \emptyset$. The transition probabilities are defined as follows: starting from the node (n, j) , the probability that the process jumps to $j_u^h(n, j)$ and $j_d^h(n, j)$ at time-step $n + 1$ are set as

$$p_u^{X,h}(n, j) = 0 \vee \frac{\mu_X(x_{n,j})h + x_{n,j} - x_{n+1,j_d^h(n,j)}}{x_{n+1,j_u^h(n,j)} - x_{n+1,j_d^h(n,j)}} \wedge 1 \quad \text{and} \quad p_d^{X,h}(n, j) = 1 - p_u^{X,h}(n, j) \quad (3.4)$$

respectively. This gives rise to a Markov chain $(\hat{X}_n^h)_{n=0,\dots,N}$ that weakly converges, as $h \rightarrow 0$, to the diffusion process $(X_t)_{t \in [0,T]}$ and turns out to be a robust tree approximation for the OU process X .

3.2 The tree for V

For the CIR volatility process V , we consider the binomial tree procedure in [2].

For $n = 0, 1, \dots, N$, consider the lattice

$$\mathcal{V}_n^h = \{v_{n,k}\}_{k=0,1,\dots,n} \quad \text{with} \quad v_{n,k} = \left(\sqrt{V_0} + \frac{\sigma}{2}(2k - n)\sqrt{h}\right)^2 \mathbb{1}_{\sqrt{V_0} + \frac{\sigma}{2}(2k - n)\sqrt{h} > 0} \quad (3.5)$$

(notice that $v_{0,0} = V_0$). For each fixed $v_{n,k} \in \mathcal{V}_n^h$, we define the “up” and “down” jump by means of

$$k_u^h(n, k) = \min\{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } v_{n,k} + \mu_V(v_{n,k})h \leq v_{n+1,k^*}\}, \quad (3.6)$$

$$k_d^h(n, k) = \max\{k^* : 0 \leq k^* \leq k \text{ and } v_{n,k} + \mu_V(v_{n,k})h \geq v_{n+1,k^*}\} \quad (3.7)$$

where the drift μ_V of V is defined in (2.6) and with the understanding $k_u^h(n, k) = n + 1$ if $\{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } v_{n,k} + \mu_V(v_{n,k})h \leq v_{n+1,k^*}\} = \emptyset$ and $k_d^h(n, k) = 0$ if $\{k^* : 0 \leq k^* \leq k \text{ and } v_{n,k} + \mu_V(v_{n,k})h \geq v_{n+1,k^*}\} = \emptyset$. The transition probabilities are defined as follows: starting from the node (n, k) the probability that the process jumps to $k_u^h(n, k)$ and $k_d^h(n, k)$ at time-step $n + 1$ are set as

$$p_u^{V,h}(n, k) = 0 \vee \frac{\mu_V(v_{n,k})h + v_{n,k} - v_{n+1,k_d^h(n,k)}}{v_{n+1,k_u^h(n,k)} - v_{n+1,k_d^h(n,k)}} \wedge 1 \quad \text{and} \quad p_d^{V,h}(n, k) = 1 - p_u^{V,h}(n, k) \quad (3.8)$$

respectively. This gives rise to a Markov chain $(\hat{V}_n^h)_{n=0,\dots,N}$ that weakly converges, as $h \rightarrow 0$, to the diffusion process $(V_t)_{t \in [0,T]}$ and turns out to be a robust tree approximation for the CIR process V .

3.3 The tree for the pair (V, X)

The tree procedure for the pair (V, X) is set by joining the trees built for V and for X . Namely, for $n = 0, 1, \dots, N$, consider the lattice

$$\mathcal{V}_n^h \times \mathcal{X}_n^h = \{(v_{n,k}, x_{n,j})\}_{k,j=0,1,\dots,n}. \quad (3.9)$$

Starting from the node (n, k, j) , which corresponds to the position $(v_{n,k}, x_{n,j}) \in \mathcal{V}_n^h \times \mathcal{X}_n^h$, we define the four possible jump by setting the four nodes at time $n+1$ following the definitions (3.2)-(3.3) and (3.6)-(3.7):

$$\begin{aligned}
(n+1, k_u^h(n, k), j_u^h(n, j)) & \text{ with probability } p_{uu}^h(n, k, j) = p_u^{V,h}(n, k) p_u^{X,h}(n, j), \\
(n+1, k_u^h(n, k), j_d^h(n, j)) & \text{ with probability } p_{ud}^h(n, k, j) = p_u^{V,h}(n, k) p_d^{X,h}(n, j), \\
(n+1, k_d^h(n, k), j_u^h(n, j)) & \text{ with probability } p_{du}^h(n, k, j) = p_d^{V,h}(n, k) p_u^{X,h}(n, j), \\
(n+1, k_d^h(n, k), j_d^h(n, j)) & \text{ with probability } p_{dd}^h(n, k, j) = p_d^{V,h}(n, k) p_d^{X,h}(n, j),
\end{aligned} \tag{3.10}$$

where the above probabilities $p_u^{V,h}(n, k)$, $p_d^{V,h}(n, k)$, $p_u^{X,h}(n, j)$ and $p_d^{X,h}(n, j)$ are defined in (3.8) and (3.4) respectively. The above factorization is due to the orthogonality of the noises driving the two processes. As a quite immediate consequence of standard results (see e.g. the techniques in [11]), one gets the following: the associated bivariate Markov chain $(\hat{V}_n^h, \hat{X}_n^h)_{n=0, \dots, N}$ weakly converges to the diffusion pair $(V_t, X_t)_{t \in [0, T]}$ solution to

$$\begin{aligned}
dV_t &= \mu_V(V_t)dt + \sigma_V \sqrt{V_t} dW_t^1, \quad V_0 > 0 \\
dX_t &= -\kappa_r X_t dt + \sigma_r dW_t^2, \quad X_0 = 0.
\end{aligned}$$

Remark 3.1. *In the case one is interested in introducing a correlation between the noises W^1 and W^2 driving the process V and X respectively, the joint tree can be constructed on the same lattice but the jump probabilities are no more of a product-type: the transition probabilities $p_{uu}^h(n, k, j)$, $p_{ud}^h(n, k, j)$, $p_{du}^h(n, k, j)$ and $p_{dd}^h(n, k, j)$ can be computed by matching (at the first order in h) the conditional mean and the conditional covariance between the continuous and the discrete processes of V and X . More precisely, for both components the conditional mean is matched by construction (this is actually the main consequence of the definition of the multiple jumps). As for the conditional covariance, assuming that $d\langle W^1, W^2 \rangle_t = \alpha dt$, with $|\alpha| < 1$, then one has $d\langle V, X \rangle_t = \alpha \sigma_V \sqrt{V_t} dt$. Therefore, the matching conditions lead to solving the following system:*

$$\begin{cases}
p_{uu}^h(n, k, j) + p_{ud}^h(n, k, j) = p_u^{V,h}(n, k) \\
p_{uu}^h(n, k, j) + p_{du}^h(n, k, j) = p_u^{X,h}(n, j) \\
p_{uu}^h(n, k, j) + p_{ud}^h(n, k, j) + p_{du}^h(n, k, j) + p_{dd}^h(n, k, j) = 1 \\
m_{uu}^h(n, k, j) p_{uu}^h(n, k, j) + m_{ud}^h(n, k, j) p_{ud}^h(n, k, j) + \\
\quad + m_{du}^h(n, k, j) p_{du}^h(n, k, j) + m_{dd}^h(n, k, j) p_{dd}^h(n, k, j) = \alpha \sigma_V \sqrt{v_{n,k}} h
\end{cases}$$

where

$$\begin{aligned}
m_{uu}^h(n, k, j) &= (v_{n+1, k_u^h(n, k)} - v_{n, k})(x_{n+1, j_u^h(n, j)} - x_{n, j}), \\
m_{ud}^h(n, k, j) &= (v_{n+1, k_u^h(n, k)} - v_{n, k})(x_{n+1, j_d^h(n, j)} - x_{n, j}), \\
m_{du}^h(n, k, j) &= (v_{n+1, k_d^h(n, k)} - v_{n, k})(x_{n+1, j_u^h(n, j)} - x_{n, j}), \\
m_{dd}^h(n, k, j) &= (v_{n+1, k_d^h(n, k)} - v_{n, k})(x_{n+1, j_d^h(n, j)} - x_{n, j}).
\end{aligned}$$

This is done in [2] in a different context but the proof of the weak convergence on the path space is analogous - this can be done by standard arguments, as in [11] or [5].

