Pricing Bermudan options via multi-level approximation methods

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1 Introduction

The following method proposed by [1] deals with the pricing of Bermudan options via multi-level approximation methods.

2 Theoretical framework

2.1 Standard approach

Let $\{X_t, 0 \leq t \leq T\}$ denote a Markov process and $0 = t_0 < t_1 < \cdots < t_J = T$ denote a finite set of exercise opportunities of the Bermudan option with payoff $g(Z)$, where $Z_j = X_{t_j}$ for all $j = 0, \cdots, J$. The price $V_j(z)$ of the Bermudan option at time t_j is given by

$$V_j^*(z) = \sup_{\tau \in \mathcal{T}_j} \mathbb{E}(g_\tau(Z_\tau)|Z_j = z), z \in \mathbb{R}^d$$

where $\mathcal{T}_j$ denotes the set of stopping times taking values in $\{j, j+1, \cdots, J\}$.
A low biased-estimate for V_0^* is

$$V_0^{n,k} = \frac{1}{n} \sum_{r=1}^{n} g_{\tau_k^{(r)}}(Z_{\tau_k^{(r)}}^{(r)})$$

where

$$\tau_k^{(r)} = \inf\{0 \leq j \leq J : g_j(Z_j^{(r)}) \geq C_{k,j}(Z_j^{(r)})\}. \tag{1}$$

$C_{k,j}(z)$ is an estimate of the continuation value

$$C_j^*(z) := \mathbb{E}(V_{j+1}^*(Z_{j+1})|Z_j = z),\ j = 0, \cdots, J-1$$

based on the set of trajectories $(Z_0^{(i)}, \cdots, Z_J^{(i)}), i = 1, \cdots, k$.

In the case of local regression method and mesh method (described later), the estimates for the continuation values are obtained via the recusrion (dynamic programming principle) :

$$C_J^*(z) = 0,$$
$$C_j^*(z) = \mathbb{E}(\max(g_{j+1}(Z_{j+1}), C_{j+1}^*(Z_{j+1}))|Z_j = z)$$

combined with Monte Carlo : at $(J-1)$th step one estimates the expectation

$$\mathbb{E}(\max(g_{j+1}(Z_{j+1}), C_{k,j+1}(Z_{j+1}))|Z_j = z)$$

via regression based on the set of paths $(Z_j^{(i)}, C_{k,j+1}(Z_{j+1}^{(i)}))$, for $i = 1, \cdots, k$, where $C_{k,j+1}(z)$ is the estimate for $C_{j+1}^*(z)$ obtained in the previous step.

2.2 Mesh method and local regression method

2.2.1 Mesh method

The continuation value C_j^* at point z is approximated via

$$C_{k,j}(z) = \frac{1}{k}\sum_{i=1}^{k}\left(\zeta_{k,j+1}(Z_{j+1}^{(i)}) \cdot w_{ij}^k(z)\right) \tag{2}$$

where $\zeta_{k,j+1} = \max\{g_{j+1}(z), C_{k,j+1}(z)\}$ and

$$w_{ij}^k(z) = \frac{p_j(z, Z_{j+1}^{(i)})}{\frac{1}{k}\sum_{l=1}^k p_j(Z_j^{(l)}, Z_{j+1}^{(i)})}$$

where $p_j(x, \cdot)$ is the conditional density of Z_{j+1} given $Z_j = x$.

2.2.2 local regression method

We still define $C_{k,j}$ by (2). $w_{ij}^k(z)$ is defined by

$$w_{ij}^k(z) = \frac{\mathbb{1}_{\{|z - Z_j^{(i)}| \le \delta_k\}}}{\frac{1}{k}\sum_{l=1}^k \mathbb{1}_{\{|z - Z_j^{(l)}| \le \delta_k\}}},$$

with $\delta_k = 100 \cdot k^{-1/(d+2)}$.

