APPROXIMATE PRICES OF BASKET AND ASIAN OPTIONS

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CONTENTS

INTRODUCTION

Routines *lowlinearprice* and *uplinearprice* compute lower and upper bound approximations proposed by [1], for the price of a Call or a Put option written on a linear combination of Black-Scholes asset prices. These routines also give approximations of the deltas of the claim. Routines *lower_basket, upper_basket, lower_asian* and *upper_asian* barely use these two general routines and just correctly initialize their parameters. Note that the lower bound approximation is far better than the upper bound one, so the former is the one to be prefered.

1. FRAMEWORK

More precisely, routines *lowlinearprice* and *uplinearprice* compute lower and upper bound approximations of $p = \mathbb{E}^{+}$, where :

$$X = \sum_{i=0}^{n} \varepsilon_i x_i \; e^{G_i\sqrt{T} - \mathrm{Var}(G_i)T/2}, \tag{1}$$

with $(G_i)_{0\leqslant i\leqslant n}$ a centered Gaussian vector, of covariance matrix Σ,
$\varepsilon_i = \pm 1$ and $x_i > 0$.

One may assume that $\forall i \neq j$, $G_i \neq G_j$; otherwise, one can group the terms with the same Gaussian random variable in the summation above. If the ε_i's are all equal to $+1$ then $\mathbb{E}(X^+) = \mathbb{E}(X)$ and if they are all equal to -1 then $\mathbb{E}(X^+) = 0$.

We therefore assume from now on that all the ε_i's are not equal.

For a good choice of the parameters ε, x, and G, p is the price of a basket or a discrete time average Asian option.

1.1. **Basket options.** In the case of a basket option characterized by the weigths w_i applied to the different assets whose volatilities are stored in vector σ and prices at T are given by $S_i(0)e^{G_i\sqrt{T}-\mathrm{Var}(G_i)T/2-(r-q_i)T}$, the notation means : $\varepsilon_i = \mathrm{sgn}(w_i)$, $x_i = |w_i|S_i(0)e^{-q_iT}$, q_i being the dividend on stock i. The strike K is included in the notation as stock 0 : $\varepsilon_0 = -1$, $x_0 = K\ e^{-rT}$, r being the rate of interest, and $\sigma_0 = 0$.

The covariance matrix of vector $(G_i)_{0\leqslant i\leqslant n}$ then is : $\Sigma_{ij} = \sigma_i\sigma_j C_{ij}$, where C_{ij} is the correlation between stocks i and j.

The price p then is given by :

$$p = e^{-rT}\mathbb{E}\left[\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^+\right].$$

1.2. **Asian options.** As for discrete-time average Asian options over n equally spaced dates, the notation will amount to the following :

$\varepsilon_0 = -1$, $x_0 = K\ e^{-rT}$, and $\varepsilon_i = 1$, $x_i = \frac{1}{n}S(0)\ e^{(r-q)\frac{iT}{n}-rT}$, q is the dividend yield on the stock. σ being the volatility of the stock, the covariance matrix of vector (G_i) is given by $\Sigma_{ij} = \frac{\min(i,j)}{n}\sigma^2$.

2. Computing the price

With the previous notation, price p is given by $p = \mathbb{E}(X^+)$.

Lower and upper bounds derive from the following observation :

$$\sup_{0\leqslant Y\leqslant 1}\mathbb{E}(XY) = \mathbb{E}(X^+) = \inf_{\substack{X=Z_1-Z_2\\ Z_1\geqslant 0\\ Z_2\geqslant 0}}\mathbb{E}(Z_1) \tag{2}$$

where X, Y, Z_1 and Z_2 are random variables.

Indeed, for $0 \leqslant Y \leqslant 1$, $\mathbb{E}(XY) = \mathbb{E}(X^+Y) - \mathbb{E}(X^-Y) \leqslant \mathbb{E}(X^+)$. And, for $Y = \mathbf{1}_{\{X\geqslant 0\}}$ the supremum is attained.

