Title : Double Heston simulation

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We consider the following model :

$$\begin{cases} \frac{dS_t}{S_t} & = & (r-\delta)dt + \sum_{j=1}^2 \sqrt{V_t^j}\left(\rho_j dW_t^j + \sqrt{1-\rho_1^2}dB_t^1\right) \\ dV_t^j & = & b_j(\theta_j - V_t^j)dt + \sigma_j\sqrt{V_t^j}dW_t^j & j=1,2 \end{cases} \tag{1}$$

1 Discretization of the spot price equation

Following [GP10], we are going to discretize the log-discounted price process $X_t = \ln(e^{-(r-\delta)t}S_t)$. Standard computations using Ito's formula gives us :

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} \sum_{j=1}^2 -\frac{1}{2}V_s^j ds + \sqrt{V_s^j}dW_s^j. \tag{2}$$

Using the equation on V^1, V^2, we finally get (assuming $\sigma_j \neq 0$) :

$$\begin{aligned} X_{t+\Delta t} =& X_t + \sum_{j=1}^2 \frac{\rho_j}{\sigma_j}\left(V_{t+\Delta t}^j - V_t^j\right) \\ &+ \sum_{j=1}^2 \int_t^{t+\Delta t} -\frac{1}{2}V_s^j ds - \frac{\rho_j}{\sigma_j}b_j(\theta_j - V_s^j)ds + \sqrt{(1-\rho_j^2)V_s^j}dB_s^j. \end{aligned}$$

Endly, we notice that by introducing $I_t^j = \int_t^{t+\Delta t} V_s^j ds$ and G_1, G_2 two standard normal variables independent with respect to everything that the following equality holds in law :

$$\begin{aligned} X_{t+\Delta t} =& X_t + \sum_{j=1}^2 \frac{\rho_j}{\sigma_j}\left(V_{t+\Delta t}^j - V_t^j\right) - \frac{\rho_j b_j}{\sigma_j}\theta_j\Delta t + \left[\frac{\rho_j b_j}{\sigma_j} - \frac{1}{2}\right] I_t^j \\ &+ \sum_{j=1}^2 \sqrt{(1-\rho_j^2)I_t^j}G_j. \end{aligned}$$

It remains to chose a way to discretize I_t^j and we choose to approximate it by $I_t^j \approx \frac{\Delta t}{2}\left(V_{t+\Delta t}^j + V_t^j\right)$. This way of doing is called the predictor corrector scheme according to [GP10]. Finally, our discretized process is the following :

$$\begin{aligned} \hat{X}_{t+\Delta t} = \hat{X}_t + \sum_{j=1}^2 \frac{\rho_j}{\sigma_j}\left(V_{t+\Delta t}^j - V_t^j\right) - \frac{\rho_j b_j}{\sigma_j}\theta_j\Delta t \\ + \left[\frac{\rho_j b_j}{\sigma_j} - \frac{1}{2}\right]\frac{\Delta t}{2}\left(V_{t+\Delta t}^j + V_t^j\right) \\ + \sqrt{(1-\rho_j^2)\frac{\Delta t}{2}\left(V_{t+\Delta t}^j + V_t^j\right)}G_j. \end{aligned} \tag{3}$$

It remains to work on the discretization of the variance process.

2 Variance process

The variance process is a CIR process. Many algorithms are known. We have implemented here four algorithms. The first one is the Exact Zhu (cf [J.Z08]), the second and the third one are respectively the 2nd and 3rd order Alfonsi schemes (cf [Alf10]), the fourth one is the Quadratic Exponential Martingale (cf [L.A08]).

2.1 Exact Zhu scheme

At each discretization step, we replace the increment of the original variance process by the square of a gaussian random variable with the same mean and the same variance as the increment.

2.2 2nd Order Alfonsi scheme

At each discretization step, we replace the increment of the original variance process by a binary random variable which matches the mean and the variance of the increment.

2.3 3rd Order Alfonsi scheme

At each discretization step, we replace the increment of the original variance process by a tri-valued random variable which matches the first three moments of the increment.

2.4 Quadratic Exponential scheme

At each discretization step, we replace the increment of the original variance process by a random variable with a proxy law close to the real one following [?].

3 Results

With the following parameters :

S_0	61.90	r	0.03
V_0^1	0.36	V_0^2	0.49
σ_1	0.1	σ_2	0.2
b_1	0.9	b_2	1.2
ρ_1	-0.5	b_2	-0.5
θ_1	0.1	θ_2	0.15

and for 10^5 simulations and 24 discretization steps by year, we obtain the following result :

Maturity Strike	1 S_0		1 $0.7S_0$		1 $1.3S_0$	
$Fourier$	19.4569		27.6092		13.9299	
Zhu	19.4579	± 0.1096	27.6030	± 0.0667	13.9275	± 0.148
$2^{nd} Alfonsi$	19.4654	± 0.1097	27.6079	± 0.0668	13.9374	± 0.1481
$3^{rd} Alfonsi$	19.4	± 0.1098	27.5906	± 0.0668	13.8667	± 0.1481
QEM	19.5646	± 0.1103	27.7	± 0.0673	14.05	± 0.1486

Maturity Strike	10 S_0		10 $0.7S_0$		10 $1.3S_0$	
$Fourier$	41.4006		45.2866		38.278	
Zhu	41.4582	± 0.1103	45.318	± 0.0786	38.3587	± 0.1394
$2^{nd} Alfonsi$	41.2883	± 0.1106	45.1998	± 0.0787	38.1606	± 0.1397
$3^{rd} Alfonsi$	41.3906	± 0.1105	45.282	± 0.0787	38.2641	± 0.1397
QEM	42.18	± 0.11	45.895	± 0.0799	39.1934	± 0.1391

References

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