3 Multi-Level approach

Fix some natural number L and let $\mathbf{k} = (k_0, k_1, \cdots, k_L)$ and $\mathbf{n} = (n_0, n_1, \cdots, n_L)$ be two sequences of natural numbers, satisfying $k_0 < k_1 < \cdots < k_L$ and $n_0 > n_1 > \cdots > n_L$. We define

$$V_0^{\mathbf{k},\mathbf{n}} = \frac{1}{n_0}\sum_{r=1}^{n_0} g_{\tau_{k_0}^{(r)}}(Z_{\tau_{k_0}^{(r)}}^{(r)}) + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{r=1}^{n_l}\left(g_{\tau_{k_l}^{(r)}}(Z_{\tau_{k_l}^{(r)}}^{(r)}) - g_{\tau_{k_{l-1}}^{(r)}}(Z_{\tau_{k_{l-1}}^{(r)}}^{(r)})\right)$$

where $\tau_k^{(r)}$ is defined in (1). For any $l = 1, \cdots, L$, both estimates $C_{k_l,j}$ and $C_{k_{l-1},j}$ are based on one set of k_l training trajectories.

4 Numerical algorithm

4.1 Mesh method and Black-Scholes model

In the case of the Black and Scholes model, i.e

$$dX_t^i = rX_t^i dt + \sigma X_t^i dB_t^i$$

where r is the risk-free interest rate and σ the volatility, we simulate the process Z using the exact formula

$$Z_j^{(i)} = Z_{j-1}^{(i)} \exp\left((r - \frac{1}{2}\sigma^2)h + \sigma\sqrt{h}\xi_j^i\right)$$

where $\xi_j^i,\ i = 1, \cdots, k$ are i.i.d. standard normal random variables. The conditional density of Z_j given Z_{j-1} is given by

$$p_j(x, y) = \prod_{i=1}^{d} p_j(x_i, y_i), x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d)$$

where

$$p_j(x_i, y_i) = \frac{x_i}{y_i \sigma \sqrt{2\pi h}} \times \exp\left(\frac{-(\log(y_i/x_i) - (r - \frac{\sigma^2}{2})h)^2}{2\sigma^2 h}\right)$$

4.2 Mesh method

In case of the standard approach, we set

$$k = \frac{2.4}{\varepsilon},\ n = \left(\frac{2.4}{\varepsilon}\right)^2$$

for any $\varepsilon > 0$.

In case of the MLMC approach, we set $\mathbf{k} = (k_0, k_1, \cdots, k_L)$ and $\mathbf{n} = (n_0, n_1, \cdots, n_L)$ such that $k_0 = 5$ and

$$k_l = k_0 \cdot 2^l,\ n_l = \frac{1}{(\varepsilon/8)^2}\left(\sum_{i=1}^{L}\sqrt{k_i^{1/2}}\right)\sqrt{k_i^{-3/2}},\ l = 0, \cdots, L$$

and

$$L = \lceil \log_2(8/(k_0 \cdot \varepsilon)) \rceil$$

4.3 Local regression

In case of the standard approach, we set

$$k = \left(\frac{1.2}{\varepsilon}\right)^6,\ n = \left(\frac{1.2}{\varepsilon}\right)^2$$

for any $\varepsilon > 0$.

K \ T	1	3
95	27.74	50.62
100	23.57	46.36
110	15.63	39.02

K \ T	1	3
95	27.92	53.53
100	23.46	49.32
110	15.07	42.39

Table 1: Left : Mesh method - Right : Local regression method

In case of the MLMC approach, we set $\mathbf{k} = (k_0, k_1, \cdots, k_L)$ and $\mathbf{n} = (n_0, n_1, \cdots, n_L)$ such that $k_0 = 100$ and

$$k_l = k_0 \cdot 2^l,\ n_l = \frac{10}{(\varepsilon/3)^2} \left(\sum_{i=1}^{L} \sqrt{k_i^{11/12}} \right) \sqrt{k_i^{-13/12}},\ l = 0, \cdots, L$$

and

$$L = \lceil 6 \log_2 (3/(k_0^{1/6} \cdot \varepsilon)) \rceil.$$

5 Numerical experiments

We test the algorithm on a Maximum Call option with payoff $e^{-rT}(\max_{1 \le i \le d} X_T^i - K)_+$ with the following parameters, for different maturities and strikes :

r	s_0	d	σ	ρ
0.05	100	4	0.2	0.1

We use $L = 3$ levels of Monte Carlo and $J = 3$ time steps. Results are given in Table 1

References

[1] D. Belomestny, F. Dickmann and T. Nagapetyan Pricing Bermudan Options via Multilevel Approximation Methods SIAM J. Finan. Math., 6(1), 448-466.

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