Coding theory as a tool in crypto

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Introduction

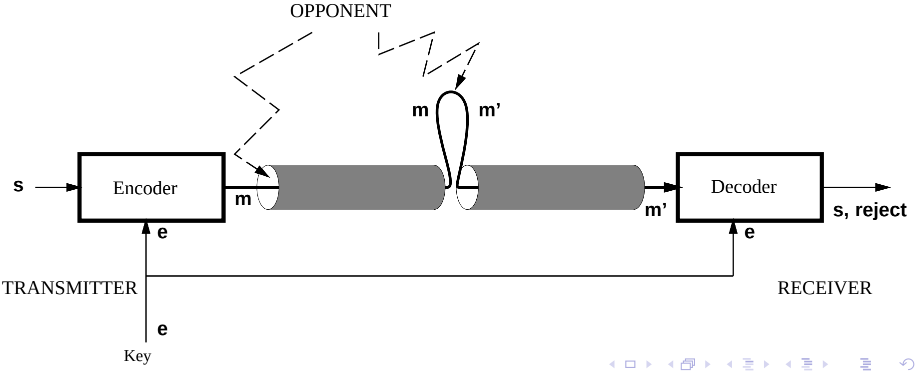
Coding theory has many applications in cryptology.

- PART I: Authentication codes
- PART II: Correlation attacks on stream ciphers

Unconditionally secure authentication

An unconditionally secure solution

- Simmons' model (Gilbert, MacWilliams, Sloane 1974)
- The transmitted information is a *source message*, s from $\mathcal{S}$.
- mapped into a (channel) *message*, $m \in \mathcal{M}$.
- the secret *key*, e and taken from the set $\mathcal{E}$.



Unconditionally secure authentication

- Encoding
$$f : \mathcal{S} \times \mathcal{E} \to \mathcal{M}, \quad (s, e) \mapsto m.$$
If $f(s, e) = m$ and $f(s', e) = m$, then $s = s'$ (injective for each $e \in \mathcal{E}$).
- The mapping f together with $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{E}$ define an *authentication code* (A-code).

- The receiver must check whether a source message s exists, such that $f(s, e) = m$.
- If such an s exists, m is accepted (m is called valid).
- Otherwise, m is not authentic and thus rejected.

Attacks

The opponent has two possible attacks at his disposal:

- The *impersonation attack*: Inserting a message m and hoping for it to be accepted as authentic.
- *substitution attack*: opponent observes the message m and replaces this with another message m', $m \neq m'$, hoping for m' to be valid.

Definitions of attack success

The opponent chooses the message that maximizes his chances of success when performing an attack.

- Success in impersonation attack:

$$P_I = \max_m P(m \text{ is valid})$$

- Success in substitution attack:

$$P_S = \max_{\substack{m,m' \\ m \neq m'}} P(m' \text{ is valid} | m \text{ is valid}).$$

Probability of deception P_D as $P_D = \max(P_I, P_S)$.

Basic bounds

Theorem

For any authentication code,

$$\begin{aligned} P_I &\geq \frac{|\mathcal{S}|}{|\mathcal{M}|}, \\ P_S &\geq \frac{|\mathcal{S}|-1}{|\mathcal{M}|-1}. \end{aligned}$$

$|\mathcal{M}|$ must be chosen much larger than $|\mathcal{S}|$.

Simmons' bounds

Theorem (Simmons' bounds)

For any authentication code,

$$\begin{aligned} P_I &\geq 2^{-I(M;E)}, \\ P_S &\geq 2^{-H(E|M)}, \quad \textit{if } |\mathcal{S}| \geq 2. \end{aligned}$$

For a good protection, i.e., P_I small, we must give away a lot of information about the key.

The square root bound

Multiply the two bounds together and get

$$P_I P_S \geq 2^{-I(M;E)-H(E|M)} = 2^{-H(E)}.$$

From $H(E) \leq \log |\mathcal{E}|$ we obtain the *square root bound*.

Theorem (Square root bound)

For any authentication code,

$$P_D \geq \frac{1}{\sqrt{|\mathcal{E}|}}.$$

On the square root bound

Theorem

The square root bound can be tight only if

$$|\mathcal{S}| \leq \sqrt{|\mathcal{E}|} + 1.$$

a large source size demands a twice as large key size. This is not very practical.

