Analytical Study of a Preconditioner for Fractured Porous Media

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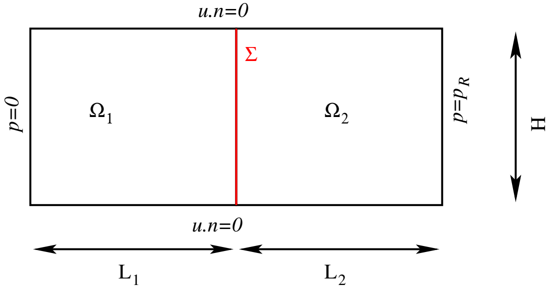
Plan

Outline

Problem setting

- Fractured porous medium : one large fracture, modeled individually.
- Fracture width is small compared to transverse dimension of porous medium : model fracture as dimension $d-1$ interface between subdomains.



- Pressure continuous through the fracture
- Flow in fracture due to velocity jump between two subdomains

Outline

Mathematical model

Flow in subdomains

$$\left.\begin{array}{r} \operatorname{div} u_i = 0 \\ u_i = -K_i \nabla p_i \end{array}\right\} \quad \text{in } \Omega_i$$

Pressure continuity

$$p_i = p_f, i = 1, 2$$

Flow in fracture

$$\left.\begin{array}{r} \operatorname{div}_f u_f = u_1.n - u_2.n \\ u_f = -d K_f \nabla p_f \end{array}\right\} \quad \text{in } \Sigma$$

d width of fracture, K_i ($i = 1, 2$), K_f permeabilities

Cf. Alboin, Jaffre, Roberts (2002)

Domain decomposition formulation

Introduce Steklov-Poincaré (DtN) operator on Ω_i :

$$S_i \bar{p} = -K_i (u_i . n)_{|\Sigma}, \text{ where :}$$

$$\left.\begin{array}{l} \text{div } u_i = 0 \\ u_i = -K_i \nabla p_i \end{array}\right. \quad \text{in } \Omega_i \qquad \begin{array}{ll} u_i . n = 0 & \text{on } \partial\Omega_i^N \\ p_i = 0 & \text{on } \partial\Omega_i^D \\ p_i = \bar{p} & \text{on } \Sigma \end{array}$$

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DD formulation

$$\left.\begin{array}{l} \text{div}_f\, u_f = S_1 p_f + S_2 p_f \\ u_f = -d K_f \nabla p_f \end{array}\right. \quad \text{in } \Sigma$$

$$u_f = 0 \qquad \text{on } \partial\Sigma.$$

Solve by conjugate gradient, solve one flow problem per subdomain at each iteration.

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Numerical example

A possible preconditioner

As Laplacian is 2nd order, and DtN operator is 1st order, preconditioning by Laplacian in the fracture should be effective (cf Amir, Jaffre, Roberts, 2005)

Influence of fracture permeability

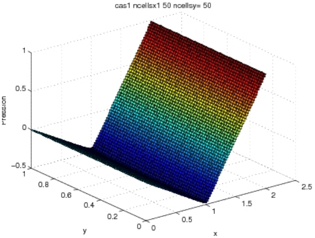
Number of iterations when K_f varies :

dK_f	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}
No prec.	68	50	50	47	36	34
Laplace prec.	7	10	16	29	38	38

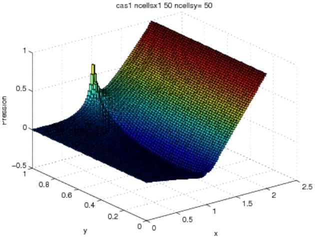
Preconditioner more effective for large values of K_f. Why ?

Influence of the fracture permeability

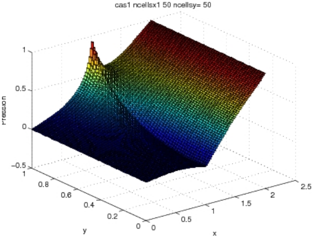
$dK_f = 1$



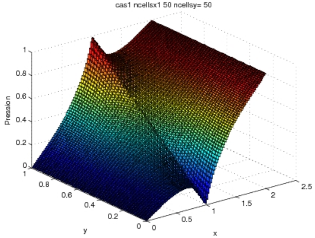
$dK_f = 10^2$



$dK_f = 10^3$



$dK_f = 10^5$

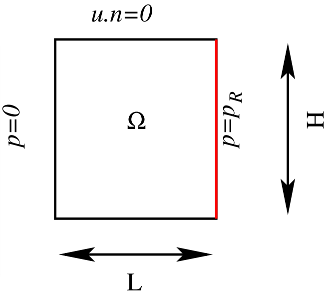


Fourier analysis of the continuous problem

Theorem (Eigenstructrure of the DtN operator)

