PARABOLIC INVERSE FOURIER TRANSFORM METHOD FOR EUROPEAN OPTION PRICING UNDER LÉVY PROCESSES

OLEG KUDRYAVTSEV

Department of Informatics, Russian Customs Academy Rostov Branch,
Budennovskiy 20, Rostov-on-Don, 344002, Russia
E-mail: koe@donrta.ru

Abstract. We describe “Conformal parabolic inverse Fourier transform method” for efficient pricing European options for a wide class of Lévy processes developed in Boyarchenko and Levendorskiǐ (2014). The method was implemented into program platform Premia for KoBoL (CGMY) model.

Premia 22

Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

In the case of pricing European options in one-factor exponential Levy models, the most popular approach is the Fourier transform method which was applied in Carr and Madan (1999), Boyarchenko and Levendorskiǐ (2002) and many others. In all these papers, as in most others, the inverse Fourier integral representation is used, and the option price is represented as the integral over an appropriate line in the complex plane parallel to the real axis. A numerical realization of the inverse Fourier transform (iFT) can be handled very efficiently by means of the Fast Fourier Transform (FFT), if we need a set of option prices at different spot/strike levels.

Boyarchenko and Levendorskiǐ (2014) give fairly simple and efficient recommendations for choosing the parameters of the numerical scheme and suggest families of the conformal contour deformations, which greatly increases the rate of convergence of the integral. The resulting pricing formula was called “parabolic iFT” because it can be described as a change of variables in the standard Fourier inversion formula, resembling the analytical expression for a fractional parabola. In cases in which the standard inverse Fourier transform realization may require thousands or even millions of terms, parabolic iFT may sufficiently reduce the number of terms in the integral sum. Notice that parabolic iFT cannot be applied in combination with the FFT technique introduced to finance in

Key words and phrases. Lévy processes, European options, Fourier transform.

Carr and Madan (1999). If prices of European options at less than one or two hundred points are needed, then parabolic iFT remains faster than the standard iFT with FFT.

It should be noted that parabolic iFT can be applied to price at-the-money (ATM) and out-the-money (OTM) European options. In-the-money (ITM) European options can be priced from OTM-prices by using the put-call parity.

1. Lévy processes: a short reminder

A Lévy process is a process with stationary independent increments (for details, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. We denote it by $F(dy)$. A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case). The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1})F(dy), \tag{1.1}$$

where σ^2 is the variance of the Gaussian component, and $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty. \tag{1.2}$$

If the jump component is a process of finite variation, equivalently, if

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, |y|\}F(dy) < +\infty, \tag{1.3}$$

then (1.1) can be simplified

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y})F(dy), \tag{1.4}$$

with a different μ, and the new μ is the drift of the Gaussian component.

Assume that under a risk-neutral measure chosen by the market, the stock has the dynamics $S_t = e^{X_t}$. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\operatorname{Im}\xi \in (-1, 0)$ and continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$. Further, if the riskless rate, r, is constant, and the stock does not pay dividends, then the discounted price process must be a martingale. Equivalently, the following condition must hold

$$r + \psi(-i) = 0, \tag{1.5}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty}(1 - e^{y} + y\mathbf{1}_{|y|\le 1})F(dy). \tag{1.6}$$

Example 1. [Tempered stable Lévy processes] The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{1.7}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (1.7) is derived in Boyarchenko and Levendorskiǐ (2000, 2002) from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$F_\mp(dy) = ce^{\lambda_\pm y}|y|^{-\nu-1}dy; \tag{1.8}$$

in the first two papers, the name extended Koponen family was used. Later, the same class of processes was used in Carr et al. (2002) under the name CGMY-model. The following relations between parameters of KoBoL model and C, G, M, Y parameters of CGMY-model is valid:

$$C = c,\ Y = \nu,\ G = \lambda_+,\ M = -\lambda_-.$$

Example 2. [Normal Inverse Gaussian processes] A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see Barndorff-Nielsen (1998))

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{1.9}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example 3. [Variance Gamma processes] The Lévy density of a Variance Gamma process is of the form (1.8) with $\nu = 0$, and the characteristic exponent is given by (see Madan et al. (1998))

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{1.10}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

2. Conformal parabolic iFT method for pricing European options under Lévy processes, [5]

Let T and $G(x)$ be the maturity and the payoff function of European claim, and the stock price $S_t = e^{X_t}$ is an exponential Lévy process under a chosen risk-neutral measure. The riskless rate r is assumed a constant. Then the no-arbitrage price of the European option at time $t < T$ and $X_t = x$ with payoff $G(X_T)$ is given by

$$V(x,t) = V(T, G; t, x) = E^{t,x}\left[e^{-r\tau}G(X_T)\right], \tag{2.1}$$

where $\tau = T - t$.

The standard Fourier transform technique gives the following pricing formula:

$$V(x,t) = (2\pi)^{-1}\int_{\operatorname{Im}\xi=\omega} \exp[ix\xi - \tau(r + \psi(\xi))]\hat{G}(\xi)d\xi, \tag{2.2}$$

where $\hat{G}$ is the Fourier transform of a function G:

$$\hat{G}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}G(x)dx.$$

Now, we briefly describe the algorithm of parabolic iFT method developed in [5] for KoBoL (CGMY) model (see Example 1). We use the notation $x = \ln(S/K)$, $x' = x + \mu\tau$, where K is the strike price, and μ is the coefficient in the linear term $-i\mu\xi$ of the characteristic exponent $\psi(\xi)$ (see (1.7)). Set $\phi(\xi) = \psi(\xi) + i\mu\xi$. We consider OTM and at-the-money (ATM) puts $x' \geq 0$ and OTM and ATM calls $x' \leq 0$. We consider only the standard puts and calls.