Systematic authentication codes

- An A-code for which the map $f : \mathcal{S} \times \mathcal{E} \to \mathcal{M}$ can be written in the form

$$f : \mathcal{S} \times \mathcal{E} \to \mathcal{S} \times \mathcal{Z}, \quad (s, e) \mapsto (s, z),$$

  where $s \in \mathcal{S}, z \in \mathcal{Z}$, is called a *systematic* (or Cartesian) A-code.
- The second part z in the message is called the *tag* (or authenticator) and is taken from the tag alphabet $\mathcal{Z}$.

Systematic authentication codes

Theorem

For any systematic A-code

$$P_S \geq P_I.$$

Constructing authentication codes

Define $\mathcal{E}(m)$ as the set of keys for which a message m is valid,

$$\mathcal{E}(m) = \{e \in \mathcal{E}; \exists s \in \mathcal{S}, f(s,e) = m\}.$$

The probability of success in a substitution attack can be written as

$$P_S = \max_{\substack{m,m' \\ m \neq m'}} \frac{|\mathcal{E}(m) \cap \mathcal{E}(m')|}{|\mathcal{E}(m)|},$$

provided that the keys are uniformly distributed.

The vector space construction:

- Let $|\mathcal{S}| = q$, $|\mathcal{Z}| = q$, and $|\mathcal{E}| = q^2$, q prime power.
- Decompose the key as $e = (e_1, e_2)$, where $s, z, e_1, e_2 \in \mathbb{F}_q$.
- For transmission of source message s, generate a message $m = (s, z)$, where

$$z = e_1 + se_2.$$

Theorem

The above construction provides $P_I = P_S = 1/q$. Moreover, it has parameters $|\mathcal{S}| = q$, $|\mathcal{Z}| = q$, and $|\mathcal{E}| = q^2$.

Proof

$$
\begin{aligned}
P_S &= \max_{\substack{m,m' \\ m\neq m'}} \frac{|\mathcal{E}(m)\cap\mathcal{E}(m')|}{|\mathcal{E}(m)|} \\
&= \max_{\substack{m,m' \\ m\neq m'}} \frac{|\{e\in\mathcal{E}; m=(s,e_1+se_2), m'=(s',e_1+s'e_2)\}|}{|\{e\in\mathcal{E}; m=(s,e_1+se_2)\}|} \\
&= \max_{\substack{m,m' \\ m\neq m'}} \frac{|\{e\in\mathcal{E}; m=(s,e_1+se_2), m-m'=(s-s',(s-s')e_2)\}|}{|\{e\in\mathcal{E}; m=(s,e_1+se_2)\}|} \\
&= \frac{1}{q}.
\end{aligned}
$$

Polynomial evaluation construction

- Let $\mathcal{S} = \{\mathbf{s} = (s_1, \ldots, s_k)\,;\, s_i \in \mathbb{F}_q\}$. Define the source message polynomial to be
$$s(x) = s_1 x + s_2 x^2 + \cdots + s_k x^k.$$
- Let $\mathcal{E} = \{e = (e_1, e_2)\,;\, e_1, e_2 \in \mathbb{F}_q\}$ and $\mathcal{Z} = \mathbb{F}_q$.
- For the transmission of source message $\mathbf{s}$, the transmitter sends $\mathbf{s}$ together with the tag
$$z = e_1 + s(e_2).$$

Theorem

The construction gives systematic A-codes with parameters

$$|\mathcal{S}| = q^k, \quad |\mathcal{E}| = q^2, \quad |\mathcal{Z}| = q, \quad P_I = 1/q, \quad P_S = k/q.$$

Proof

$$
\begin{aligned}
P_S &= \max_{\substack{m,m' \\ m\neq m'}} \frac{|\mathcal{E}(m)\cap\mathcal{E}(m')|}{|\mathcal{E}(m)|} \\
&= \max_{\substack{\mathbf{s},\mathbf{s}',z,z' \\ \mathbf{s}\neq\mathbf{s}'}} \frac{|\{e\in\mathcal{E}; z=e_1+s(e_2), z'=e_1+s'(e_2)\}|}{|\{e\in\mathcal{E}; z=e_1+s(e_2)\}|} \\
&= \max_{\substack{\mathbf{s},\mathbf{s}',z,z' \\ \mathbf{s}\neq\mathbf{s}'}} \frac{|\{e\in\mathcal{E}; z=e_1+s(e_2), z-z'=s(e_2)-s'(e_2)\}|}{|\{e\in\mathcal{E}; z=e_1+se_2)\}|} \\
&= \frac{k}{q}.
\end{aligned}
$$