4 Approximating the Y -component: the finite difference approach

We go now back to (2.3), that is

$$\begin{aligned} dY_t &= \mu_Y(V_t, X_t, t)dt + \sqrt{V_t}(\rho_1 dW_t^1 + \rho_2 dW_t^2 + \rho_3 dW_t^3), \quad Y_0 = \ln S_0, \\ dV_t &= \mu_V(V_t)dt + \sigma_V \sqrt{V_t} dW_t^1, \quad V_0 > 0 \\ dX_t &= \mu_X(X_t)dt + dW_t^2, \quad X_0 = 0, \end{aligned}$$

where μ_Y , μ_V and μ_X are given in (2.4), (2.5) and (2.6) respectively. By isolating $\sqrt{V_t}dW_t^1$ in the second line and dW_t^1 in the third one, we obtain

$$dY_t = \frac{\rho_1}{\sigma_V} dV_t + \rho_2 \sqrt{V_t} dX_t + \mu(V_t, X_t, t)dt + \rho_3 \sqrt{V_t} dW_t^3 \quad (4.1)$$

with

$$\begin{aligned} \mu(v, x, t) &= \mu_Y(v, x, t) - \frac{\rho_1}{\sigma_V} \mu_V(v) - \rho_2 \sqrt{v} \mu_X(x) \\ &= \sigma_r x + \varphi_t - \eta - \frac{1}{2} v - \frac{\rho_1}{\sigma_V} \kappa_V(\theta_V - v) + \rho_2 \kappa_r x \sqrt{v}. \end{aligned} \quad (4.2)$$

The main point is that the noise W^3 is independent of the processes V and X .

4.1 The approximating scheme for the triple (Y, V, X)

We consider an approximating process Y^h for Y turning out by freezing the coefficients in (4.1): we define $Y_0^h = Y_0$ and for $t \in [nh, (n+1)h]$ with $n = 0, 1, \dots, N-1$ we set

$$Y_t^h = Y_{nh}^h + \frac{\rho_1}{\sigma_V} (V_t - V_{nh}) + \rho_2 \sqrt{V_{nh}} (X_t - X_{nh}) + \mu(V_{nh}, X_{nh}, nh)(t - nh) + \rho_3 \sqrt{V_{nh}} (W_t^3 - W_{nh}^3).$$

We consider now the approximating tree $(\hat{V}_n^h, \hat{X}_n^h)_{n \in \{0, \dots, N\}}$ and we call $(\bar{V}_t^h, \bar{X}_t^h)_{t \in [0, T]}$ the associated time-continuous approximating process for the pair (V, X) , that is

$$\bar{V}_t^h = \hat{V}_{[t/h]}^h \quad \text{and} \quad \bar{X}_t^h = \hat{X}_{[t/h]}^h.$$

Of course, if one is interested in using continuous paths, then the linear interpolated path can be preferred to the piecewise constant one (both of them weakly converge to the pair (V, X)). We then assume that the noise driving the pair $(\bar{V}_t^h, \bar{X}_t^h)_{t \in [0, T]}$ is independent of the Brownian motion W^3 and we insert this discretization for (V, X) in the above discretization scheme for Y . So, we obtain our final approximating process \bar{Y}_t^h by setting $\bar{Y}_0^h = Y_0$ and for $t \in [nh, (n+1)h]$ with $n = 0, 1, \dots, N-1$ then

$$\bar{Y}_t^h = Y_{nh}^h + \frac{\rho_1}{\sigma_V} (\bar{V}_t^h - \bar{V}_{nh}^h) + \rho_2 \sqrt{\bar{V}_{nh}^h} (\bar{X}_t^h - \bar{X}_{nh}^h) + \mu(\bar{X}_{nh}^h, \bar{V}_{nh}^h, nh)(t - nh) + \rho_3 \sqrt{\bar{V}_{nh}^h} (W_t^3 - W_{nh}^3). \quad (4.3)$$

Notice that if we set

$$\bar{Z}_t^h = \bar{Y}_t^h - \frac{\rho_1}{\sigma_V} (\bar{V}_t^h - \bar{V}_{nt}^h) - \rho_2 \sqrt{\bar{V}_{nh}^h} (\bar{X}_t^h - \bar{X}_{nt}^h), \quad t \in [nh, (n+1)h] \quad (4.4)$$

then we have

$$\begin{aligned} d\bar{Z}_t^h &= \mu(\bar{X}_{nh}^h, \bar{V}_{nh}^h, nh)dt + \rho_3 \sqrt{\bar{V}_{nh}^h} dW_t^3, \quad t \in (nh, (n+1)h] \\ \bar{Z}_{nh}^h &= \bar{Y}_{nh}^h \end{aligned} \quad (4.5)$$

that is \bar{Z}^h solves a SDE with constant coefficients and at time nh its starts from \bar{Y}_{nh}^h . Take now a function f : we are interested in approximating

$$\mathbb{E}(f(Y_{(n+1)h}) \mid Y_{nh} = y, V_{nh} = v, X_{nh} = x).$$

By using our scheme and the process \bar{Z}^h in (4.4), we approximate it with

$$\begin{aligned} & \mathbb{E}(f(\bar{Y}_{(n+1)h}^h) \mid \bar{Y}_{nh}^h = y, \bar{V}_{nh}^h = v, \bar{X}_{nh}^h = x) \\ &= \mathbb{E}(f(\bar{Z}_{(n+1)h}^h + \frac{\rho_1}{\sigma_V}(\bar{V}_{(n+1)h}^h - \bar{V}_{nh}^h) + \rho_2\sqrt{\bar{V}_{nh}^h}(\bar{X}_{(n+1)h}^h - \bar{X}_{nh}^h)) \mid \bar{Z}_{nh}^h = y, \bar{V}_{nh}^h = v, \bar{X}_{nh}^h = x). \end{aligned}$$

Since (\bar{V}^h, \bar{X}^h) is independent of the Brownian noise W^3 driving \bar{Z}^h in (4.4), we can write

$$\begin{aligned} & \mathbb{E}(f(\bar{Y}_{(n+1)h}^h) \mid \bar{Y}_{nh}^h = y, \bar{V}_{nh}^h = v, \bar{X}_{nh}^h = x) \\ &= \mathbb{E}\left(\Psi_f\left(\frac{\rho_1}{\sigma_V}(\bar{V}_{(n+1)h}^h - v) + \rho_2\sqrt{v}(\bar{X}_{(n+1)h}^h - x); y, v, x\right) \mid \bar{V}_{nh}^h = v, \bar{X}_{nh}^h = x\right), \end{aligned} \quad (4.6)$$

in which

$$\Psi_f(\xi; y, v, x) = \mathbb{E}(f(\bar{Z}_{(n+1)h}^h + \xi) \mid \bar{Z}_{nh}^h = y, \bar{V}_{nh}^h = v, \bar{X}_{nh}^h = x). \quad (4.7)$$

Now, in order to compute the above quantity $\Psi_f(\xi)$, consider a generic function g and set

$$v(z, v, x) = \mathbb{E}(g(\bar{Z}_{(n+1)h}^h) \mid \bar{Z}_{nh}^h = z, \bar{V}_{nh}^h = v, \bar{X}_{nh}^h = x).$$

By (4.5) and the Feynmac-Kac representation formula we can state that

$$v(z, v, x) = u(nh, z; v, x)$$

where, for every fixed $x \in \mathbb{R}$ and $v \geq 0$, the function $(s, z) \mapsto u(s, z; v, x)$ is the solution to

$$\begin{cases} \partial_s u + \mu(v, x, s)\partial_z u + \frac{1}{2}\rho_3^2 v \partial_z^2 u = 0, & s \in [nh, (n+1)h), \quad z \in \mathbb{R}, \\ u((n+1)h, z; v, x) = g(z), \end{cases} \quad (4.8)$$

μ being given in (4.2). In order to solve the above PDE problem, we use a finite difference approach.