2.1. **Deformation and change of variable for put options.** Let $x\prime \geq 0$. For $\alpha \in [1;2]$, consider the conformal map χ_α defined on the half plane $\mathrm{Im}\,\eta < \lambda_+$ by

$$\chi_\alpha(\eta) = i\lambda_+ - i\lambda_+^{1-\alpha}(\lambda_+ + i\eta)^\alpha. \tag{2.3}$$

Fix $\omega \in (0, \lambda_+)$ and let L be the image of line $\mathrm{Im}\,\xi = \omega$ under the mapping χ_α. Consider the deformation of line $\mathrm{Im}\,\xi = \omega$ in the initial pricing formula (2.2) for the put into the contour L:

$$V_{put}(x,t) = -\frac{Ke^{-r\tau}}{2\pi}\int_L \frac{\exp[ix'\xi - \tau\phi(\xi)]}{\xi(\xi+i)}d\xi. \tag{2.4}$$

In (2.4), change the variable $\xi = \chi_\alpha(\eta + i\omega)$, where $\eta \in \mathbf{R}$:

$$V_{put}(x,t) = -\frac{Ke^{-r\tau}}{2\pi}\int_{\mathbf{R}} \frac{\exp[ix'\chi_\alpha(\eta+i\omega) - \tau\phi(\chi_\alpha(\eta+i\omega))]}{\chi_\alpha(\eta+i\omega)(\chi_\alpha(\eta+i\omega)+i)}\chi'_\alpha(\eta+i\omega)d\eta. \tag{2.5}$$

Further simplification as in Carr and Madan (1999) leads to the final formula

$$V_{put}(x,t) = -\frac{Ke^{-r\tau}}{\pi}\,\mathrm{Re}\int_0^{+\infty} \frac{\exp[ix'\chi_\alpha(\eta+i\omega) - \tau\phi(\chi_\alpha(\eta+i\omega))]}{\chi_\alpha(\eta+i\omega)(\chi_\alpha(\eta+i\omega)+i)}\chi'_\alpha(\eta+i\omega)d\eta. \tag{2.6}$$

According recommendations in [5], we set $\alpha = \min\{2, 1 + 1/\nu\}$ in (2.3) for the case of KoBoL (CGMY) model. For typical KoBoL parameters values, the choice $\omega = \lambda_+/2$ is typically optimal.

An efficient numerical realization of (2.6) starts with a discretization of the integral using the infinite trapezoid rule, denote the discretization step by ζ. Then we truncate the sum from the up; we denote by N the number of terms in the truncated sum. Thus, there are two sources of the errors: discretization and truncation.

Assuming that the error tolerance $\epsilon > 0$ is small, we set

$$\zeta = 1.5\pi d/\ln(1/\epsilon_1),$$

where $d = \lambda_+/2$ and $\epsilon_1 = 2\pi\epsilon e^{r\tau}/K$.

More detailed recommendations about the most optimal choice of the algorithm parameters can be found in [5].

3. Implementation to the Premia

We implemented parabolic iFT-method for two types of European options (call and put) under CGMY model (see Example 1). One can use the routine for the other types of Lévy processes by replacing the corresponding part with the computation of the characteristic exponent provided that a justification of the contour transformation is done.

Note that in the program implemented to Premia one can manage by the parameter N of the algorithm. To improve the truncation error one should increase N.

References

[1] Barndorff-Nielsen, O. E., 1998, "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2, 41–68.

[2] Boyarchenko, S. I., and S. Z. Levendorskiǐ, 2000, "Option pricing for truncated Lévy processes", *International Journal of Theoretical and Applied Finance*, 3, 549–552.

[3] Boyarchenko, S. I., and S. Z. Levendorskiǐ, 2002, *Non-Gaussian Merton-Black-Scholes theory*, World Scientific, New Jercey, London, Singapore, Hong Kong.

[4] Carr, P., and Madan, D. B., 1999, "Option valuation using the fast Fourier transform", *The Journal of Computational Finance* 2(4), 61–73.

[5] Boyarchenko, S. I., and S. Z. Levendorskiǐ, 2014, Efficient variations of the Fourier transform in applications to option pricing, *Journal of Computational Finance*, 18(2), 57–90. 3, 4

[6] Carr, P., H. Geman, D.B. Madan, and M. Yor, 2002, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75, 305-332.

[7] Cont, R., and P.Tankov, 2004, *Financial modelling with jump processes*, Chapman & Hall/CRC Press.

[8] S.G. Kou, 2002, "A jump-diffusion model for option pricing", *Management Science*, 48, 1086-1101

[9] Madan, D.B., Carr, P., and E. C. Chang, 1998, "The variance Gamma process and option pricing", *European Finance Review*, 2, 79–105.

[10] Sato, K., 1999, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge.