Wishart manual

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Premia 22

1 Introduction

1.1 Presentation

The goal of "Disc_Wishart.c" is to simulate the paths of the Wishart process and also to estimate its parameters. This implementation has been led following the article by Alfosni, Keabaier and Rey [2]. Let us be more specific. Let $d \in \mathbb{N}^*$ the dimension, $\mathcal{M}_d$ the set of squared real d-dimensional matrices, $\mathcal{S}_d^+$ (resp. $\mathcal{S}_d^{+,*}$) the subset of (semidefinite) positive (resp. definite positive) matrices and $\mathcal{S}_d$ the subset of symetric matrices. The Wishart processes are defined with the following Stochastic Differential Equation

$$\begin{cases} dX_t = \left[\alpha a^\top a + bX_t + X_t b^\top\right] dt + \sqrt{X}_t dW_t a + a^\top dW_t^\top \sqrt{X}_t, \quad t > 0 \\ X_0 = x \in \mathcal{S}_d^+, \end{cases} \tag{1}$$

where $\alpha \geqslant d-1$, $a \in \mathcal{M}_d$, $b \in \mathcal{M}_d$ et $(W_t)_{t\geqslant 0}$ is a matrix that belongs to $\mathcal{M}_d$ made with independent standard brownian motions. The parameters to estimate are a, α and b. The estimation of a will be done using the quadratic variation method. We will use the Maximum Likelihood Estimator for α and b. The estimator of α is defined only for $\alpha \geqslant d+1$ and we give the estimator b only if $b \in \mathcal{S}_d$. In those cases, we provide the estimation of the couple (α, b), α when $b(\in \mathcal{S}_d)$ is known, b when $\alpha(\geqslant d-1)$ is known and b when $\alpha(\geqslant d-1)$ is knwon and we known that b is diagonal.

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1.2 Content of the implementation

1.2.1 Disc_Wishart.c

The first part of this code is dedicated to the simulation of such processes. Eventually we have to simulate $(X_t)_{t\geqslant 0}$ but also $(\int_0^t X_s ds)_{t\geqslant 0}$, $(\int_0^t X_s^{-1} ds)_{t\geqslant 0}$ and $(\langle X\rangle_t)_{t\geqslant 0}$ (which are the processes that appear in the estimation of the parameters). In order to do it, we will use the exact simulation of the Wishart processes provided by Ahdida and Alfonsi [1]. The integrals, which can not be simulated exactly, will be approached on a time grid.

The other part of "Disc_Wishart.c" is devoted to the estimation of the parameters of Wishart processes. More particularly, in "Disc_Wishart.c", we estimate the parameters (a, α, b) for a Wishart process given by (1). Notice that it is possible to estimate each parameters on its own when the others are known but also to estimate all 3 parameters together. Moreover, the estimation of a is quite different from the estimation of α and b. Indeed, the parameter aa^T can be estimated exactly using the quadratic variation of X. However the estimation of α and b, together or each on its own, will be performed using the Maximum Likelihood Estimator. The method consists in estimating aa^T first and then scaling the process with the obtained value in order to estimate the parameter α and b (apart when one of them is known or together) of a Wishart process where $a = I_d$. The implementation treat only the two dimensional case for the MLEs.

1.2.2 Main.cxx

The implementation "Main.cxx" is a verification procedure for the estimation provided by "Disc_Wishart.c". In "main.cxx", we simulate a chosen number of realization of the Wishart process (for some chosen dates $t > 0$) and we estimate the parameters a, α and b for each of these realizations. More particularly we estimate a first with the total variation method. We provide the estimation of α when b is known, of b when α is known, of b when b is known a priori diagonal and α is known, and of the couple (α, b) using the Maximum Likelyhood Estimators. For the first realization (at each chosen date $t > 0$), the result is printed out. Moreover, the normalized errors (with a speed that depends on the regime, ergodic or not, and is a function of t) committed between the real value of the parameter and its estimator are printed in some histogram files. For instance "histo_b22_t100.txt" gather the normalized errors (their number is obviously equal to the number of realizations of the Wishart process) for the estimation of component $b_{2,2}$ when we estimate the couple (α, b) at date $t = 100$.

2 Implemented functions

All the implementations which are done here require the Pnl Library.