Authentication codes using coding theory

- Let $m = (s, z)$ and write

$$z = e(s),$$

i.e., every key describes a map $\mathcal{S} \to \mathcal{Z}$.

- Let $n = |\mathcal{E}|$, $M = |\mathcal{S}|$ and $q = |\mathcal{Z}|$.
- Write $\mathcal{E}(s, z) = \{e \in \mathcal{E} : e(s) = z\}$.
- We restrict to A-codes for which

$$\mathcal{E}(m) = |\mathcal{E}| \, / \, |\mathcal{Z}| = n/q, \quad \forall m \in \mathcal{M}.$$

or $P_I = 1/q$.

Interpret the A-code as a code

- $\mathcal{S} = \{s_1, s_2, \ldots s_M\}$.
- A q-ary code C of length $|\mathcal{E}|$ with $|\mathcal{S}|$ codewords by

$$\mathbf{c}^{(i)} = (c_1^{(i)}, c_2^{(i)}, \ldots c_n^{(i)})$$

where

$$c_j^{(i)} = e_j(s_i).$$

$$\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & \ldots & e_n & \\
\hline
s_1 & (\; c_1^{(1)} & c_2^{(1)} & c_3^{(1)} & \ldots & c_n^{(1)} &) \\
s_2 & (\; c_1^{(2)} & c_2^{(2)} & c_3^{(2)} & \ldots & c_n^{(2)} &) \\
\vdots & \vdots & \vdots & & \ddots & \vdots & \\
s_M & (\; c_1^{(M)} & c_2^{(M)} & c_3^{(M)} & \ldots & c_n^{(M)} &)
\end{array}$$

Interpret the A-code as a code

- $$C = \{\mathbf{c}^{(i)}; i = 1, 2, \ldots, M\}.$$
- Define
$$\gamma(\mathbf{c}, \mathbf{c}') = \max_{\alpha,\beta \in \mathcal{Z}} \left|\{j; c_j = \alpha, c'_j = \beta\}\right|$$
- Define the *a-distance*
$$D(\mathbf{c}, \mathbf{c}') = n - q\gamma(\mathbf{c}, \mathbf{c}').$$
- Define the *minimum a-distance*
$$D(C) = \min_{\mathbf{c},\mathbf{c}' \in C; \mathbf{c} \neq \mathbf{c}'} D(\mathbf{c}, \mathbf{c}').$$
- Triangle inequality does not hold!

Interpret the A-code as a code

$$P_S = \max_{\substack{s,z,s',z' \\ s\neq s'}} \frac{|\mathcal{E}(s,z)\cap\mathcal{E}(s',z')|}{|\mathcal{E}(s,z)|} = \max_{\substack{\mathbf{c},\mathbf{c}' \\ \mathbf{c}\neq\mathbf{c}'}} \max_{\alpha,\beta\in\mathcal{Z}} \frac{\left|\{j;\, c_j=\alpha,\, c_j'=\beta\}\right|}{|\{j;\, c_j=\alpha\}|}$$

- Relation

$$D(C) = n(1-P_S).$$

Trivially, we have $D(C)\leq d_H(C)$, where $d_H(C)$ is the minimum (Hamming) distance of C.

Bounds on authentication codes

- $m(n, \epsilon, q)$ maximal number of source messages in an A-code with n keys, $P_S \leq \epsilon$, tag size q.
- $A_q(n, d)$ maximal number of codewords in a q-ary code of length n and minimum Hamming distance d
- $A_q^*(n, d)$ the same but assuming equal symbol composition.

