The Heston Stochastic-Local Volatility Model: Efficient Particle Method

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1 Introduction

The following method proposed by [1] deals with an efficient particle method for computing the price of a stochastic volatility model.

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2 Theoretical framework

We consider the following local stochastic volatility model

$$dS_t = rS_t dt + S_t \sigma(t, S_t) a_t dW_t$$

where a is a stochastic process. From Dupire, this model is exactly calibrated to market smiles if we have

$$\sigma_{LV}^2(t,s) = \sigma^2(t,s)\mathbb{E}[a_t^2 | S_t = s].$$

where σ_{LV} represents the Dupire local volatility. The volatility function depends on the joint PDF $p(t,s,a)$ of (s_t, a_t)

$$\sigma(t,s) = \sigma_{LV}(t,s)\sqrt{\frac{\int p(t,s,a')da'}{\int (a')^2 p(t,s,a')da'}} \tag{1}$$

3 Particle method

Let us introduce this method by considering the following McKean SDE

$$dX_t = b(t, X_t, \mathbb{P}_t)dt + \sigma(t, X_t, \mathbb{P}_t)dW_t,\ \mathbb{P}_t = Law(X_t)$$

The simulation of such an equation consists of replacing the law $\mathbb{P}_t$ by the empirical distribution

$$\mathbb{P}_t^M := \frac{1}{M}\sum_{i=1}^{M} \delta_{X_t^{i,M}}$$

where $(X_t^{i,M})_{1\leq i\leq M}$ are the solutions to the $\mathbb{R}^M$ dimensional linear SDE

$$\begin{aligned}dX_t^{i,M} &= b(t,X_t^{i,M},\mathbb{P}_t^M)dt + \sigma(t,X_t^{i,M},\mathbb{P}_t^M)dW_t^i,\\ Law(X_0^{i,M}) &= \mathbb{P}_0\end{aligned}$$

where $(W^i)_{1\leq i\leq M}$ are M independent Brownian motions and $\mathbb{P}_t^M$ is a random measure on $\mathbb{R}$. We get

$$dX_t^{i,M} = \frac{1}{M}\sum_{j=1}^M b(t,X_t^{i,M},X_t^{j,M})dt + \frac{1}{M}\sum_{j=1}^M \sigma(t,X_t^{i,M},X_t^{j,M})dW_t^i.$$

From (Snitzman, 1991), we know that if at $t=0$ $X_0^{i,M}$ are independent r.v., then as M tends to infinity, for any fixed $t>0$, $(X_t^{i,M})_{1\leq i\leq M}$ are asymptotically independent and their empirical measure $\mathbb{P}_t^M$ converges in distribution to the true measure $\mathbb{P}_t$.

4 Numerical algorithm

4.1 Particle method to compute $\mathbb{E}[a_t^2|S_t=s]$

We approximate $\mathbb{E}[a_t^2|S_t=s]$ by $\mathbb{E}^{\mathbb{P}_t^M}[a_t^2|S_t=s]$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_t^M}[a_t^2|S_t=s] &= \frac{\int (a')^2 p_M(t,s,a')}{\int p_M(t,s,a')da'}\\ &= \frac{\sum_{i=1}^M (a_t^{i,M})^2\delta(S_t^{i,M}-s)}{\sum_{i=1}^M \delta(S_t^{i,M}-s)}\end{aligned}$$

We use a regularizing kernel $\delta_{t,M}(\cdot)$ that converges to the Dirac function as M tends to infinity. It is natural to choose

$$\delta_{t,M}(x) = \frac{1}{h_{t,M}}K\left(\frac{x}{h_{t,M}}\right)$$

where K is a fixed, symmetric kernel with a bandwitdh $h_{t,M}$ tending to 0 as M tends to infinity. For example we can choose $K(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $K(x)=\frac{15}{16}(1-x^2)^2\mathbb{1}_{\{|x|\leq 1\}}$ or an indicator function $K(x)=\mathbb{1}_{\{|x|\leq 1\}}$. Using this method, we approximate $\sigma(t,s)$ defined in (1) by

$$\sigma_M(t,s) = \sigma_{LV}(t,s)\sqrt{\frac{\sum_{i=1}^M \delta(S_t^{i,M}-s)}{\sum_{i=1}^M (a_t^{i,M})^2\delta(S_t^{i,M}-s)}}. \tag{2}$$

4.2 Simulation scheme

We consider the following dynamics of the Heston SLV model expressed in terms of independent Brownian motions:

$$\begin{aligned}dS_t &= rS_t\,dt + \sigma(t,S_t)S_t\sqrt{a_t}\left(\rho_{x,v}d\tilde{W}_t^v + \sqrt{1-\rho_{x,v}^2}d\tilde{W}_t^x\right),\\ da_t &= \kappa(\overline{v}-a_t)dt + \gamma\sqrt{a_t}d\tilde{W}_t^v\end{aligned}$$

where $\tilde{W}^x$ and $\tilde{W}^v$ are independent Brownian motions. We discretize $[0, T]$ on a regular grid of size N, with step size $\Delta = \frac{T}{N}$. An Euler scheme gives the following approximation

$$\begin{aligned} s_{i+1,j} =& s_{i,j} + r s_{i,j}\Delta + \sigma_M(t_i, s_{i,j}) s_{i,j}\sqrt{(a_t)_+}\sqrt{\Delta} Z_x, \; s_{0,j} = S_0, \\ a_{i+1,j} =& a_{i,j} + a_v(t_i, a_{i,j})\Delta + a_v(t_i, a_{i,j})\sqrt{\Delta} Z_v, \; a_{0,j} = a_0, \end{aligned}$$

for $j = 1, \cdots, M$ and $i = 0, \cdots, N$ and $\sigma_M(t, s)$ is given by (2). $Z_x = Z_1$ and $Z_v = \rho_{x,v} Z_1 + \sqrt{(1 - \rho_{x,v}^2)} Z_2$ where Z_1 and Z_2 are two independent standard normal variables.

5 Numerical experiments

We test the algorithm on a Call option with payoff $e^{-rT}(S_T - K)_+$ with the following parameters, for different maturities and strikes :

r	s_0	a_0	κ	γ	$\rho_{x,v}$
0	1	0.0945	1.05	0.95	-0.315

We use $N = 100$ times steps and $M = 5000$ particles and the local volatility σ_{LV} is given by

$$\sigma_{LV}(t, x) = 0.01 + 0.1e^{-x/s_0} + 0.01t.$$

K T	2	8
0.7	0.300711	0.310436
0.9	0.107903	0.152974
1.1	0.005151	0.053920
1.3	0.000345	0.021565
1.5	0.000227	0.012228

References

[1] J. Guyon, P.H. Labordère. Being particular about calibration. Risk, January 2012.

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