Moreover, if $X = Z_1 - Z_2$ with Z_1 and Z_2 positive, $Z_1 \geqslant X^+$, leading to $\mathbb{E}(Z_1) \geqslant \mathbb{E}(X^+)$. And, for $Z_1 = X^+$ $(X = X^+ - X^-)$, the infimum is attained.

3. Lower bound

3.1. **Closed formula for the price.** A closed formula is obtained for the lower bound by restricting the supremum in (2) over $\{Y,\ \exists\ u \in \mathbb{R}^{n+1}$ and $d \in \mathbb{R},\ Y = \mathbf{1}_{\{u\cdot G\leqslant d\}}\}$. Letting

$$p_* = \sup_{u,d}\mathbb{E}(X\mathbf{1}_{u\cdot G\leqslant d}),$$

and rewriting $\sigma_i = \sqrt{\mathrm{Var}(G_i)} = \sqrt{\Sigma_{ii}}$, one gets the computationally efficient formula :

$$p_* = \sup_{d\in\mathbb{R}}\sup_{\|v\|=1}\sum_{i=0}^{n}\varepsilon_i x_i\Phi(d + \sigma_i(\sqrt{C}v)_i\sqrt{T}),$$

where

- $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}$ denotes the cumulative distribution function of the Normal law;
- C is the correlation matrix ($C_{ij} = \frac{\Sigma_{ij}}{\sigma_i\sigma_j}$);
- $\sqrt{C}$ is such as $\sqrt{C}\sqrt{C}^T = C$.

PROOF :
By conditioning and linearity,

$$p_* = \sup_{d\in\mathbb{R}} \sup_{u\in\mathbb{R}^{n+1}} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{E}\left(\mathbb{E}(e^{G_i\sqrt{T}-\mathrm{Var}(G_i)T/2}|u\cdot G)\mathbf{1}_{\{u\cdot G\leqslant d\}}\right).$$

Since $(G_i, u\cdot G)$ forms a centered Gaussian vector, $\mathbb{E}(G_i|u\cdot G) = \frac{\mathrm{Cov}(G_i,u\cdot G)}{\mathrm{Var}(u\cdot G)}u\cdot G$, and $\mathbb{E}(e^{G_i\sqrt{T}-\mathrm{Var}(G_i)T/2}|u\cdot G) = e^{\mathbb{E}(G_i|u\cdot G)\sqrt{T}-\mathrm{Var}(\mathbb{E}(G_i|u\cdot G))T/2}$

Since $\mathrm{Var}(u\cdot G) = u\cdot\Sigma u$,

$$\begin{aligned}
p_* &= \sup_{u,d}\sum_{i=0}^{n}\varepsilon_i x_i\mathbb{E}\left(\exp\left(\frac{\mathrm{Cov}(G_i,u\cdot G)}{u\cdot\Sigma u}u\cdot G\sqrt{T} - \frac{\mathrm{Cov}(G_i,u\cdot G)^2}{u\cdot\Sigma u}T/2\right)\mathbf{1}_{\{u\cdot G\leqslant d\}}\right)\\
&= \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma u=1}\sum_{i=0}^{n}\varepsilon_i x_i\mathbb{E}(e^{\mathrm{Cov}(G_i,u\cdot G)u\cdot G\sqrt{T}-\mathrm{Cov}(G_i,u\cdot G)^2T/2}\mathbf{1}_{\{u\cdot G\leqslant d\}})\\
&= \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma u=1}\sum_{i=0}^{n}\varepsilon_i x_i\mathbb{E}(e^{(\Sigma u)_i u\cdot G\sqrt{T}-(\Sigma u)_i^2T/2}\mathbf{1}_{\{u\cdot G\leqslant d\}})\\
&= \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma u=1}\sum_{i=0}^{n}\varepsilon_i x_i\Phi(d+(\Sigma u)_i\sqrt{T})
\end{aligned}$$