Theorem

We have

$$q(q-1)m(n, \epsilon, q) \leq A_q^*(n, (1-\epsilon)n),$$

and

$$q(q-1)m(n, \epsilon, q) + q \leq A_q(n, (1-\epsilon)n).$$

Bounds on authentication codes

We prove

$$q(q-1)m(n,\epsilon,q) = A_q^*(n,(1-\epsilon)n),$$

by constructing

$$C' = \{\mathbf{c}'; \mathbf{c}' = a\mathbf{c} + b\mathbf{1}, \mathbf{c} \in C, a \neq 0, b \in \mathbb{F}_q\}.$$

and then argue that $d_H(C') \geq D(C)$.

Bounds on authentication codes

Theorem

Any systematic A-code for which $P_I = P_S = 1/q$ satisfies

$$(q-1)\,|\mathcal{S}| \leq n-1.$$

- The code will have parameters

$$(n, M, d) = (n, q(q-1)\,|\mathcal{S}| + q, \theta \cdot n),$$

  where $\theta = 1 - P_S = (q-1)/q$.
- The Plotkin bound gives

$$A_q(n, \theta n) \leq q A_q(n-1, \theta n) \leq q \frac{\theta n}{\theta n - \theta(n-1)} = qn.$$

Bounds on authentication codes

Theorem

Any systematic A-code satisfies

$$P_S \geq \frac{\log_q((q-1)\,|\mathcal{S}|+1)}{n}.$$

- Apply the Singleton bound $n \geq d+k-1$, $k = \log_q M$.

Bounds on authentication codes

The previous result can be strengthened.

Theorem

Any systematic A-code satisfies

$$P_S \geq \frac{q\lfloor \log_q |\mathcal{S}| \rfloor}{n},$$

provided $\log_q |\mathcal{S}| < \sqrt{(2n(1-1/q)/q)} - 1/2$.

- The proof is more complicated and involves the Johnson bound on *binary constant weight codes*.

Going the other way: Constructing A-codes from codes

Theorem

Let a code C with parameters (n, M, d) be given, with the special property that if $\mathbf{c} \in C$ then $\mathbf{c} + \lambda\mathbf{1} \in C$, for all $\lambda \in \mathbb{F}_q$. Then there exists an A-code with parameters

$$|\mathcal{S}| = Mq^{-1}, |\mathcal{E}| = nq, P_I = 1/q, P_S = 1 - d/n.$$

- 1. Form a "quotient code" $C/\mathbf{1}$ with parameters $(n, M/q, d)$.
- 2. Expand $\mathbf{c}$ to length nq by

$$\mathbf{c} \mapsto (\mathbf{c}, \mathbf{c} + \alpha_1\mathbf{1}, \mathbf{c} + \alpha_2\mathbf{1}, \ldots, \mathbf{c} + \alpha_{q-1}\mathbf{1}).$$

Going the other way: Constructing A-codes from codes

- For a linear code the condition if $\mathbf{c} \in C$ then $\mathbf{c} + \lambda\mathbf{1} \in C$, for all $\lambda \in \mathbb{F}_q$ simply means

$$\mathbf{1} \in C$$

.

- Finding such codes we can now construct A-codes...

Existence bounds on A-codes

- $A_q^\bullet(n,d)$ maximal number of codewords in a q-ary code of length n and minimum Hamming distance d such that if $\mathbf{c} \in C$ then $\mathbf{c} + \lambda\mathbf{1} \in C$, for all $\lambda \in \mathbb{F}_q$.
- Modified Gilbert bound:

$$A_q^\bullet(n,d) \geq \frac{q^n}{V_q(n,d-1)},$$

where $V_q(n,d-1) = \sum_{i=0}^{d-1} \binom{n}{i}(q-1)^i$ is the size of a Hamming sphere.

Theorem

The maximal number of source messages satisfies

$$m(nq,\epsilon,q) \geq \frac{q^{n-1}}{V_q(n,(1-\epsilon)n-1)},$$

where $P_S \leq \epsilon$.