4.2 Finite differences

Following [3], at each time step n we need to numerically solve (4.8) at time $s = nh$. So, we briefly describe the finite difference method we apply to problem (4.8), outlining some key properties that allow one to prove the convergence result following the technique in [3].

We fix a grid on the z -axis $\mathcal{Y}_M = \{z_i = Z_0 + i\Delta z\}_{i \in \mathcal{J}_M}$, with $\mathcal{J}_M = \{-M, \dots, M\}$ and $\Delta z = z_i - z_{i-1}$. For fixed n , $v \geq 0$ and $x \in \mathbb{R}$, we set $u_i^n = u(nh, z_i; v, x)$ the discrete solution of (4.8) at time nh on the point z_i of the grid \mathcal{Y}_M - for simplicity of notations, we do not stress in u_i^n the dependence on v and x (from the coefficients of the PDE).

As already explained in [3], for fixed values of $v \geq 0$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we establish to fix a small real threshold $\epsilon > 0$ and to solve the case $v < \epsilon$ and $v > \epsilon$ by applying an explicit in time and an implicit in time approximation respectively. It is indeed well known that for a big enough diffusion coefficient ($\rho_3^2 v / 2$), to avoid over-restrictive conditions on the grid steps, it is suggested to apply implicit finite differences to problem (4.8). On the other hand, when the diffusion coefficient is small compared with the reaction one, it is suggested to apply an explicit in time approximation coupled with a forward or backward finite difference for the first order term u_z depending on the sign of the reaction coefficient.

4.2.1 The case $v > \epsilon$

In the case $v > \epsilon$, the discrete solution u^n at time nh is computed in terms of the solution u^{n+1} at time $(n+1)h$ by solving the following discrete problem:

$$\frac{u_i^{n+1} - u_i^n}{h} + \mu(v, x, nh) \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta z} + \frac{1}{2}\rho_3^2 v \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta z^2} = 0. \quad (4.9)$$

Of course, (4.9) has to be coupled with suitable numerical boundary relations. As in [3] we assume that the boundary values are defined by the following Neumann-type conditions:

$$u_{-M-1}^n = u_{-M+1}^n, \quad u_{M+1}^n = u_{M-1}^n. \quad (4.10)$$

Then, by applying the implicit finite difference (4.9) coupled with the boundary conditions (4.10), we get the solution $u^n = (u_{-M}^n, \dots, u_M^n)^T$ by solving the following linear system

$$A u^n = u^{n+1}, \quad (4.11)$$

where $A = A(v, x)$ is the $(2M+1) \times (2M+1)$ tridiagonal real matrix given by

$$A = \begin{pmatrix} 1+2\beta & -2\beta & & & \\ \alpha-\beta & 1+2\beta & -\alpha-\beta & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha-\beta & 1+2\beta & -\alpha-\beta \\ & & & -2\beta & 1+2\beta \end{pmatrix}, \quad (4.12)$$

with

$$\alpha = \frac{h}{2\Delta z} \mu(v, x, nh) \quad \text{and} \quad \beta = \frac{h}{2\Delta z^2} \rho_3^2 v, \quad (4.13)$$

μ being defined in (4.2). We stress on that at each time step n , the quantities v and x are constant and known values (defined by the tree procedure for the pair (V, X)) and then α and β are constant parameters too. We assume that

$$\beta > |\alpha|. \quad (4.14)$$

Then the following result guarantees the solution to (4.11).

Proposition 4.1. *Under (4.14), A is invertible and A^{-1} is a stochastic matrix, that is all entries are non negative and, for $\mathbf{1} = (1, \dots, 1)^T$, $A\mathbf{1} = \mathbf{1}$.*

The proof is identical to that given in [3]. We shall see later that condition (4.14) is natural.

4.2.2 The case $v < \epsilon$

Consider now the case for v close to 0, that is $v < \epsilon$. Here, as usual, we split the explicit finite difference scheme according to the sign of the reaction coefficient: for $\mu(v, x, (n+1)h) > 0$ we solve the problem by the following approximation scheme

$$\frac{u_i^{n+1} - u_i^n}{h} + \mu(v, x, (n+1)h) \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta z} + \frac{1}{2}\rho_3^2 v \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta z^2} = 0; \quad (4.15)$$

if instead $\mu(v, x, (n+1)h) < 0$ we solve the problem by

$$\frac{u_i^{n+1} - u_i^n}{h} + \mu(v, x, (n+1)h) \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta z} + \frac{1}{2}\rho_3^2 v \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta z^2} = 0; \quad (4.16)$$

Our Neumann-type boundary conditions look like

$$u_{-M-1}^{n+1} = u_{-M+1}^{n+1}, \quad u_{M+1}^{n+1} = u_{M-1}^{n+1}. \quad (4.17)$$

Therefore, it easily follows that the solution u^n of the explicit in time scheme satisfies the condition

$$u^n = Cu^{n+1}$$

where

$$C = \begin{pmatrix} 1 - 2\beta - 2|\alpha| & 2\beta + 2|\alpha| & & & \\ \beta + 2|\alpha|\mathbb{1}_{\{\alpha < 0\}} & 1 - 2\beta - 2|\alpha| & \beta + 2|\alpha|\mathbb{1}_{\{\alpha > 0\}} & & \\ & \ddots & \ddots & \ddots & \\ & & \beta + 2|\alpha|\mathbb{1}_{\{\alpha < 0\}} & 1 - 2\beta - 2|\alpha| & \beta + 2|\alpha|\mathbb{1}_{\{\alpha > 0\}} \\ & & & 2\beta + 2|\alpha| & 1 - 2\beta - 2|\alpha| \end{pmatrix}, \quad (4.18)$$

with

$$\alpha = \frac{h}{2\Delta z} \mu(v, x, (n+1)h) \quad \text{and} \quad \beta = \frac{h}{2\Delta z^2} \rho_3^2 v \quad (4.19)$$

and $\mathbb{1}$ denoting the indicator function ($\mathbb{1}_\Gamma = 1$ on Γ and $\mathbb{1}_\Gamma = 0$ otherwise). We remark that $C = C(v, x)$ is a stochastic matrix if and only if

$$2\beta + 2|\alpha| \leq 1. \quad (4.20)$$

4.2.3 The final finite difference scheme

At each time step n , for each fixed $v \geq 0$ and $x \in \mathbb{R}$ the finite difference procedure described above defines the following operator

$$\Pi^h(v, x) = \begin{cases} C(v, x) & \text{if } v < \epsilon, \\ A^{-1}(v, x) & \text{if } v > \epsilon. \end{cases}$$

$\Pi^h(v, x)$ gives the discrete solution $\{u_i^n\}_{i \in \mathcal{J}_M}$ of (4.8) at time nh in terms of the solution $\{u_i^{n+1}\}_{i \in \mathcal{J}_M}$ at time $(n+1)h$:

$$u^n = \Pi^h(v, x)u^{n+1}.$$

Of course, in order that $\Pi^h(v, x)$ is well-defined and is a stochastic matrix too, we must assess all the parameters h , ϵ , M and Δz such that if $v > \epsilon$ then (4.14) holds and if $v < \epsilon$ then (4.20) holds. This can be done by writing all these parameters as functions of h , by developing the same arguments described at the beginning of Section 3.2 in [3] (see also Proposition 3.8 therein). Once all these things are well set, we can reproduce the proof of the convergence of Theorem 3.9 in [3].