Let $w_n(y) = \cos(\frac{n\pi y}{H})$, *then*

$$Sw_n = \frac{K}{H} \frac{n\pi}{\tanh n\pi L/H} w_n$$



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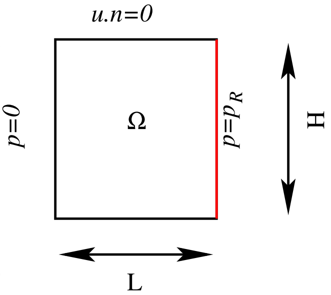


Eigenvalues of the global operator

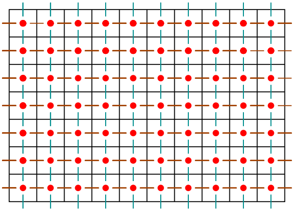
$$\lambda_n^{ex} = \frac{K_1}{H}\frac{n\pi}{\tanh n\pi L_1/H} + \frac{K_2}{H}\frac{n\pi}{\tanh n\pi L_2/H} + dK_f\frac{n^2\pi^2}{H^2}$$

Last part (fracture Laplacian) always dominant

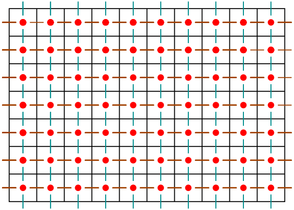
Outline

Mixed finite elements

- Lowest order RT element
- Uniform mesh → difference equations



Numerical discretization



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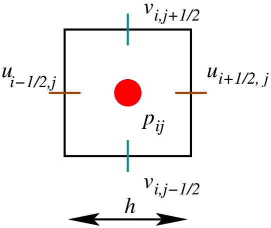

Analysis of the discrete Steklov-Poincaré operator

- S operator maps $\bar{p}_j = p_{Nj}$ to $u_{N+1/2j}$
- Determine eigenstructure of S, analysis similar to Chan (87)

Discrete equations (1)

$$\begin{cases} \frac{1}{6}\frac{h}{k}\left(u_{i-1/2,j} + 4u_{i+1/2,j} + u_{i+3/2,j}\right) & = p_{ij} - p_{i+1,j} \\ \frac{1}{6}\frac{h}{k}\left(2u_{1/2,j} + u_{3/2,j}\right) & = -p_{1j} \\ \frac{1}{6}\frac{h}{k}\left(u_{N-1/2,j} + 2u_{N+1/2,j}\right) & = p_{Nj} - p_j \end{cases}$$

$$\begin{cases} \frac{1}{6}\frac{h}{k}\left(u_{ij-1/2} + 4u_{ij+1/2} + u_{ij+3/2}\right) & = p_{ij} - p_{ij+1} \\ u_{i1/2} = u_{iM+1/2} & = 0 \end{cases}$$

$$u_{i+1/2j} - u_{i-1/2j} + u_{ij+1/2} - u_{ij-1/2} = 0$$

Discrete equations (2)

Special form for solution

$$\begin{cases} p_{i\,j} & = p_i \cos(n\pi jh/H) \\ u_{i+1/2\,j} & = u_{i+1/2} \cos(n\pi jh/H) \\ u_{i\,j+1/2} & = v_i \sin(n\pi(j+1/2)h/H) \end{cases}$$

Eliminate p_i, v_i, difference equation for $u_{i+1/2}$, with $\sigma_n = \sin^2(n\pi h/(2H))$:

$$(3-2\sigma_n)(u_{i+3/2} - 2u_{i+1/2} + u_{i-1/2}) - 2\sigma_n(u_{i+3/2} + 4u_{i+1/2} + u_{i-1/2}) = 0$$

with boundary conditions

$$\begin{aligned} (3-4\sigma_n)u_{3/2} - (3+2\sigma_n)u_{1/2} &= 0 \\ (3-4\sigma_n)u_{N-1/2} - (3+2\sigma_n)u_{N+1/2} &= 12K\sigma_n\bar{p}_n \end{aligned}$$

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The discrete Steklov Poincaré operator

Characteristic equation

$$(3 - 4\sigma_n)r^2 - 2(3 + 2\sigma_n)r + (3 - 4\sigma_n) = 0$$

Roots $0 < r^- < 1 < r^+, \quad \gamma_n = \frac{r^-}{r^+}$.

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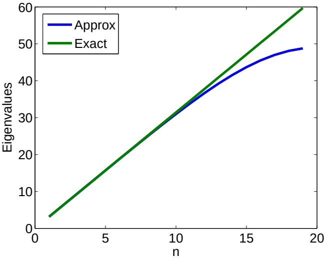
Theorem (Eigenvalues of Steklov-Poincaré operator)

$$\lambda_n = \frac{K}{h}\sqrt{\frac{12\sigma_n}{3 - \sigma_n}\frac{1 + \gamma_n^{N+1}}{1 - \gamma_n^{N+1}}},$$

Remark

For fixed n, $\lim_{h \to 0} \lambda_n = \lambda_n^{ex}$.