Then defining D as the diagonal matrix with diagonal coefficients σ_i, $\Sigma = D\sqrt{C}\sqrt{C}^T D$, so that $u\cdot\Sigma u = u\cdot D\sqrt{C}\sqrt{C}^T Du$
$= \|\sqrt{C}^T Du\|^2 = 1$, taking $v = \sqrt{C}^T Du$ leads to $(\Sigma u)_i = (D\sqrt{C}v)_i = \sigma_i(\sqrt{C}v)_i$, and finally :

$$p_* = \sup_{d\in\mathbb{R}}\sup_{\|v\|=1}\sum_{i=0}^{n}\varepsilon_i x_i\Phi(d+\sigma_i(\sqrt{C}v)_i\sqrt{T}),$$

3.2. **Implementation.** The goal of routine lowlinearprice is therefore to compute the maximum of the function $(v,d)\mapsto\sum_{i=0}^{n}\varepsilon_i x_i\Phi(d+\sigma_i(\sqrt{C}v)_i\sqrt{T})$, under the constraint $\|v\| = 1$. Rather than optimizing under this constraint, routine lowlinearprice computes the unconstrained maximum of the function $F(v,d) = \sum_{i=0}^{n}\varepsilon_i x_i\Phi(d+\sigma_i\frac{(\sqrt{C}v)_i}{\|v\|}\sqrt{T})$.

The lower bound approximation of the price will therefore be :

$$p_* = \sum_{i=0}^{n}\varepsilon_i x_i\Phi\left(d^*+\sigma_i\frac{(\sqrt{C}v^*)_i}{\|v^*\|}\sqrt{T}\right), \tag{3}$$

where d^* and v^* are the solution of the unconstrained problem.

Optimization in lowlinearprice routine uses a simple conjugate gradient method. First-order derivatives must therefore be known. One can check that :

$$\frac{\partial F}{\partial v_j} = \sum_{i=0}^{n}\frac{\varepsilon_i x_i\sigma_i\sqrt{T}}{\|v\|}\times\varphi\left(d+\frac{\sigma_i(\sqrt{C}v)_i\sqrt{T}}{\|v\|}\right)\times\left(\sqrt{C}_{ij}-\frac{v_j}{\|v\|^2}(\sqrt{C}v)_i\right)$$

$$\frac{\partial F}{\partial d} = \sum_{i=0}^{n}\varepsilon_i x_i\varphi\left(d+\frac{\sigma_i(\sqrt{C}v)_i\sqrt{T}}{\|v\|}\right),$$

with

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

The matrix $\sqrt{C}$ which is a parameter of this problem, is computed by Cholesky decomposition, and Φ is obtained thanks to the incomplete Gamma function, itself computed as a series. The other parameters ε, x, σ, T and dimension n are known (cf. 1), so there is no need for further computation to implement the algorithm.

Finally note that the correlation between two distinct stocks is not necessarily constant in routines *lowlinearprice* and *uplinearprice*, but it is supposed so in routines that specifically price basket or Asian options.

3.3. **Computing the deltas.** The point (d^*, v^*) where the function F reaches its maximum depends on the x_i's. But because of the Euler equations of optimality for d^* and v^*, one simply has :

$$\frac{\partial p_*}{\partial x_i} = \varepsilon_i \Phi\left(d^* + \sigma_i \frac{(\sqrt{C}v^*)_i\sqrt{T}}{\|v\|}\right)$$

3.3.1. *Basket options.* In the case of basket options, $x_i = |w_i| S_i(0) e^{-q_i T}$ for $i > 0$, thus leading to :

$$\delta_i = \frac{\partial p_*}{\partial S_i(0)} = \frac{\partial p_*}{\partial x_i} \times \frac{x_i}{S_i(0)} = \varepsilon_i \Phi\left(d^* + \sigma_i \frac{(\sqrt{C}v^*)_i\sqrt{T}}{\|v\|}\right) \times \frac{x_i}{S_i(0)}.$$