Constructions of A-codes

- An $[n = q, k + 1, d]$ Reed-Solomon code C over $\mathbb{F}_q$ can be described as

$$C = \{(f(0), f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_{q-1})\}; f \in L\},$$

  where L is the set of all polynomials of degree $< k + 1$ in $\mathbb{F}_q[x]$ and $\mathbb{F}_q = \{0, \alpha_1, \ldots, \alpha_{q-1}\}$.
- $\mathbf{1} \in C$ so apply the construction. We get:

Polynomial evaluation construction

Let $\mathcal{S} = \{\mathbf{s} = (s_1, \ldots, s_k) \,;\, s_i \in \mathbb{F}_q\}$. Define the source message polynomial to be $s(x) = s_1 x + s_2 x^2 + \cdots + s_k x^k$. Let $\mathcal{E} = \{e = (e_1, e_2) \,;\, e_1, e_2 \in \mathbb{F}_q\}$ and $\mathcal{Z} = \mathbb{F}_q$. For the transmission of source message $\mathbf{s}$, the transmitter sends $\mathbf{s}$ together with the tag

$$z = e_1 + s(e_2).$$

Theorem

The construction gives systematic A-codes with parameters

$$|\mathcal{S}| = q^k, \quad |\mathcal{E}| = q^2, \quad |\mathcal{Z}| = q, \quad P_I = 1/q, \quad P_S = k/q.$$

More on the polynomial evaluation construction

- An A-code is *weakly optimal* if for fixed $|\mathcal{S}|, |\mathcal{E}|, |\mathcal{Z}|, P_I$ we have P_S at its lowest value.

Theorem

The polynomial evaluation construction gives weakly optimal A-codes.

Recall

$$P_S \geq \frac{q\lfloor \log_q |\mathcal{S}| \rfloor}{n},$$

Constructions through concatenation of codes

- Very large source sizes ($\log |\mathcal{S}| = 2^{30}$) requires very large codes.
- We can get that by concatenating codes, for example RS codes.

Construction: Let $Q = 2^{r+s}$ and $q = 2^r$. The source message is a polynomial $s(x) \in \mathbb{F}_Q[x]$ of degree $\leq 2^s$. Let $e_1, e_2 \in \mathbb{F}_Q$ and $e_3 \in \mathbb{F}_q$. The tag is

$$z = e_3 + [s(e_1)e_2],$$

where $[x]$ returns the last r bits of x.
Parameters:

$$\log |\mathcal{S}| = (r+s)(1+2^s), \quad \log |\mathcal{E}| = 3r+2s, \quad P_S < 2/2^r.$$

More recent developments

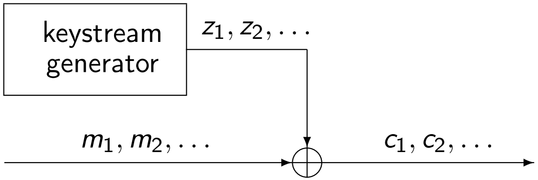
Focus on efficient impementation rather than minimum key size.

- LFSR-based Hashing and Authentication (Krawczyk)
- Bucket hashing (Rogaway)
- UMAC (Black, Halevi, Krawczyk, Krovetz, Rogaway)
- The Poly1305 MAC (Bernstein)
- and others...

In standards we have

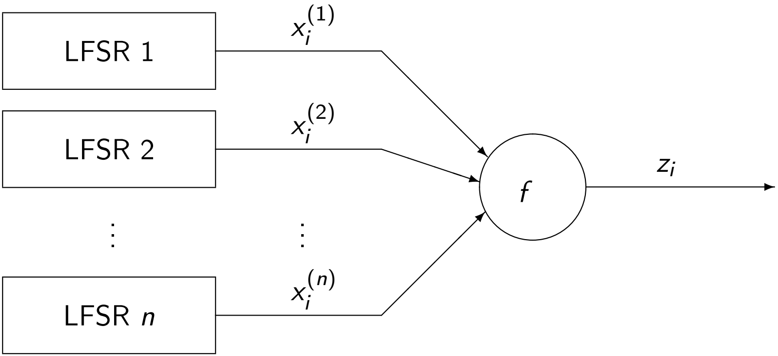
- NISTs GCM mode
- ETSI (3GPP) UIA2 mode.

PART II: Correlation attacks on stream ciphers



- The keystream generator contains one or several LFSRs.
- Observed keystream sequence $z_1, z_2, \ldots, z_N$.

