APPLICATION OF THE IMPROVED FAST GAUSS TRANSFORM TO EUROPEAN OPTION PRICING IN THE MERTON MODEL

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Abstract. We describe a numerical method for pricing European options under a wide class of jump-diffusion processes. The method uses the Crank-Nicolson scheme to solve the partial integrodifferential equation for the option value and the improved Fast Gauss Transform for an efficient computing the nonlocal integral term. The method implemented into Premia 18 is based on the one developed in T. Sakuma and Y. Yamada (2014).

1. Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

By now, there exist several large groups of relatively universal numerical methods for pricing options under exponential Lévy processes. The number of publications is huge, and, therefore, an exhaustive list is virtually impossible. We concentrate on the one-dimensional case.

Existing numerical methods in literature can be categorized into three groups: Monte Carlo simulation, partial-(integro) differential equation (PIDE) methods, and backward induction methods.

Monte Carlo methods are typically time consuming. The backward induction methods are based on the fact that the risk-neutral valuation formula for the European option can be seen as a convolution of the payoff function with the transition density. The key idea is to set up a time lattice and view the option as of European type between two adjacent dates. Hence, the backward induction method requires the transition density to be known in closed-form, which is the case in e.g. the Black-Scholes model and Merton's jump-diffusion model. The approximation proposed by Geske and Johnson (1984) used the discretization of the time parameter and the backward induction for pricing American

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options in the GBM model. The method was extended in Boyarchenko and Levendorskiĭ (2002) for some Lévy models, and its applications can be founded e.g. in Kudryavtsev and Levendorskiĭ (2006) and Levendorskiĭ et al. (2006). If there is no an explicit formula for the probability density, it can be recovered by inverting the characteristic function, so the method can be used for a wide range of Lévy models.

Since convolutions can be handled very efficiently by means of the Fast Fourier Transform (FFT), an overall complexity of the method is $O(NM \ln M)$, where N and M are the numbers of points on the grid in time and space, respectively. The FFT-based backward induction method was applied in Jackson et al. (2008), see also Lord et al. (2008).

Finite difference schemes are the standard tools to solve partial integrodifferential equations for option prices. In the latter case, one need to compute the nonlocal integral terms related to the corresponding PIDE numerically. In a finite difference scheme, derivatives are replaced by finite differences. In the presence of jumps, one needs to discretize the integral term as well. Finite difference schemes were applied to pricing European and barrier options in Cont and Voltchkova (2005), and to pricing American options in Carr and Hirsa (2003), Hirsa and Madan (2003) and Levendorskiĭ et al. (2005). Wang et al (2007) calculate prices of American options using the penalty method and a finite difference scheme. Construction of any finite difference scheme involves discretization in space and time, truncation of large jumps and approximation of small jumps. Truncation of large jumps is necessary because an infinite sum cannot be calculated; approximation of small jumps is needed when Lévy measure diverges at zero. Thus, in the case of jump-diffusion models the latter step is not needed.

The result is a linear system that needs to be solved at each time step, starting from payoff function. In the general case, solution of the system on each time step by a linear solver requires $O(M^2)$ operations (M is a number of space points), which is too time consuming. In Cont and Voltchkova (2005) the integral part represented as a discrete integral kernel sum is computed using the solution from the previous time step, while the differential term is treated implicitly. This leads to the explicit-implicit scheme, with tridiagonal system which can be solved in $O(M \ln M)$ operations provided that the discretized integral sum is computed by means of the Fast Fourier Transform. Levendorskiĭ et al. (2005) use the implicit scheme and the iteration method at each time step.

In the fields of machine learning and computational physics, an efficient computation of Gaussian kernel sums was originally proposed by Greengard and Rokhlin (1987) and Greengard and Strain (1991). The computational cost of these methods called the fast Gauss transform (FGT) is $O(M)$, where M is a number of space points, which is less than that of the FFT. However, in the application to computational finance indicated above, the FGT approach requires an impractically large number of grid points in order to achieve the same accuracy as the FFT method (see e.g. d'Halluin et al (2005)).