We can finally give the numerical solution to (4.8) on the grid \mathcal{Y}_M through the above discretization procedure: for $z_i \in \mathcal{Y}_M$,

$$u(nh, z_i; v, x) \simeq \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v, x) g(z_\ell), \quad \ell \in \mathcal{J}_M. \quad (4.21)$$

Remark 4.2. We emphasize that other numerical boundary conditions can surely be selected, for example the two boundary values u_{-M}^n and u_M^n may be a priori fixed by a known constant (this typically appears in financial problems). However, in order to prove the convergence by means of a Markov chain approach as in [3], different boundary conditions have to guarantee that $\Pi^h(v, x)$ (and mainly $A^{-1}(v, x)$) is a stochastic matrix and a suitable boundary decay is satisfied. A discussion on this topic is given in Appendix A.1 of [3].

4.3 The scheme on the Y -component

We can now come back to our original problem, that is the computation of the function $\Psi_f(\xi; y, v, x)$ in (4.7) allowing one to numerically compute the expectation in (4.6). Of course, we consider the approximating process $(\bar{Y}^h, \bar{V}^h, \bar{X}^h)$ as described in Section 4.1. This means that the pair (v, x) at time-step n is chosen on the lattice $\mathcal{V}_n \times \mathcal{X}_n$: $v = v_{n,k}$ and $x = x_{n,j}$, for $0 \leq k, j \leq n$. We set

$$y_i = z_i, \quad i \in \mathcal{J}_M, k, j = 1, \dots, n.$$

Then, (4.21) gives the approximation

$$\Psi_f(\xi; y_i, v_{n,k}, x_{n,j}) \simeq \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) f(y_\ell + \xi), \quad i \in \mathcal{J}_M.$$

Therefore, the expectation in (4.6) is computed on the approximating tree for (V, X) by means of the above approximation:

$$\mathbb{E}(f(\bar{Y}_{(n+1)h}^h) \mid \bar{Y}_{nh}^h = y_i, \bar{V}_{nh}^h = v_{n,k}, \bar{X}_{nh}^h = x_{n,j}) \simeq \sum_{a,b \in \{d,u\}} \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) T_{n,k,j} f(\ell, a, b) p_{ab}^h(n, k, j) \quad (4.22)$$

where

$$T_{n,k,j} f(\ell, a, b) = f\left(y_\ell + \frac{\rho_1}{\sigma_V}(v_{n+1,k_a(n,k)} - v) + \rho_2 \sqrt{v}(x_{n+1,j_b(n,j)} - x)\right)$$

and the jump probabilities $p_{ab}^h(n, k, j)$ are given in (3.10) (or in Remark 3.1 if a correlation is assumed between the noises driving V and X).

Similar arguments can be used in order to compute the conditional expectation in the left hand side of (4.22) when the function f depends on the variables v and x also. Then one gets

$$\begin{aligned} & \mathbb{E}(f(\bar{Y}_{(n+1)h}^h, \bar{V}_{(n+1)h}^h, \bar{X}_{(n+1)h}^h) \mid \bar{Y}_{nh}^h = y_i, \bar{V}_{nh}^h = v_{n,k}, \bar{X}_{nh}^h = x_{n,j}) \\ & \simeq \sum_{a,b \in \{d,u\}} \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) T_{n,k,j} f(\ell, a, b) p_{ab}^h(n, k, j) \end{aligned} \quad (4.23)$$

where

$$\begin{aligned} T_{n,k,j} f(\ell, a, b) &= \\ &= f\left(y_\ell + \frac{\rho_1}{\sigma_V}(v_{n+1,k_a(n,k)} - v_{n,k}) + \rho_2 \sqrt{v_{n,k}}(x_{n+1,j_b(n,j)} - x_{n,j}), v_{n+1,k_a(n,k)}, x_{n+1,j_b(n,j)}\right). \end{aligned} \quad (4.24)$$

We stress that, at each time step n , the conditional expectation in (4.23) is computed on the grid

$$\mathcal{D}_{n,M}^h = \left\{ (y, v, x) : (v, x) \in \mathcal{V}_n^h \times \mathcal{X}_n^h \text{ and } y \in \mathcal{Y}_M \right\}. \quad (4.25)$$

5 The algorithm for the pricing of American options

The natural application of the hybrid tree-finite difference approach arises in the pricing of American options. Consider an American option with maturity T and payoff function $(\Phi(S_t))_{t \in [0, T]}$. First of all, we consider the log-price, so the obstacle will be given by

$$\Psi(Y_t) = \Phi(e^{Y_t}), \quad t \in [0, T].$$

The price at time 0 of such an option is then approximated by a backward dynamic programming algorithm, working as follows. First, consider a discretization of the time interval $[0, T]$ into N subintervals of length $h = T/N$: $[0, T] = \cup_{n=0}^{N-1} [nh, (n+1)h]$. Then the price $P(0, Y_0, V_0, X_0)$ of such an American option is numerically approximated through the quantity $P_h(0, Y_0, V_0, X_0)$ which is iteratively defined as follows: for $(y, v, x) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, by recalling formulas (2.1) and (2.2) for the interest rate, we have

$$\begin{cases} P_h(T, y, v, x) = \Psi(y) & \text{and as } n = N - 1, \dots, 0 \\ P_h(nh, y, v, x) = \max \left\{ \Psi(y), e^{-\int_{nh}^{(n+1)h} (\sigma_r X_t^{nh,x} + \varphi_t) dt} \mathbb{E} \left(P_h((n+1)h, Y_{(n+1)h}^{nh,y,v,x}, V_{(n+1)h}^{nh,v}, X_{(n+1)h}^{nh,x}) \right) \right\}. \end{cases}$$

From the financial point of view, this means to allow the exercise at the fixed times nh , $n = 0, \dots, N$.

We consider the discretization scheme $(\bar{Y}^h, \bar{V}^h, \bar{X}^h)$ discussed in Section 4 and we use the approximation (4.23) for the conditional expectations that have to be computed at each time step n . So, we recall the grid $\mathcal{D}_{n,M}^h$ defined in (4.25) and for every point $(y_i, v_{n,k}, x_{n,j}) \in \mathcal{D}_{n,M}^h$, by (4.23) we have

$$\begin{aligned} & \mathbb{E} \left(P_h((n+1)h, Y_{(n+1)h}^{nh,y_{n,i,k,j},v_{n,k},x_{n,j}}, V_{(n+1)h}^{nh,v_{n,k}}, X_{(n+1)h}^{nh,x_{n,j}}) \right) \\ & \simeq \sum_{a,b \in \{d,u\}} \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) p_{ab}^h(n, k, j) T_{n,k,j} P_h(\ell, a, b) \end{aligned} \quad (5.1)$$

where, by (4.24), for a general function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\begin{aligned} T_{n,k,j} f(\ell, a, b) &= \\ &= f \left(y_\ell + \frac{\rho_1}{\sigma_V} (v_{(n+1),ka(n,k)} - v_{n,k}) + \rho_2 \sqrt{v_{n,k}} (x_{(n+1),jb(n,j)} - x_{n,j}), v_{n+1,ka(n,k)}, x_{n+1,jb(n,j)} \right) \end{aligned}$$