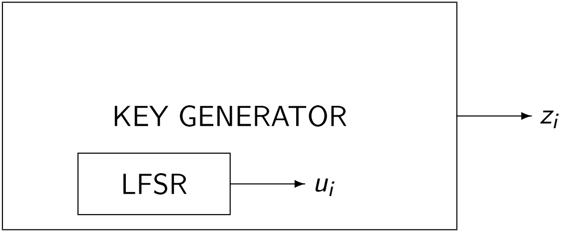
Correlation attacks



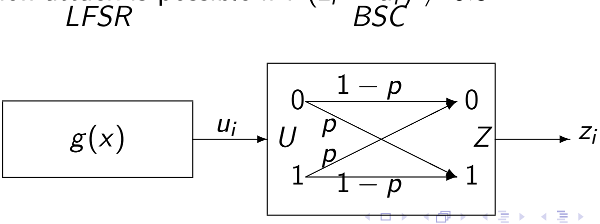
A nonlinear combining generator

Siegenthaler introduced *correlation attacks*.

Correlation attacks



- A correlation attack is possible if $P(z_i = u_i) \neq 0.5$.



A coding theory problem

- l is the length of the LFSR.
- The set of all 2^l possible LFSR sequences $\mathbf{u} = u_1, u_2, \ldots, u_N$ form a linear $[N, l]$ code, call it C.
- Assume that we know N binary keystream symbols

$$\mathbf{z} = z_1, z_2, \ldots, z_N.$$

- Then $\mathbf{z}$ corresponds to a received word, obtained by sending an unknown codeword through the BSC.
- Our problem is to *decode* $\mathbf{z}$ to the correct codeword.
- Typical characteristics:
  - Code length N is very large.
  - The noise is very strong (p close to $1/2$).

Meier-Staffelbach original approach

- Better than exhaustively searching LFSR – fast correlation attacks.
- Assume a *low weight* of $g(x)$.

Finding parity checks:

- The feedback polynomial

$$g(x) = 1 + g_1x^1 + g_2x^2 + \ldots + g_lx^l.$$

- $t =$ the number of taps, i.e., the weight of $g(x)$ is $t + 1$.
- Recurrence relation

$$u_n = g_1u_{n-1} + g_2u_{n-2} + \ldots + g_lu_{n-l}.$$

- We get in this way $t + 1$ different parity check equations for u_n.

Fast correlation attacks

- $g(x)^j = g(x^j)$ for $j = 2^i$, low degree multiples of $g(x)$.
- We create new weight $t + 1$ parity checks by

$$g_{k+1}(x) = g_k(x)^2.$$

- This squaring is continued until the degree of $g_k(x)$ is greater than the length N of the observed keystream.
- Each $g_k(x)$ gives $t + 1$ new parity check equations for a fixed position u_n.

Fast correlation attacks

- Write m equations for position u_n as,

$$
\begin{aligned}
u_n + b_1 &= 0, \\
u_n + b_2 &= 0, \\
&\vdots \\
u_n + b_m &= 0,
\end{aligned}
$$

  where each b_i is the sum of t different positions of $\mathbf{u}$.
- Applying the same to the keystream

$$
\begin{aligned}
z_n + y_1 &= L_1 \\
z_n + y_2 &= L_2 \\
&\vdots \\
z_n + y_m &= L_m.
\end{aligned}
$$

  where y_i is the sum of the positions in the keystream corresponding to the positions in b_i.

Fast correlation attacks

- Assume that h out of the m equations hold, i.e., $h = |\{i : L_i = 0, 1 \leq i \leq m\}|$.
- Then it is possible to calculate the probability

$$p^* = P(u_n = z_n | h \text{ equations hold})$$

as

$$p^* = \frac{ps^h(1-s)^{m-h}}{ps^h(1-s)^{m-h} + (1-p)(1-s)^h s^{m-h}},$$

where $p = P(u_n = z_n)$, and $s = P(b_i = y_i)$.

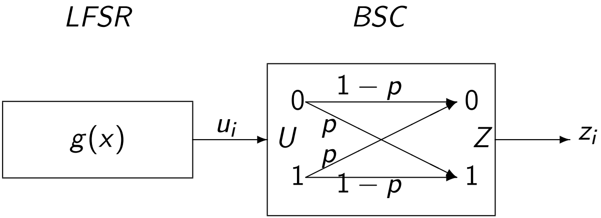
Fast correlation attacks

Algorithm B: the probabilities are calculated iteratively.
Two parameters p_{thr} and N_{thr}.