2.1 Disc_Wishart.c

Part 1 : Simulation of the Wishart process

```
int Sq_Mat(PnlMat * x)
Description : Compute the square root matrix of x and store it in x. If x is
not semi definite, it stores the null matrix in x.

double Sim_Cir_Glass(PnlMat * x)
Description : Simulation of the Cir process :
```

$$X_t = x + \int_0^t (a - kX_s)ds + \sigma\sqrt{X}_s dW_s,$$

```
using the Glasserman algorithm.
```

The next functions from Algo1 to Algo3 are directly inspired from the algorithm provided by Alfonsi and Ahdida [1]. Here, we have adopted the same segmentation of the whole algorithm into three algorithms as it is done in their article.

```
PnlMat * Algo1(const PnlMat *x, double alpha, double t, PnlRng * rng)
Description : Compute the Wishart process with parameter a(i,j)=1 if i=j=1,
a(i,j)= 0 elsewhere, b=0 and alpha starting from x (see (1)) at date t.

PnlMat * Algo2(const PnlMat *x, double alpha, double t, int n, PnlRng * rng)
Description : Compute the Wishart process with parameter a(i,j)=1 if i=j and
i<n+1, a(i,j)= 0 elsewhere, b=0 and alpha starting from x (see (1)) at date t.

PnlMat * Algo3(const PnlMat *x, double alpha, PnlMat *b, PnlMat *a, double t,
int N_disc, PnlRng * rng)
Description : Compute the Wishart process with parameter a,b and alpha starting
 from x (see (1)) at date t.
Input :
int N_disc : Number of time step for the approximation of the integral (using
trapezoidal method)
```

$$\int_0^t \exp(sb)a^T a \exp(sb^T)ds$$

In order to be able to use our Maximum Likelyhood Estimators, we introduce a variant of Algo3. In this way ,we obtain a realization of the Wishart process but also an approximation of the quantities $(\int_0^t X_s ds)_{t\geqslant 0}$, $(\int_0^t X_s^{-1} ds)_{t\geqslant 0}$ and $(\langle X\rangle_t)_{t\geqslant 0}$ (more particularly with one step of the discrete approximation) and then we update the values IntW, Int_invW and Crochet_W which contain initial values.

```
int Algo3_adapt(PnlMat * W, PnlMat *IntW, PnlMat * Int_invW,
PnlMat * Crochet_W, const PnlMat *x, double alpha, PnlMat *b, PnlMat *a,
 double t, int N_disc, PnlRng * rng)
Description : Compute the Wishart process with parameter a,b and alpha
starting from x (see (1)) at date t, store it in W, and update the values
of IntW, Int_invW and Crochet_W adding one step of size t of the
trapezoidal approximation for IntW, Int_invW and of the discrete
approximation of the quadratic variation for Crochet_W).
Input :
W_emv : Wishart process at time t.
IntW_emv : Initial value of the integral of the Wishart pocess.
Int_invW : Initial value of the integral of the inverse of the Wishart
 process.
PnlMat * Crochet_W :Initial value of the quadratic variation of the Wishart
 process.
int N_disc : Sale as Algo3.
```

Part 2 : Estimation of the parameters of the Wishart process

```
double Trace(PnlMat *x)
Description : Compute the trace of the matrix x.

double Det(PnlMat *x)
Description : Compute the determinant of the matrix x.

PnlMat * Lyap_modif(PnlMat* X, double a, PnlMat * Y)
Description : Compute the Lyapunov modified function defined by
```

$$\text{For } X \in \mathcal{S}_d, a \in \mathbb{R}, \quad \mathcal{L}_{X,a} : \begin{array}{l} \mathcal{S}_d \to \mathcal{S}_d \\ Y \mapsto YX + XY - 2a\,\mathrm{Tr}[Y] I_d. \end{array}$$

```
PnlMat * Inv_Lyap_modif(PnlMat * X, double a, PnlMat * Z)
Description : Compute the inverse of Lyapunov modified function for the
 two-dimensional case.
Output :
```