3.3.2. *Asian options.* In this case, $x_i = \frac{1}{n} S(0)\ e^{(r-q)\frac{iT}{n} - rT}$, and

$$\frac{\partial p_*}{\partial S(0)} = \sum_{i=1}^{n} \left(\frac{\partial p_*}{\partial x_i} \times \frac{x_i}{S(0)}\right) = \sum_{i=1}^{n}\left[\varepsilon_i \Phi\left(d^* + \sigma_i \frac{(\sqrt{C}v^*)_i\sqrt{T}}{\|v\|}\right) \times \frac{\partial x_i}{\partial S(0)}\right],$$

4. Upper bound

4.1. **Additional definitions and computation.** For $0 \leqslant i, k \leqslant n$, let $\sigma_i^k = \sqrt{\Sigma_{ii} - 2\Sigma_{ik} + \Sigma_{kk}} = \sqrt{\mathrm{Var}(G_i - G_k)}$. $\sigma_i^k = 0$ only for $i = k$.

Then choosing $(\lambda_i^k)_{i \neq k}$ such as $\sum_{i \neq k} \lambda_i^k = -\varepsilon_k$, and $\lambda_i^k \varepsilon_i > 0$ for all $i \neq k$ (it is possible because all the ε_i do not have the same sign), X in (1) can be rewritten as :

$$X = \sum_{i \neq k} \left(\varepsilon_i x_i\ e^{G_i\sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k x_k\ e^{G_k\sqrt{T} - \mathrm{Var}(G_k)T/2}\right) \tag{4}$$

for every $k = 0, \ldots, n$.

4.2. **Closed formula.** A closed formula is obtained for the upper bound by restricting the infimum in (2) over $\{\sum_{i \neq k}(\varepsilon_i x_i\ e^{G_i\sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k x_k\ e^{G_k\sqrt{T} - \mathrm{Var}(G_k)T/2})^+\}$, with the same notations as in (4).

Letting

$$p^* = \min_{0 \leqslant k \leqslant n} \inf_{\sum_{i \neq k} \lambda_i^k = -\varepsilon_k} \mathbb{E}\left[\sum_{i \neq k}\left(\varepsilon_i x_i e^{G_i\sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k \tilde{x}_k e^{G_k\sqrt{T} - \mathrm{Var}(G_k)T/2}\right)^+\right],$$

the efficient computational formula is :

$$p^* = \min_{0 \leqslant k \leqslant n}\left(\sum_{i=0}^{n} \varepsilon_i x_i \Phi(d^k + \varepsilon_i \sigma_i^k \sqrt{T})\right),$$

with d^k being the one solution of

$$\sum_{i=0}^{n} \varepsilon_i x_i \varphi(d^k + \varepsilon_i \sigma_i^k \sqrt{T}) = 0$$

Proof :

Price is given by :

$$p^* = \min_{0 \leqslant k \leqslant n} \inf_{\sum_{i \neq k} \lambda_i^k = -\varepsilon_k} \mathbb{E}\left[\sum_{i \neq k}\left(\varepsilon_i x_i e^{G_i\sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k x_k e^{G_k\sqrt{T} - \mathrm{Var}(G_k)T/2}\right)^+\right],$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(\varepsilon_i x_i e^{G_i\sqrt{T}-\mathrm{Var}(G_i)T/2} - \lambda_i^k x_k e^{G_k\sqrt{T}-\mathrm{Var}(G_k)T/2}\right)^+\right] \\
&= \mathbb{E}\left[e^{G_k\sqrt{T}-\mathrm{Var}(G_k)T/2}\left(\varepsilon_i x_i e^{(G_i-G_k)\sqrt{T}-(\mathrm{Var}(G_i)-\mathrm{Var}(G_k)T/2} - \lambda_i^k x_k\right)^+\right] \\
&= \mathbb{E}^k\left[\left(\varepsilon_i x_i e^{(G_i-G_k)\sqrt{T}-(\sigma_i^2-\sigma_k^2)T/2} - \lambda_i^k x_k\right)^+\right]
\end{aligned}$$