1. For all symbols in the keystream, calculate p^* and determine the number of positions N_w with $p^* < p_{thr}$.
2. If $N_w < N_{thr}$ repeat step 1 with $P(u_i = z_i) = p$ replaced by $P(u_i = z_i) = p^*$.
3. Complement the bits with $p^* < p_{thr}$ and reset the probabilities to p.
4. If not all equations are satisfied go to step 1.

Resembles iterative decoding a lot.

Correlation attacks



- General case: $g(x)$ is not of low weight.
- How to efficiently decode the "LFSR code" when transmitted over a very noisy BSC?

Correlation attacks using convolutional codes

(Johansson, Jönsson, Eurocrypt'99)

- Transform a part of $\mathcal{C}$ into a convolutional code.
- A rate $R = 1/(m+1)$ convolutional code with memory B has codeword symbols

$$\mathbf{v}_n = (v_n^{(0)}, v_n^{(1)}, \ldots, v_n^{(m)})$$

  where

$$\mathbf{v}_n = u_n G_0 + u_{n-1} G_1 + \ldots u_{n-B} G_B,$$

  and each G_i is a vector of length $(m+1)$.
- Generator matrix

$$\mathbf{G} = \begin{pmatrix} \ddots & \ddots & & \ddots & & & \\ & G_0 & G_1 & \ldots & G_m & & \\ & & G_0 & G_1 & \ldots & G_m & \\ & & & \ddots & \ddots & & \ddots \end{pmatrix},$$

Correlation attacks using convolutional codes

- Idea: Find parity checks that include u_n, an arbitrary linear combination of $u_{n-1}, \ldots, u_{n-B}$, together with at most t other symbols. say $t = 2$.
-

$$\begin{array}{c} u_n + \sum_{i=1}^{B} c_{i1} u_{n-i} + b_1 = 0, \\ u_n + \sum_{i=1}^{B} c_{i2} u_{n-i} + b_2 = 0, \\ \vdots \\ u_n + \sum_{i=1}^{B} c_{im} u_{n-i} + b_m = 0, \end{array}$$

where $b_k = \sum_{i=1}^{\leq t} u_{j_{ik}}$, $1 \leq k \leq m$ is the sum of (at most) t positions in $\mathbf{u}$.

Correlation attacks using convolutional codes

- We get a rate $R = \frac{1}{m+1}$ bi-infinite convolutional code V.
- For $v_n^{(i)}$ we create an estimate from $\mathbf{z}$.
- $v_n^{(0)} = u_n$ and $P(v_n^{(0)} = z_n) = 1 - p$. Otherwise, if $v_n^{(i)} = u_{j_1} + u_{j_2}$ then

$$P(v_n^{(i)} = z_{j_1} + z_{j_2}) = (1-p)^2 + p^2.$$

- Using these estimates we can construct a sequence

$$\mathbf{r} = \ldots r_n^{(0)} r_n^{(1)} \ldots r_n^{(m)} r_{n+1}^{(0)} r_{n+1}^{(1)} \ldots r_{n+1}^{(m)} \ldots,$$

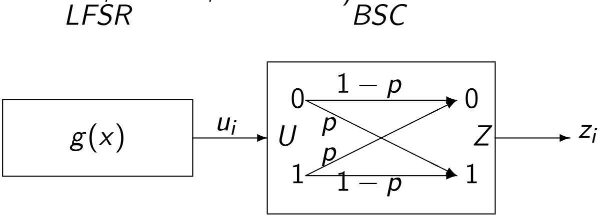
where $r_n^{(0)} = z_n$ and $r_n^{(i)} = z_{j_{1i}} + z_{j_{2i}}$, $1 \leq i \leq m$, that plays the role of a received sequence for the convolutional code.

Correlation attacks using convolutional codes

- To recover the initial state of the LFSR we need to decode l consecutive information bits correctly.
- Optimal decoding (ML decoding) of convolutional codes uses the Viterbi algorithm.
- There is neither a starting state, nor an ending state. Start by assigning the metrics $\log P(\mathbf{s} = z_1, z_2, \ldots, z_B)$ to each state $\mathbf{s}$ in the trellis. We then proceed to decode from $n = B$ as usual.