Finally, we can summarize the backward induction giving our approximating algorithm as follows. For $n = 0, 1, \dots, N$, we define $\tilde{P}_h(nh, y, v, x)$ for $(y, v, x) \in \mathcal{D}_{n,M}^h$ as

$$\begin{cases} \tilde{P}_h(T, y_i, v_{N,k}, x_{N,j}) = \Psi(y) & (y_i, v_{N,k}, x_{N,j}) \in \mathcal{D}_{N,M} \text{ and as } n = N - 1, \dots, 0 \\ \tilde{P}_h(nh, y_i, v_{n,k}, x_{n,j}) = \max \left\{ \Psi(y_i), e^{-(\sigma_r x_{n,j} + \varphi_{nh})h} \sum_{a,b \in \{d,u\}} \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) p_{ab}^h(n, k, j) T_{n,k,j} P_h(\ell, a, b) \right\}, \\ \text{for } (y_i, v_{n,k}, x_{n,j}) \in \mathcal{D}_{n,M}^h \end{cases} \quad (5.2)$$

Notice that, at time step n , for every fixed $(y_i, v_{n,k}, x_{n,j}) \in \mathcal{D}_{n,M}^h$ the sum in (5.2) can be seen as an integral w.r.t. the measure

$$\mu^h(y_i, v_{n,k}, x_{n,j}; A) = \sum_{a,b \in \{u,d\}} \sum_{\ell \in \mathcal{J}_M} \Pi_{i\ell}^h(v_{n,k}, x_{n,j}) p_{ab}^h(n, k, j) \delta_{\{(y_\ell, v_{n+1,ka(n,k)}, x_{n+1,jb(n,j)})\}}(A) \quad (5.3)$$

for every Borel set A in \mathbb{R}^3 , $\delta_{\{a\}}$ denoting the Dirac mass in a , so that $\mu^h(y_i, v_{n,k}, x_{n,j}; \cdot)$ is a discrete measure on $\mathcal{D}_{n+1,M}^h$. Now, starting from the construction in Section 4.2 and following the techniques developed in [3], one can prove that for each h small then $\mu^h(y_i, v_{n,k}, x_{n,j}; \cdot)$ is actually a probability measure, that can be interpreted as a transition probability measure. Thus, one is actually doing expectations on a Markov chain $(\hat{Y}_n^h, \hat{V}_n^h, \hat{X}_n^h)_{n=0,1,\dots,N}$, whose state-space, at time step n , is given by $\mathcal{D}_{n,M}^h$ and whose transition probability measure at time step n is given by $\mu^h(y_i, v_{n,k}, x_{n,j}; \cdot)$ in (5.3). Then, one can prove, as in [3], that under appropriate conditions on Δz_h and M_h such that, as $h \rightarrow 0$, $\Delta z_h \rightarrow 0$ and $M_h \Delta z_h \rightarrow \infty$, the family of Markov chains $(\hat{Y}_n^h, \hat{V}_n^h, \hat{X}_n^h)_h$ weakly converges to the diffusion process (Y, V, X) . And this gives the convergence of our hybrid tree-finite difference algorithm approximating the Heston-Hull-White model.

6 Generalization to the Heston-Hull-White2d model

The Heston-Hull-White2d model generalizes the previous model in the fact that the quantity η is assumed to be stochastic and to follow a diffusion model itself. So, the underlying process is now 4-dimensional and is given by: the share price S , the volatility process V , the interest rate r and the continuous dividend rate η . Actually, here the process η has not necessarily the meaning of a dividend rate, being for example a further interest rate process. In fact, the Heston-Hull-White2d model occurs in multi-currency models with short-rate interest rates, see e.g. [9].

Under the risk neutral measure, the dynamics are governed by the stochastic differential equation system

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t - \eta_t)dt + \sqrt{V_t}dZ_t, \\ dV_t &= \kappa_V(\theta_V - V_t)dt + \sigma_V\sqrt{V_t}dW_t^1, \\ dr_t &= \kappa_r(\theta_r(t) - r_t)dt + \sigma_r dW_t^2, \\ d\eta_t &= \kappa_\eta(\theta_\eta(t) - \eta_t)dt + \sigma_\eta dW_t^3,\end{aligned}$$

with initial data $S_0, V_0, r_0, \eta_0 > 0$, where Z , W^1 , W^2 and W^3 are suitable and possibly correlated Brownian motions. Note that the process η evolves as a generalized OU process: θ_η is a deterministic function of the time. We consider non null correlations between the Brownian motions driving the pairs (S, V) , (S, r) and (S, η) , that is

$$d\langle Z, W^1 \rangle_t = \rho_1 dt, \quad d\langle Z, W^2 \rangle_t = \rho_2 dt, \quad d\langle Z, W^3 \rangle_t = \rho_3 dt.$$

Correlations among the processes V , r and η can be surely inserted and can be handled as in Remark 3.1 (see next Remark 6.1).

As done in Section 2, we consider the transformations (2.1)-(2.2) for the generalized OU processes r and η : we set

$$r_t = \sigma_r X_t^r + \varphi_t^r \quad \text{and} \quad \eta_t = \sigma_\eta X_t^\eta + \varphi_t^\eta \quad (6.1)$$

where

$$\begin{aligned}X_t^r &= -\kappa_r \int_0^t X_s^r ds + W_t^2, & \varphi_t^r &= r_0 e^{-\kappa_r t} + \kappa_r \int_0^t \theta_r(s) e^{-\kappa_r(t-s)} ds \\ X_t^\eta &= -\kappa_\eta \int_0^t X_s^\eta ds + W_t^3, & \varphi_t^\eta &= \eta_0 e^{-\kappa_\eta t} + \kappa_\eta \int_0^t \theta_\eta(s) e^{-\kappa_\eta(t-s)} ds\end{aligned} \quad (6.2)$$

So, by considering the log-price process, we reduce to the 4-dimensional process (Y, V, X^r, X^η) whose dynamics is given by

$$\begin{aligned}dY_t &= \mu_Y(V_t, X_t^r, X_t^\eta, t)dt + \sqrt{V_t}(\rho_1 dW_t^1 + \rho_2 dW_t^2 + \rho_3 dW_t^3 + \rho_4 dW_t^4), \\ dV_t &= \mu_V(V_t)dt + \sigma_V\sqrt{V_t}dW_t^1, \\ dX_t^r &= \mu_{X^r}(X_t^r)dt + dW_t^2, \\ dX_t^\eta &= \mu_{X^\eta}(X_t^\eta)dt + dW_t^3, \\ \text{with } Y_0 &= \ln S_0 \in \mathbb{R}, \quad V_0 > 0, \quad X_0^r = 0, \quad X_0^\eta = 0\end{aligned} \quad (6.3)$$

where

$$\rho_4 = \sqrt{1 - \rho_1^2 - \rho_2^2 - \rho_3^2}, \quad (\rho_1, \rho_2, \rho_3) \in B_1(0) \quad (6.4)$$

$$\mu_Y(v, x_1, x_2, t) = \sigma_r x_1 + \varphi_t^r - \sigma_\eta x_2 - \varphi_t^\eta - \frac{1}{2}v, \quad (6.5)$$

$$\mu_V(v) = \kappa_V(\theta_V - v) \quad (6.6)$$

$$\mu_{X^r}(x) = -\kappa_r x, \quad (6.7)$$

$$\mu_{X^\eta}(x) = -\kappa_\eta x, \quad (6.8)$$

$B_1(0)$ denoting here the unit ball centered at 0 in \mathbb{R}^3 .

Starting from (6.3), we set-up an approximating procedure similar to the one developed in Section 3 and Section 4. In the following, we briefly describe how to extend such algorithms to the Heston-Hull-White2d model.

6.1 Approximation of (V, X^r, X^η)

Concerning the triple (V, X^r, X^η) , we build an approximating tree on \mathbb{R}^3 as follows:

- we apply the procedure in Section 3.1 to the process X^r ;
- we apply the procedure in Section 3.1 to the process X^η ;
- we apply the procedure in Section 3.2 to the process V .