In Sakuma and Yamada (2014) the Crank-Nicolson finite difference scheme was used to solve the PIDEs under jump-diffusion models and the improved fast Gauss transform (IFGT) (see Yang et al (2005) and Raykar et al (2005)) was applied to compute the corresponding nonlocal integral terms. Numerical results in Sakuma and Yamada (2014) show that the IFGT evaluation is more efficient than FFT evaluation and can achieve the same accuracy with a practical number of grid points. We implemented the method into program platform Premia for Merton's jump-diffusion model and we will refer on this method as the FD-IFGT-method. The computational complexity of the FD-IFGT-method is $O(NM)$, where N and M are the numbers of points on the grid in time and space, respectively.

2. Lévy processes: basic facts

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\le 1})\nu(dy), \tag{2.1}$$

where $\sigma^2 \ge 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2.2}$$

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = S_0 e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\operatorname{Im}\xi \in (-1, 0)$ and continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$.

If X is a jump-diffusion process with a finite Lévy measure $\nu(dy)$, then the infinitesimal generator of X, denote it L, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x))\nu(dy). \tag{2.3}$$

Further, if the riskless rate, r, is constant, and the stock pays dividends d, then the following condition (the EMM-requirement) must hold

$$r - d + \psi(-i) = 0, \tag{2.4}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - d - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty}(1 - e^y)\nu(dy). \tag{2.5}$$

Example 1. If Lévy measure of a jump diffusion process is given by normal distribution:

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}}\exp\left(-\frac{(x-\gamma)^2}{2\delta^2}\right)dx,$$

then we obtain Merton model. The parameter λ characterizes the intensity of jumps. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\left(1 - \exp\left(-\frac{\delta^2\xi^2}{2} + i\gamma\xi\right)\right), \tag{2.6}$$

where $\sigma, \delta, \lambda \geq 0$, $\mu, \gamma \in \mathbf{R}$.

Due (2.5), the drift term $\mu = r - d - \frac{\sigma^2}{2} + \lambda\left(1 - e^{\gamma + \frac{\delta^2}{2}}\right)$ is fixed by the EMM-requirement. Hence, the infinitesimal generator may be rewritten as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) - \lambda u(x) + \int_{\mathbf{R}} u(x+y)\nu(dy). \tag{2.7}$$

3. The combined finite difference and fast Gauss transform methods for European options

Let T and K be the maturity and the strike and the stock price $S_t = Ke^{X_t}$ is an exponential Lévy process under a chosen risk-neutral measure. The riskless rate r and dividend rate d are assumed constant. Then the payoff at maturity is $G(X_T)$, where $G(x) = (K - Ke^x)_+$ in the case of the put option, and the no-arbitrage price of the European option at time $t < T$ and $X_t = x$ is given by

$$V(x,t) = E^{t,x}\left[e^{-r(T-t)}G(X_T)\right]. \tag{3.1}$$

The price $V(x,t)$ with maturity T satisfies the following partial integro-differential equation

$$(\partial_t + r - L)V(x,t) = 0 \tag{3.2}$$

subject to the following terminal condition

$$V(x,T) = (K - Ke^x)_+. \tag{3.3}$$

Sakuma and Yamada (2014) suggest to solve (3.2)-(3.3) via Crank-Nicolson scheme. We approximate $V(x,t_k)$ in (3.2) by the price $v^k(x)$ of the European option at the corresponding equally spaced dates t_k, $k = 0, 1, \ldots, N$, where $t_0 = 0$, $t_N = T$. Set $\Delta\tau := T/N$ and $\Delta x := (x_{max} - x_{min})/M$, where (x_{max}, x_{min}) is a localization domain in x-space, and M is the number of space discretization points. Then we have

$$v^N(x) = (K - Ke^x)_+, \tag{3.4}$$

and for $k = N-1, N-2, \ldots,$ the price $v^k(x)$ at the space discretization points $x_i = x_{min} + i\Delta x$, $i = 1, \ldots, M$, can be found as the solution to the problem:

$$\frac{v^{k+1}(x_i) - v^k(x_i)}{\Delta\tau} = \frac{1}{2}D(v^{k+1} + v^k)(x_i) - \frac{1}{2}(r+\lambda)(v^{k+1} + v^k)(x_i) + I_k(x_i), \tag{3.5}$$

where

$$\begin{aligned} Dv(x_i) &= \frac{\sigma^2}{2}\cdot\frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{2(\Delta x)^2} + \mu\cdot\frac{v(x_{i+1}) - v(x_{i-1})}{2(\Delta x)}, \\ I_k(x_i) &= \frac{\lambda}{\delta\sqrt{2\pi}}\int_{-\infty}^{+\infty} v^k(z)\exp\left(-\frac{(z - x_i - \gamma)^2}{2\delta^2}\right)dz \end{aligned}$$