$$\mathcal{L}_{X,a}^{-1}(Z).$$

The following functions give the estimation of the parameters. The estimation of α and b are valid when $a = I_d$. When this is not the case we use a transformation of the Wishart process (done in "main.cxx") which involves the estimator or the real value (if it is known) of a. Since the inversion of the Lyapunov functions is valid only for the two dimensional case, the estimation remains also valid only for the two dimensional case.

```
double EMV_alpha_Wishart(PnlMat * W_emv, PnlMat *  Int_invW_emv, PnlMat * x,
 PnlMat * b ,double t)
Description : Compute the MLE of alpha when b is known.
Input :
W_emv : Wishart process at time t.
Int_invW_emv : Integral of the inverse of the Wishart process between 0 and t.

PnlMat * EMV_b_Wishart(PnlMat * W_emv, PnlMat *  IntW_emv,PnlMat * x,
double alpha ,double t))
Description : Compute the MLE of b when alpha is known.
Input :
W_emv : Wishart process at time t.
IntW_emv : Integral of the Wishart process between 0 and t.

PnlMat * EMV_diag_b_Wishart(PnlMat * W_emv, PnlMat *  IntW_emv, PnlMat * x,
 double alpha, double t)
PnlMat * EMV_b_Wishart(PnlMat * W_emv, PnlMat *  IntW_emv,PnlMat * x,
double alpha ,double t))
Description : Compute the MLE of b when alpha is known. and b is known
a priori diagonal.
Input :
W_emv : Wishart process at time t.
IntW_emv : Integral of the Wishart process between 0 and t.

double EMV_global_alpha_Wishart(PnlMat * W_emv, PnlMat *  IntW_emv, double QT,
PnlMat * x,double t)
Description : Compute the estimator of alpha for the MLE of (alpha,b).
Input :
W_emv : Wishart process at time t.
IntW_emv : Integral of the Wishart process between 0 and t.
QT : Inverse of the trace of the integral of the inverse of the Wishart
process between 0 and t.

PnlMat * EMV_global_b_Wishart(PnlMat * W_emv, PnlMat *  IntW_emv, double QT,
PnlMat * x,double t)
Description : Compute the estimator of b for the MLE of (alpha,b).
Input :
W_emv : Wishart process at time t.
IntW_emv : Integral of the Wishart process between 0 and t.
QT : Inverse of the trace of the integral of the inverse of the Wishart
process between 0 and t.

PnlMat * Estim_Crochet_a_Wishart(PnlMat * W_emv, PnlMat *  IntW_emv,
PnlMat * Crochet_W,PnlMat * x, double t)
Description : Compute the estimator of a with total variation method.
```

```
Input :
W_emv : Wishart process at time t.
IntW_emv : Integral of the Wishart process between 0 and t.
PnlMat * Crochet_W :Quadratic variation of the Wishart process :
```

$$\text{Crochet\_W}_{i,j} = \langle X_{i,j}, X_{i,j} \rangle_t$$

2.2 Main.cxx

Here, we are going to describe the miscellaneous parameters involved in the simulation and estimation as well as for the outputs.

Parameters of the Wishart process

The parameter x,α,b and a for the Wishart process defined in (1) are given by the file with the same names.

Parameters of the simulation

```
long long int i0,i_max : The simulation of the Wishart process takes place for
all maturity T= i0,i0+1,...,i_max.

int M :  Number of simulation (or realization) of the Wishart process

int N_disc : Number of time step between 0 and t=i_max for the approximation
(using trapezoidal rule for Lebesgue integrals) of
```

$$(\int_0^t X_s ds)_{t \geqslant 0}, (\int_0^t X_s^{-1} ds)_{t \geqslant 0} \text{ and } (\langle X \rangle_t)_{t \geqslant 0}.$$

Outputs

As we already precised, the first realization (for each T=i0,i0+1,...,i_max), the result is printed out. More precisely we print the estimator of a using the total variation method. The estimator is exact if the Wishart process is known for any $t \geqslant 0$. Since we simulate a discretization of the Wishart process, we have an approximation of the estimator of a which depends on the parameter N_disc. We also give the MLE of α when b is known, of b when α is known, of b when b is known a priori diagonal and α is known, and of the couple (α, b). Moreover, the normalized errors committed between the real value of the parameter and its estimator are printed in some histogram files. The speed of convergence depends of the regime and can be modified in "main.cxx" in the section "Scaling". The reader may refer to [2] in order to obtain the speed of convergence for any regime. We describe the content of the histogram files (all with size M).

histo_alpha_seul_tT.txt for $i, j \in 1, 2$, $T > 0$: Normalized errors for the estimation of α when we use the MLE of α at date $t = T$ when b is known.

histo_alpha_tT.txt for $i, j \in 1, 2,\ T > 0:$ Normalized errors for the estimation of α when we use the MLE of the couple (α, b) at date $t = T$.

histo_b_seul_tT.txt for $i, j \in 1, 2,\ T > 0:$ Normalized errors for the estimation of component $b_i j$ when we use the MLE of b at date $t = T$ when α is known.
histo_bij_tT.txt for $i, j \in 1, 2,\ T > 0:$ Normalized errors for the estimation of component $b_i j$ when we use the MLE of the couple (α, b) at date $t = T$.
histo_bii_tT.txt for $i \in 1, 2,\ T > 0:$ Normalized errors for the estimation of component $b_i i$ when we use the MLE the couple b at date $t = T$ when α is known and b is a priori known to be diagonal.

histo_aij_tT.txt for $i, j \in 1, 2,\ T > 0:$ Errors for the estimation of component $a_i j$ when we use the total variation of the Wishart process alog the time grid of size N_disc, at date $t = T$.

References

[1] Ahdida, A. and Alfonsi, A. (2013). Exact and high order discretization schemes for Wishart processes and their affine extensions *Ann. Appl. Probab. 23(2013), no.3, 1025-1073* 2, 3

[2] Alfonsi, A., A.Kebaier, C.Rey (2015). Maximum Likelihood Estimation for Wishart processes Exact and high order discretization schemes for Wishart processes and their affine extensions *arXiv:1508.03323* 1, 6