We then get three approximating trees:

$$\hat{X}^{r,h} \text{ for } X^r, \quad \hat{X}^{\eta,h} \text{ for } X^\eta, \quad \hat{V}^h \text{ for } V.$$

Then, we use the null correlation between any two of V , X^r and X^η : we concatenate the above three trees in order to get a 3-dimensional approximating tree $(\hat{V}^h, \hat{X}^{r,h}, \hat{X}^{\eta,h})$ for (V, X^r, X^η) by introducing product-type jump probabilities. In other words, we generalize the probabilities in (3.10) for all the $2^3 = 8$ possible jumps.

Remark 6.1. *One might include correlations between any two of the Brownian motions driving the processes V , X^r and X^η . As described in Remark 3.1, the jump probabilities are no more of a product-type but they solve a linear system of equations that must include the matching of the local cross-moments up to order one in h .*

6.2 The scheme on the Y -component and the approximating 4-dimensional process

We repeat the reasonings in Section 4.1 in order to define an approximating time-continuous process $(\bar{Y}^h, \bar{V}^h, \bar{X}^{r,h}, \bar{X}^{\eta,h})$ for (Y, V, X^r, X^η) - roughly speaking, it suffices to replace the one-dimensional process X in Section 4.1 with the 2-dimensional process (X^r, X^η) . So, we start from

$$dY_t = \frac{\rho_1}{\sigma_V} dV_t + \rho_2 \sqrt{V_t} dX_t^r + \rho_3 \sqrt{V_t} dX_t^\eta + \mu(V_t, X_t^r, X_t^\eta, t) dt + \rho_4 \sqrt{V_t} dW_t^4 \quad (6.9)$$

with

$$\mu(v, x_1, x_2, t) = \mu_Y(v, x_1, x_2, t) - \frac{\rho_1}{\sigma_V} \mu_V(v) - \rho_2 \sqrt{v} \mu_{X^r}(x_1) - \rho_3 \sqrt{v} \mu_{X^\eta}(x_2) \quad (6.10)$$

Then, we apply the finite difference method in Section 4.2 and we obtain a final difference scheme given by

$$\Pi^h(v, x_1, x_2) = \begin{cases} A^{-1}(v, x_1, x_2) & \text{if } v > \epsilon, \\ C(v, x_1, x_2) & \text{if } v < \epsilon \end{cases}$$

where, $\mu(\cdot)$ being defined in (6.10),

- A is given in (4.12) with

$$\alpha = \frac{h}{2\Delta z} \mu(v, x_1, x_2, nh) \quad \text{and} \quad \beta = \frac{h}{2\Delta z^2} \rho_4^2 v; \quad (6.11)$$

- C is given in (4.18) with

$$\alpha = \frac{h}{2\Delta z} \mu(v, x_1, x_2, (n+1)h) \quad \text{and} \quad \beta = \frac{h}{2\Delta z^2} \rho_4^2 v. \quad (6.12)$$

Finally, we extend the approximation scheme (4.23) to the case in which $X = (X^r, X^\eta)$ and the algorithm for the pricing of European or American options described in Section 5.

7 The hybrid Monte Carlo algorithm

The approximation we have set-up for the Heston-Hull-White processes can be used to construct a Monte Carlo algorithm. Let us see how one can simulate a single path by using the tree approximation and the standard Euler scheme for the Y -component. We call it “hybrid” because two different noise sources are considered: we simulate a continuous process in space (the component Y) starting from a discrete process in space (the 3-dimensional tree for (V, X^r, X^η)).

Concerning the Heston-Hull-White dynamics in Section 2, consider the triple (Y, V, X) as in (2.3). Let $(\hat{V}_n^h, \hat{X}_n^h)_{n=0,1,\dots,N}$ denote the Markov chain that approximates the pair (V, X) . We construct a sequence $(\hat{Y}_n^h)_{n=0,1,\dots,N}$ approximating Y at times $n = 0, 1, \dots, N$ by means of the Euler scheme defined in (4.3): we set $\hat{Y}_0^h = Y_0$ and for $t \in [nh, (n+1)h]$ with $n = 0, 1, \dots, N-1$ then

$$\hat{Y}_{n+1}^h = \hat{Y}_n^h + \frac{\rho_1}{\sigma_V} (\hat{V}_{n+1}^h - \hat{V}_n^h) + \rho_2 \sqrt{\hat{V}_n^h} (\hat{X}_{n+1}^h - \hat{X}_n^h) + \mu(\hat{V}_n^h, \hat{X}_n^h, nh)h + \rho_3 \sqrt{h \hat{V}_n^h} \Delta_{n+1}, \quad (7.1)$$

where μ is defined in (4.2) and $\Delta_1, \dots, \Delta_N$ denote i.i.d. standard normal r.v.'s, independent of the noise driving the chain (\hat{V}, \hat{X}) . So, the simulation algorithm is very simple: at each time step $n \geq 1$, one let the pair (V, X) evolve on the tree and simulate the process Y at time nh by using (7.1).

A similar algorithm can be considered to simulate the Heston-Hull-White2d dynamics in Section 6, that can be seen as a function of the triple (Y, V, X^r, X^η) in (6.3). Here, we apply the Euler scheme to (6.9). So, let $(\hat{V}_n^h, \hat{X}_n^{r,h}, \hat{X}_n^{\eta,h})_{n=0,1,\dots,N}$ denote the Markov chain approximating (V, X^r, X^η) , as described in Section 6.1. Starting from (6.9), we approximate the component Y at times nh , $n = 0, 1, \dots, N$, as follows: we set $\hat{Y}_0^h = Y_0$ and for $n = 1, \dots, N$, $n = 0, 1, \dots, N-1$ then

$$\begin{aligned} \hat{Y}_{n+1}^h = \hat{Y}_n^h + \frac{\rho_1}{\sigma_V} (\hat{V}_{n+1}^h - \hat{V}_n^h) + \rho_2 \sqrt{\hat{V}_n^h} (\hat{X}_{n+1}^{r,h} - \hat{X}_n^{r,h}) + \rho_3 \sqrt{\hat{V}_n^h} (\hat{X}_{n+1}^{\eta,h} - \hat{X}_n^{\eta,h}) \\ + \mu(\hat{V}_n^h, \hat{X}_n^{r,h}, \hat{X}_n^{\eta,h}, nh)h + \rho_4 \sqrt{h \hat{V}_n^h} \Delta_{n+1} \end{aligned} \quad (7.2)$$

where μ is defined in (6.10) and $\Delta_1, \dots, \Delta_N$ denote i.i.d. standard normal r.v.'s, independent of the noise driving the chain $(\hat{V}^h, \hat{X}^{r,h}, \hat{X}^{\eta,h})$. And again, the simulation algorithm is straightforward.

8 Numerical results

In this section we provide numerical results in order to assess the efficiency and the robustness of our hybrid numerical approaches in the case of plain vanilla options in the Heston-Hull-White and Heston-Hull-White2d models.

8.1 European and American options in the Heston-Hull-White model

In the European and American option contracts we are dealing with, we consider the following set of parameters:

- initial share price $S_0 = 100$, strike price $K = 100$, maturity $T = 1$, dividend rate $\eta = 0.03$;
- initial interest rate $r_0 = 0.04$, speed of mean-reversion $\kappa_r = 1$, interest rate volatility $\sigma_r = 0.2$;
- initial volatility $V_0 = 0.1$, long-mean $\theta_V = 0.1$, speed of mean-reversion $\kappa_V = 2$, volatility of volatility $\sigma_V = 0.3$.