In Sakuma and Yamada (2014), the integral term $I_k(x_i)$ is approximated by the trapezoid rule:

$$I_k(x_i) \approx \frac{\lambda}{\delta\sqrt{2\pi}}\sum_{k=1}^{M} v^k(x_k)\exp\left(-\frac{(x_k - x_i - \gamma)^2}{2\delta^2}\right)w_k\Delta x, \tag{3.6}$$

where the weights $w_1 = w_M = 0.5$ and $w_k = 1$, for $k = 2, \dots, M-1$. At each time step the integral term $I_k(x_i)$ is calculated explicitly by means of the improved fast Gauss transform (IFGT) developed in Yang et al. (2005) and Raykar et al. (2005). The advanced technique for automatic parameters tuning for the IGFT can be found in Morariu et al. (2009). We briefly summarize the IFGT, which was developed for an efficient calculation the discrete Gauss transform:

$$G(y_j) = \sum_{i=1}^{M_1} q_i e^{-(y_j-x_i)^2/h^2}, \tag{3.7}$$

where q_i are weight coefficients, x_i, $i = 1, ..., M_1$ are the centers of the Gaussians ("source" points), y_j, $j = 1, ..., M_2$ are the "target" points), h is the bandwidth of the Gaussians.

In straightforward computations with M_1 "source" points, and M_2 "target" points, we need to evaluate and sum $M_1 \times M_2$ square exponentials. The Improved Fast Gauss Transform is an ϵ-exact approximation that reduces complexity from $O(M_1 M_2)$ to $O(M_1 + M_2)$. In our option pricing problem $M_1 = M_2 = M$.

To achieve the desired error bound, the points are divided into clusters by using k-center clustering (the farthest point algorithm). The sum of Gaussians considered is approximated as the following sum of monomials:

$$\begin{aligned} G(y_j) &= \sum_{i=1}^{N} q_i e^{-(y_j-x_i)^2/h^2} \\ &\approx \sum_{|y_j - c_k| \le h\rho_y} \sum_{\alpha \le p_{max}} C_\alpha^k e^{-(y_j-c_k)^2/h^2} \left(\frac{y_j - c_k}{h}\right)^\alpha, \end{aligned}$$

where c_k, are the centers of the clusters S_k, $k = 1, ..., k_{max}$, ρ_y is the maximal cluster radius, p_{max} is the maximum truncation number and

$$C_\alpha^k = \frac{2^\alpha}{\alpha!} \sum_{x_i \in S_k} q_i e^{-(x_i-c_k)^2/h^2} \left(\frac{x_i - c_k}{h}\right)^\alpha.$$

The clustering step can be performed in $O(Mk_{max})$ time using a simple algorithm developed in Gonzalez (1985).

4. Implementation to the Premia 18

We implemented FD-IFGT-method for European call and put options under the Merton model. One can use the routine for the other types of European options by replacing the payoff. The method can be applicable to the barrier option pricing with a slight modification of the finite difference scheme.

Helper functions are taken with a slight modification from the file figtree.cpp belonging to the FIGTree library (a library for fast computation of Gauss transforms in multiple dimensions, using the Improved Fast Gauss Transform and Approximate Nearest Neighbor searching). The full version of the FIGTree library can be found at http://www.umiacs.umd.edu/ morariu/figtree/

Note that in the program implemented to Premia 18 one can manage by two parameters of the algorithm: the number of space points M and the number of time steps N. To improve the results one should increase M and/or N.

The other algorithm's parameters are fixed (the error bound $\epsilon = 0.00001$, the truncation numbers $p_{max} = 35$, the upper limit on the number of clusters $k_{max} = 6$) and can be modified inside the routine's code.

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