Under probability $\mathbb{P}^k$ given by $\frac{\mathrm{d}\mathbb{P}^k}{\mathrm{d}\mathbb{P}} = e^{G_k\sqrt{T}-\mathrm{Var}(G_k)T/2}$, $\frac{G_k}{\sigma_k} - \sigma_k\sqrt{T} = g_k \sim \mathcal{N}(0,1)$. Letting $g_i = \frac{G_i}{\sigma_i} - \frac{\Sigma_{ik}}{\sigma_i\sigma_k}g_k - \frac{\Sigma_{ik}}{\sigma_i}\sqrt{T}$, g_i and g_k are independent and are both normally distributed under $\mathbb{P}^k$, and :

$$(G_i - G_k)\sqrt{T} - \frac{\sigma_i^2-\sigma_k^2}{2}T = \left(\sigma_i g_i - \left(\sigma_k - \frac{\Sigma_{ik}}{\sigma_k}\right)g_k\right)\sqrt{T} - \frac{(\sigma_i^k)^2 T}{2}.$$

Now,

$$\mathrm{Var}\left(\sigma_i g_i - \left(\sigma_k - \frac{\Sigma_{ik}}{\sigma_k}\right)g_k\right) = \sigma_i^2 + \sigma_k^2 - 2\times\sigma_k\frac{\Sigma_{ik}}{\sigma_k} = (\sigma_i^k)^2.$$

Hence,

$$\begin{aligned}
&\mathbb{E}^k\left[\left(\varepsilon_i x_i e^{(G_i-G_k)\sqrt{T}-(\sigma_i^2-\sigma_k^2)T/2} - \lambda_i^k x_k\right)^+\right] \\
&= \mathbb{E}^k\left[\left(\varepsilon_i x_i e^{\sigma_i^k g\sqrt{T}-(\sigma_i^k)^2T/2} - \lambda_i^k x_k\right)^+\right],
\end{aligned}$$

with $g \sim \mathcal{N}(0,1)$, and

$$\begin{aligned}
&\varepsilon_i x_i e^{\sigma_i^k g\sqrt{T}-(\sigma_i^k)^2T/2} - \lambda_i^k x_k \geqslant 0 \\
\Leftrightarrow &\left\{\begin{array}{ll}
\varepsilon_i > 0 & \text{and } \ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) \geqslant -\sigma_i^k\sqrt{T}g + (\sigma_i^k)^2T/2 \\
 & \text{or} \\
\varepsilon_i < 0 & \text{and } \ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) \leqslant -\sigma_i^k\sqrt{T}g + (\sigma_i^k)^2T/2
\end{array}\right. \\
\Leftrightarrow &-\varepsilon_i g \leqslant \frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}.
\end{aligned}$$

This leads to :

$$\begin{aligned}
&\mathbb{E}^k\left[\left(\varepsilon_i x_i e^{(G_i-G_k)\sqrt{T}-(\sigma_i^2-\sigma_k^2)T/2} - \lambda_i^k x_k\right)^+\right] \\
=&\varepsilon_i x_i\mathbb{E}^k\left[e^{-\frac{(g-\sigma_i^k\sqrt{T})^2}{2}+\frac{g^2}{2}}\mathbf{1}_{\left\{-\varepsilon_i g\leqslant\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right)-\frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right\}}\right] - \lambda_i^k x_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right),
\end{aligned}$$

and finally :

$$\begin{aligned}
&p^* = \min_{0\leqslant k\leqslant n}\inf_{\sum_{i\neq k}\lambda_i^k=-\varepsilon_k} \\
&\sum_{i\neq k}\left[\varepsilon_i x_i\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) + \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right) - \lambda_i^k x_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right)\right]
\end{aligned}$$