The time-varying long-term mean $\theta_r(t)$ fit the theoretical bond prices to the yield curve observed on the market. We have chosen for this purpose the following interest rate curve $P_r(0, T) = e^{-0.04T}$. We consider varying correlations: concerning the pair (S, V) , we set $\rho_1 = \rho_{SV} = -0.5, 0.5$; as for (S, r) , we study the cases and $\rho_2 = \rho_{Sr} = -0.5, 0, 0.5$. No correlation is assumed to exist between r and V .

The numerical study of the hybrid tree-finite difference method **HTFD** is split in two cases:

- **HTFD1** refers to the (fixed) number of time steps $N_t = 50$ and varying number of space steps $N_S = 50, 100, 150, 200$;
- **HTFD2** refers to $N_t = N_S = 50, 100, 150, 200$.

Concerning the Monte Carlo method, we compare the results by using the hybrid simulation scheme in Section 7, that we call **HMC**, and by simulating paths through the accurate third-order Alfonsi [1] discretization scheme for the CIR stochastic volatility process and an exact scheme for the interest rate, the latter being reported as **AMC**. In both cases, we consider varying number of time discretization steps $N_t = 50, 100, 150, 200$ and two cases for the number of Monte Carlo iterations:

- **HMC1** and **AMC1** refer to 50 000 iterations,
- **HMC2** and **AMC2** refer to 200 000 iterations.

The benchmark value **B-AMC** is obtained using the Alfonsi Monte Carlo method **AMC** with a huge number of Monte Carlo simulations (1 million iterations) and $N_t = 300$ discretization time steps. In the American case, in absence of reliable numerical methods, we consider the Longstaff-Schwartz [10] algorithm **MC-LS** with 20 exercise dates. All Monte Carlo results report the 95% confidence intervals.

Tables 1-(a) and 1-(b) report European call option prices. In Tables 2-(a) and 2-(b) we provide results for American call option prices. Table 3 refers to the computational time cost (in seconds) of the different algorithms in the call European case.

The numerical results show that **HTFD** is accurate, reliable and efficient for pricing European and American options in the Heston-Hull-White model. Moreover, our hybrid Monte Carlo algorithm **HMC** appears to be competitive with **AMC**, that is the one from the accurate simulations by Alfonsi [1]: the numerical results are similar in term of precision and variance but **HMC** is definitely better from the computational times point of view. Additionally, because of its simplicity, **HMC** represents a real and interesting alternative to **AMC**.

(a)

$\rho_{SV} = -0.5$	N_S	HTFD1	HTFD2	B-AMC	HMC1	HMC2	AMC1	AMC2
$\rho_{Sr} = -0.5$	50	11.202744	11.202744	11.34±0.04	11.30±0.16	11.32±0.08	11.34±0.16	11.37±0.08
	100	11.319814	11.331040		11.41±0.16	11.38±0.08	11.31±0.16	11.36±0.08
	150	11.340665	11.349902		11.36±0.16	11.36±0.08	11.35±0.16	11.38±0.08
	200	11.346972	11.355772		11.34±0.16	11.37±0.08	11.44±0.16	11.39±0.08
$\rho_{Sr} = 0$	50	12.526779	12.526779	12.77±0.04	12.66±0.18	12.69±0.09	12.68±0.18	12.79±0.09
	100	12.720651	12.705772		12.74±0.18	12.79±0.09	12.63±0.18	12.78±0.09
	150	12.754610	12.749526		12.74±0.18	12.79±0.09	12.68±0.18	12.81±0.09
	200	12.760365	12.766836		12.74±0.18	12.80±0.09	12.75±0.18	12.79±0.09
$\rho_{Sr} = 0.5$	50	13.853193	13.853193	14.04±0.04	13.88±0.19	13.92±0.10	13.97±0.20	14.05±0.10
	100	14.011537	14.013063		13.91±0.19	14.01±0.10	13.89±0.19	14.06±0.10
	150	14.031598	14.038361		13.94±0.19	14.07±0.10	13.92±0.20	14.08±0.10
	200	14.038235	14.045612		13.99±0.19	14.07±0.10	13.90±0.19	14.06±0.10

(b)

$\rho_{SV} = 0.5$	N_S	HTFD1	HTFD2	B-AMC	HMC1	HMC2	AMC1	AMC2
$\rho_{Sr} = -0.5$	50	11.208938	11.208938	11.54±0.05	11.65±0.21	11.43±0.10	11.45±0.20	11.53±0.10
	100	11.488205	11.468278		11.51±0.20	11.54±0.10	11.61±0.21	11.53±0.10
	150	11.513593	11.521964		11.44±0.20	11.54±0.10	11.54±0.20	11.52±0.10
	200	11.523943	11.536257		11.41±0.20	11.60±0.10	11.58±0.21	11.49±0.10
$\rho_{Sr} = 0$	50	12.594415	12.594415	12.96±0.05	13.01±0.22	12.83±0.11	12.82±0.22	12.94±0.11
	100	12.884160	12.855993		12.81±0.22	12.95±0.11	12.86±0.22	12.92±0.11
	150	12.937141	12.921218		12.74±0.22	12.95±0.11	12.85±0.22	12.93±0.11
	200	12.948888	12.947088		12.78±0.22	13.00±0.11	12.85±0.22	12.86±0.11
$\rho_{Sr} = 0.5$	50	13.969027	13.969027	14.23±0.05	14.19±0.23	14.09±0.12	14.10±0.23	14.22±0.12
	100	14.199994	14.189402		13.94±0.23	14.18±0.12	14.07±0.23	14.19±0.12
	150	14.221938	14.229056		13.89±0.23	14.20±0.12	14.15±0.24	14.20±0.12
	200	14.230252	14.239789		13.97±0.23	14.23±0.12	14.05±0.23	14.13±0.12

Table 1: *Prices of European call options.* $S_0 = 100$, $K = 100$, $T = 1$, $r_0 = 0.04$, $\kappa_r = 1$, $\sigma_r = 0.2$, $\eta = 0.03$, $V_0 = 0.1$, $\theta_V = 0.1$, $\kappa_V = 2$, $\sigma_V = 0.3$, $\rho_{Sr} = -0.5, 0, 0.5$, $\rho_{SV} = -0.5, 0.5$.

(a)

$\rho_{SV} = -0.5$	N_S	HTFD1	HTFD2	MC-LS
$\rho_{Sr} = -0.5$	50	12.090433	12.090433	12.18 ± 0.01
	100	12.205014	12.212884	
	150	12.224432	12.231392	
	200	12.230288	12.237054	
$\rho_{Sr} = 0$	50	12.912708	12.912708	13.14 ± 0.01
	100	13.119121	13.101073	
	150	13.156492	13.149182	
	200	13.162893	13.168602	
$\rho_{Sr} = 0.5$	50	13.944266	13.944266	14.15 ± 0.01
	100	14.125059	14.122918	
	150	14.146240	14.152060	
	200	14.153288	14.160288	

(b)

$\rho_{SV} = 0.5$	N_S	HTFD1	HTFD2	MC-LS
$\rho_{Sr} = -0.5$	50	12.044761	12.044761	12.33 ± 0.01
	100	12.327173	12.306911	
	150	12.352117	12.364256	
	200	12.362528	12.379805	
$\rho_{Sr} = 0$	50	12.910530	12.910530	13.29 ± 0.01
	100	13.234103	13.203037	
	150	13.293507	13.278025	
	200	13.306270	13.308165	
$\rho_{Sr} = 0.5$	50	14.029398	14.029398	14.32 ± 0.02
	100	14.284281	14.273324	
	150	14.307465	14.316216	
	200	14.316138	14.327803	

Table 2: *Prices of American call options.* $S_0 = 100$, $K = 100$, $T = 1$, $r_0 = 0.04$, $\kappa_r = 1$, $\sigma_r = 0.2$, $\eta = 0.03$, $V_0 = 0.1$, $\theta_V = 0.1$, $\kappa_V = 2$, $\sigma_V = 0.3$, $\rho_{Sr} = -0.5, 0, 0.5$, $\rho_{SV} = -0.5, 0.5$.