Another coding based approach

(Chepychov, Johansson, Smeets, FSE 2000)



- The LFSR code $\mathcal{C}$ has the $(N \times l)$-generator matrix

$$G = \begin{pmatrix} h_1^1 & h_2^1 & \cdots & h_N^1 \\ h_1^2 & h_2^2 & \cdots & h_N^2 \\ \vdots & & \cdots & \\ h_1^l & h_2^l & \cdots & h_N^l \end{pmatrix}.$$

Another coding based approach

- Fix a $k < l$. Look for pairs of columns of G such that,

$$h_i^{k+1} = h_j^{k+1}, \ldots, h_i^l = h_j^l,\ 1 \leq i \neq j \leq N.$$

- The indices of all such pairs: $\{i_1, j_1\}, \ldots, \{i_{n_2}, j_{n_2}\}$.
- The sum $c_i + c_j$ is independent of $u_{k+1}, u_{k+2}, \ldots, u_l$,

$$c_i + c_j = \left(h_i^1 + h_j^1\right) u_1 + \ldots + \left(h_i^k + h_j^k\right) u_k.$$

- This means that the words

$$(C_1, C_2, \ldots C_{n_2}) = (c_{i_1} + c_{j_1}, c_{i_2} + c_{j_2}, \ldots, c_{i_{n_2}} + c_{j_{n_2}})$$

form an $[n_2, k]$-code, referred to as $\mathcal{C}_2$.

Another coding based approach

- Calculate

$$Z_1 = z_{i_1} + z_{j_1}, \ldots, Z_{n_2} = z_{i_{n_2}} + z_{j_{n_2}},$$

  a word acting as a received word for $\mathcal{C}_2$.
- Decode the code $\mathcal{C}_2$ using exhaustive search through all the 2^k codewords of $\mathcal{C}_2$.
- New much worse error probability $p_2 = P(C_i \neq Z_i) = 2p(1-p)$, but dimension is smaller.

The general attack

Precomputation.

- Choose a $k < l$ and a $t \geq 2$. Construct generator matrix G.
- Find all sets of t indices $\{i(1), i(2), \ldots, i(t)\}$ that satisfy

$$\sum_{j=1}^{t} h_{i(j)}^{m} = 0, \text{ for } m = k+1, k+2, \ldots, l.$$

Store the indices $i(1), i(2), \ldots, i(t)$ together with the value of

$$\left(\sum_{j=1}^{t} h_{i(j)}^{1}, \sum_{j=1}^{t} h_{i(j)}^{2}, \ldots, \sum_{j=1}^{t} h_{i(j)}^{k} \right).$$

(Parity checks for the code $\mathcal{C}_t$)

The general attack

Decoding.

Input: The received vector $(z_1, z_2, \ldots, z_N)$.

Step 1. Compute

$$(Z_1 = \sum_{j=1}^{t} z_{i_1(j)}, \ldots, Z_n = \sum_{j=1}^{t} z_{i_n(j)}).$$

Step 2. Decode the code $\mathcal{C}_t$ using exhaustive search through the all 2^k codewords of $\mathcal{C}_t$.

Theoretical results

Theorem

With given k, l, t, $p = 1/2 - \varepsilon$, the required length N of the observed sequence $\mathbf{z}$ for the attack to succeed is

$$N \approx 1/4 \cdot (2kt! \ln 2)^{1/t} \cdot \varepsilon^{-2} \cdot 2^{\frac{l-k}{t}},$$

assuming $N >> n_t$.

Correlation attacks: more recent attacks

- Canteaut and Trabbia: Use parity check equations of weight 4 and 5, decode with Gallager iterative decoding algorithm.
- Johansson, Jönsson: reconstruction of linear polynomials
- Mihaljevic, Fossorier and Imai: combine exhaustive search over the first B bits with list decoding algorithm.
- Chose, Joux and Mitton: new methods for efficient implementations, better decoding algorithm.
- Golic: vectorial approach to fast correlation attacks
- and many more

Conclusions

- We saw two examples of coding in crypto:
  - authentication codes
  - correlation attacks
- Many other: Boolean functions/S-boxes; McEliece PKC etc.;
- New application areas may come...