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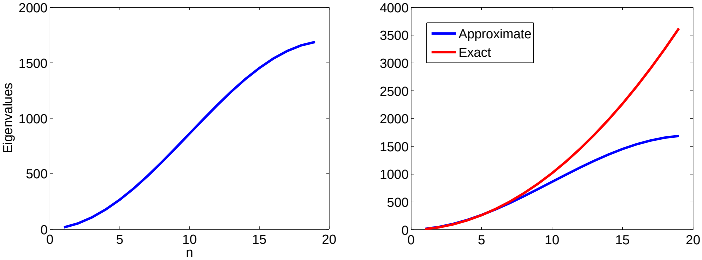
Outline

Eigenvalues of the unpreconditioned operator

Eigenvalues of the "fracture operator"

$$\lambda_n = \frac{K_1}{h}\sqrt{\frac{12\sigma_n}{3-\sigma_n}}\frac{1+\gamma_n^{N_1+1}}{1-\gamma_n^{N_1+1}} + \frac{K_2}{h}\sqrt{\frac{12\sigma_n}{3-\sigma_n}}\frac{1+\gamma_n^{N_2+1}}{1-\gamma_n^{N_2+1}} + 4dK_f\frac{\sigma_n}{h^2}.$$



Two preconditioners

$A = S_1 + S_2 + L$, S_i is Steklov-Poincaré for subdomain i, L is flow in the fracture $Lp = -dK_f \frac{d^2p}{dx^2}$.

Condition number, no preconditioning

$$\kappa^{\text{NP}} = \frac{Kh\sqrt{\frac{12\sigma_N}{3-\sigma_N}\frac{1+\gamma_N^{N+1}}{1-\gamma_N^{N+1}}} + 2dK_f\sigma_N}{Kh\sqrt{\frac{12\sigma_1}{3-\sigma_1}\frac{1+\gamma_1^{N+1}}{1-\gamma_1^{N+1}}} + 2dK_f\sigma_1}$$

- $\lim_{K_f\to\infty} \kappa^{\text{NP}} = \frac{\sigma_N}{\sigma_1}$, like for Laplace eqn.
- $\lim_{K_f\to 0} \kappa^{\text{NP}} = O(N)$, like DD

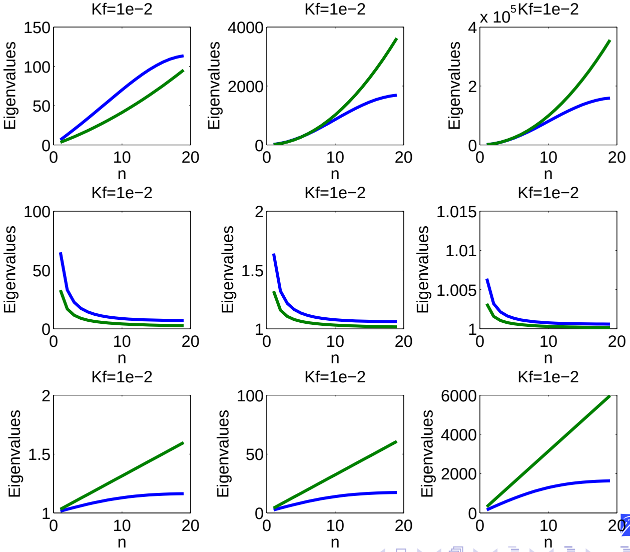
Laplacian preconditioning Use L as a preconditioner

DD preconditioning Use $S_1 + S_2$ as a preconditioner.

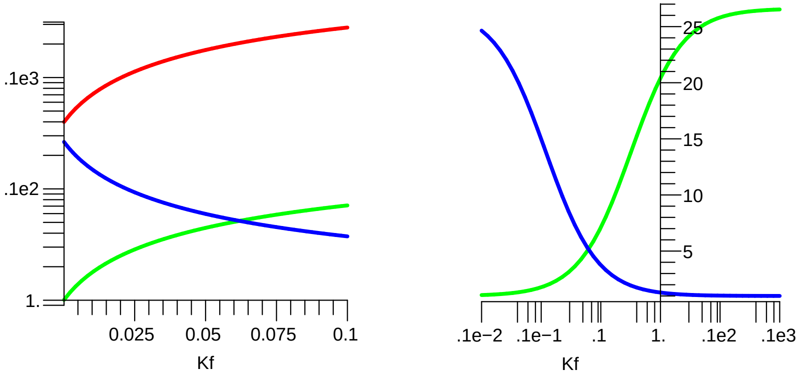
Top : No prec.,

Middle : Lap. prec.,

Bottom : DD prec.



Condition numbers when K_f varies



Cond. number vs K_f, no prec., Laplacian prec., DD prec.

Conclusions – perspectives

- Analysis of discrete Steklov – Poincaré operator for mixed finite elements on a regular grid
- Condition number for discrete fracture problem, asymptotic behavior

- Better understanding of low K_f values
- Use for designing a better preconditioner, for all values of K_f.