N_S	HTFD1	HTDF2	B-AMC	HMC1	HMC2	AMC1	AMC
50	1.24	1.24	223.67	0.77	3.05	2.16	7.48
100	2.53	21.64		1.59	6.11	4.00	14.61
150	3.97	128.6		2.33	9.13	5.87	21.64
200	5.74	471.4		3.11	12.73	7.61	28.85

Table 3: *Computational times (in seconds) for European call options.*

8.2 European and American options in the Heston-Hull-White2d model

In the European and American option contracts we are dealing with, we consider the following set of parameters :

- $S_0 = 100, K = 100, T = 1$;
- $r_0 = 0.04, \eta_0 = 0.03, \kappa_r = \kappa_\eta = 1, \sigma_r = \sigma_\eta = 0.2$;
- $V_0 = 0.1, \theta_V = 0.1, \kappa_V = 2, \sigma_V = 0.3$;
- $\rho_{Sr} = -0.5, 0, 0.5, \rho_{SV} = -0.5, \rho_{S\eta} = -0.5, 0.5, \rho_{Vr} = \rho_{V\eta} = \rho_{r\eta} = 0$;
- $P_r(0, T) = e^{-0.04T}, P_\eta(0, T) = e^{-0.03T}$.

Similarly as for the interest rate, the time-varying long-term mean $\theta_\eta(t)$ fit the theoretical bond prices $P_\eta(0, T) = e^{-0.03T}$ to the yield curve observed on the market. We make this choice because this model occurs for example in multi-currency models with short-rate interest rates (see [9]). We consider here only the number of space steps $N_S = 30, 50, 100$ because the cases $N_S = 150, 200$ need huge computational times. Tables 4 and 5 report European and American call option prices. Table 6 refers to the computational time cost (in seconds) of the different algorithms in the call European case.

The numerical results confirm the good numerical behavior of **HTFD** and **HMC** in the Heston-Hull-White2d model as well.

(a)								
$\rho_{SV} = -0.5,$ $\rho_{S\eta} = -0.5$ $\rho_{Sr} = -0.5$	N_S	HTFD1	HTFD2	B-AMC	HMC1	HMC2	AMC1	AM2
	30	13.470572	13.470572	13.79 ± 0.04	13.82±0.20	13.74±0.10	13.83±0.20	13.79±0.10
	50	13.688842	13.671173		13.96±0.20	13.81±0.10	13.88±0.20	13.80±0.10
	100	13.790205	13.781519		14.00±0.20	13.80±0.10	13.68±0.20	13.73±0.10
$\rho_{Sr} = 0$	30	14.736242	14.736242	15.04 ± 0.05	15.10±0.22	14.99±0.11	14.95±0.22	15.03±0.11
	50	14.958094	14.946029		15.23±0.22	15.04±0.11	14.98±0.22	15.01±0.11
	100	15.019204	15.032709		15.23±0.22	15.04±0.11	14.80±0.21	14.97±0.11
$\rho_{Sr} = 0.5$	30	15.805046	15.805046	16.19 ± 0.03	15.21±0.22	15.04±0.11	16.04±0.23	16.17±0.12
	50	16.052315	16.032043		16.13±0.23	16.06±0.11	16.09±0.23	16.13±0.12
	100	16.155354	16.145308		16.33±0.23	16.10±0.11	15.93±0.23	16.12±0.12
(b)								
$\rho_{SV} = -0.5,$ $\rho_{S\eta} = 0.5$ $\rho_{Sr} = -0.5$	N_S	HTFD1	HTFD2	B-AMC	HMC1	HMC2	AMC1	AMC2
	30	9.418513	9.418513	9.61 ± 0.03	9.57±0.13	9.62±0.07	9.64±0.13	9.66±0.07
	50	9.552565	9.532194		9.57±0.13	9.61±0.07	9.65±0.13	9.66±0.07
	100	9.633716	9.607339		9.66±0.13	9.62±0.07	9.63±0.13	9.63±0.07
$\rho_{Sr} = 0$	30	10.916753	10.916753	11.18 ± 0.03	11.15±0.15	11.16±0.08	11.07±0.15	11.22±0.08
	50	11.117050	11.100343		11.18±0.15	11.16±0.08	11.14±0.15	11.22±0.08
	100	11.178119	11.173631		11.16±0.15	11.18±0.08	11.08±0.15	11.20±0.08
$\rho_{Sr} = 0.5$	30	12.203271	12.203271	12.55 ± 0.04	12.44±0.17	12.43±0.09	12.47±0.17	12.60±0.09
	50	12.443197	12.411406		12.54±0.17	12.44±0.09	12.53±0.17	12.59±0.09
	100	12.552842	12.522237		12.45±0.17	12.55±0.09	12.45±0.17	12.58±0.09

Table 4: *Prices of European call options.* $S_0 = 100, K = 100, T = 1, r_0 = 0.04, \kappa_r = 1, \sigma_r = 0.2, \eta_0 = 0.03, \kappa_\eta = 1, \sigma_\eta = 0.2, V_0 = 0.1, \theta_V = 0.1, \kappa_V = 2, \sigma_V = 0.3, \rho_{Sr} = -0.5, 0, 0.5, \rho_{SV} = -0.5, \rho_{S\eta} = -0.5, 0.5$.

(a)				
$\rho_{SV} = -0.5,$ $\rho_{S\eta} = -0.5$	N_S	HTFD1	HTFD2	MC-LS
$\rho_{Sr} = -0.5$	30	14.057963	14.057963	14.37 ± 0.01
	50	14.290597	14.263254	
	100	14.400377	14.381552	
$\rho_{Sr} = 0$	30	14.989844	14.989844	15.32 ± 0.01
	50	15.253011	15.229151	
	100	15.320569	15.331744	
$\rho_{Sr} = 0.5$	30	15.826696	15.826696	16.28 ± 0.02
	50	16.146080	16.111559	
	100	16.270439	16.248656	
(b)				
$\rho_{SV} = -0.5,$ $\rho_{S\eta} = 0.5$	N_S	HTFD1	HTFD2	MC-LS
$\rho_{Sr} = -0.5$	30	11.598655	11.598655	11.64 ± 0.008
	50	11.707669	11.681873	
	100	11.775632	11.743388	
$\rho_{Sr} = 0$	30	12.400256	12.400256	12.55 ± 0.01
	50	12.579124	12.561214	
	100	12.634969	12.629401	
$\rho_{Sr} = 0.5$	30	13.137621	13.137621	13.43 ± 0.01
	50	13.380571	13.341882	
	100	13.497053	13.459978	

Table 5: *Prices of American call options.* $S_0 = 100$, $K = 100$, $T = 1$, $r_0 = 0.04$, $\kappa_r = 1$, $\sigma_r = 0.2$, $\eta_0 = 0.03$, $\kappa_\eta = 1$, $\sigma_\eta = 0.2$, $V_0 = 0.1$, $\theta_V = 0.1$, $\kappa_V = 2$, $\sigma_V = 0.3$, $\rho_{Sr} = -0.5, 0, 0.5$, $\rho_{SV} = -0.5$, $\rho_{S\eta} = -0.5, 0.5$.

N_S	HTFD1	HTDF2	B-AMC	HMC1	HMC2	AMC1	AMC2
30	7.19	7.19	284.84	0.60	2.61	1.79	6.03
50	12.00	94.34		1.14	4.19	2.73	9.58
100	28.31	3603		2.02	8.06	5.05	18.70

Table 6: *Computational times (in seconds) for European call options.*

9 Conclusions

We have introduced new hybrid tree-finite difference and a Monte Carlo methods for pricing plain vanilla options in a stochastic volatility framework with stochastic interest rates. The numerical comparisons show that our methods provide good approximation of option prices with efficient time computations.

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