Using the Lagrangian :

$$\begin{aligned}
\mathcal{L} =& \sum_{i\neq k}\left[\varepsilon_i x_i\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) + \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right) - \lambda_i^k x_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right)\right] \\
&-\mu\left(\sum_{i\neq k}\lambda_i^k + \varepsilon_k\right),
\end{aligned}$$

the first-order conditions give :

$$\frac{\partial\mathcal{L}}{\partial\lambda_i^k} = -x_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right) - \mu = 0,$$

implying that the arguments of Φ are all equal : for each $i \neq k$,

$$\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2} = d^k.$$

Consequently,

$$\lambda_i^k x_k = \varepsilon_i x_i e^{-\varepsilon_i\sigma_i^k\sqrt{T}d^k - \frac{{\sigma_i^k}^2 T}{2}},$$

this leading to :

$$\sum_{i\neq k}\varepsilon_i x_i e^{-\varepsilon_i\sigma_i^k\sqrt{T}d^k - \frac{{\sigma_i^k}^2 T}{2}} = -\varepsilon_k x_k e^{-\varepsilon_k\sigma_k^k\sqrt{T}d^k - \frac{{\sigma_k^k}^2 T}{2}},$$

because $\sigma_k^k = 0$. As a consequence,

$$\sum_{i=0}^{n}\varepsilon_i x_i e^{-\varepsilon_i\sigma_i^k\sqrt{T}d^k - {\sigma_i^k}^2 T/2} = 0 = \sum_{i=0}^{n}\varepsilon_i x_i\varphi(d^k + \varepsilon_i\sigma_i^k\sqrt{T}) \times e^{\frac{{d^k}^2}{2}}.$$

The left-hand term is a decreasing function of d^k. Since not all the ε_i have the same sign, its limits at $\pm\infty$ are $\pm\infty$, and d^k is the only solution of :

$$\sum_{i=0}^{n}\varepsilon_i x_i\varphi(d^k + \varepsilon_i\sigma_i^k\sqrt{T}) = 0.$$

Moreover,

$$\begin{aligned}
p^* &= \min_{0\leqslant k\leqslant n}\sum_{i\neq k}\left(\varepsilon_i x_i\Phi(d^k + \varepsilon_i\sigma_i^k\sqrt{T}) - \lambda_i^k x_k\Phi(d^k)\right)\\
&= \min_{0\leqslant k\leqslant n}\sum_{i\neq k}\varepsilon_i x_i\Phi(d^k + \varepsilon_i\sigma_i^k\sqrt{T}) + \varepsilon_k x_k\Phi(d^k)\\
&= \min_{0\leqslant k\leqslant n}\sum_{i=0}^{n}\varepsilon_i x_i\Phi(d^k + \varepsilon_i\sigma_i^k\sqrt{T}).
\end{aligned}$$

4.3. **Implementation.** For each k, d^k is computed by a bisection method.
The minimum in k then is computed, as well as the optimal $k = k^*$.
Upper bound approximation of the price therefore is :

$$p^* = \sum_{i=0}^{n}\varepsilon_i x_i\Phi(d^{k^*} + \varepsilon_i\sigma_i^{k^*}\sqrt{T}). \tag{5}$$

4.4. **Computing the deltas.** The same calculus as before shows that :

$$\frac{\partial p^*}{\partial x_i} = \varepsilon_i\Phi(d^{k^*} + \varepsilon_i\sigma_i^{k^*}\sqrt{T})$$

4.4.1. *Basket options.* Just as in the case of the lower bound :

$$\delta_i = \frac{\partial p^*}{\partial S_i(0)} = \varepsilon_i\Phi(d^{k^*} + \varepsilon_i\sigma_i^{k^*}\sqrt{T}) \times \frac{x_i}{S_i(0)}.$$

4.4.2. *Asian options.* Likewise :

$$\frac{\partial p^*}{\partial S(0)} = \sum_{i=1}^{n}\left(\frac{\partial p^*}{\partial x_i} \times \frac{x_i}{S(0)}\right).$$

References

[1] R. Carmona and V. Durrleman. "Generalizing the Black-Scholes formula to multivariate contingent claims". *Journal of Computational Finance, Volume 9/Number 2, Winter 2005/06.* 1