When should one stop the iterations in a domain decomposition method ?

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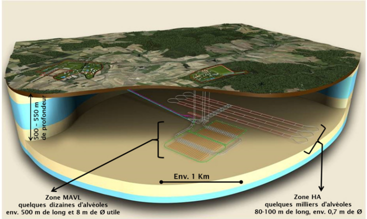
Outline

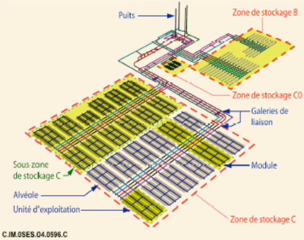
1. Motivations and problem setting
2. Space time domain decomposition
3. A posteriori estimates

Outline

Geological repository for nuclear waste

Deep underground repository
(High-level waste)





Challenges

- Different materials → strong heterogeneity, different time scales.
- Large differences in spatial scales.
- Long-term computations.

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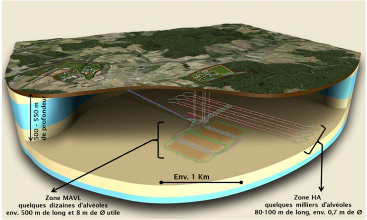

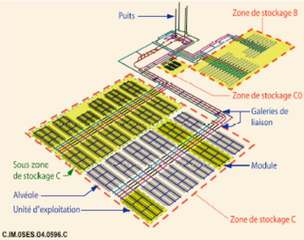

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⇒ Use space-time DD methods

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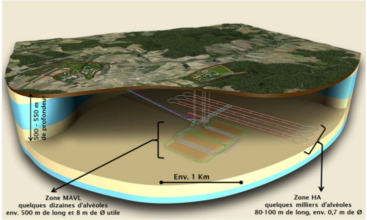

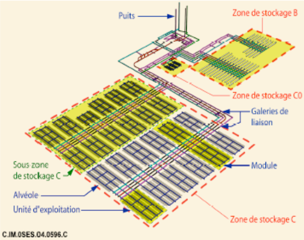

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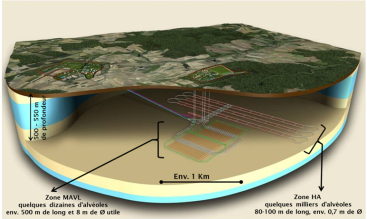
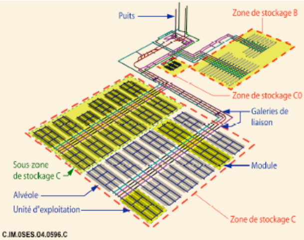




Challenges

- Different materials → strong heterogeneity, different time scales.
- Large differences in spatial scales.
- Long-term computations.

⇒ Use space-time DD methods

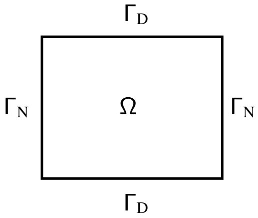
⇒ Estimate the error at DD iterations

⇒ Develop stopping criteria to stop DD iterations as soon as the discretization error is reached

Model problem: one phase unsteady flow

Time-dependent diffusion equation

$$
\begin{aligned}
\mathbf{u} &= -\boldsymbol{S}\nabla p, && \text{in} \quad \Omega \times (0,T),\\
\phi\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} &= f, && \text{in} \quad \Omega \times (0,T),\\
p &= g_{\mathrm{D}} && \text{on} \quad \Gamma_{\mathrm{D}} \cap \partial\Omega \times (0,T),\\
-\mathbf{u}\cdot\boldsymbol{n} &= g_{\mathrm{N}} && \text{on} \quad \Gamma_{\mathrm{N}} \cap \partial\Omega \times (0,T),\\
p(\cdot,0) &= p_0 && \text{in} \quad \Omega.
\end{aligned}
$$

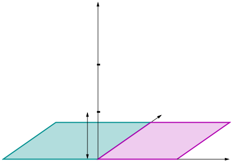


- $\mathbf{u}$ Darcy velocity,
- p pressure,
- $\boldsymbol{S}$ permeability,
- $f \in L^2(\Omega)$ the source term,
- ϕ porosity

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$,
- Γ_{D} Dirichlet boundaries,
- Γ_{N} Neumann boundaries,
- $\boldsymbol{n}$: unit normal vector outward from Ω.

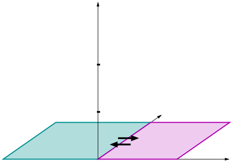
Outline

Domain decomposition in space



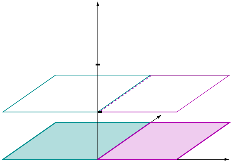
- Discretize in time and apply the DD algorithm at each time step:

Domain decomposition in space



- Discretize in time and apply the DD algorithm at each time step:
  - Solve stationary problems in the subdomains
  - Exchange information through the interface

Domain decomposition in space



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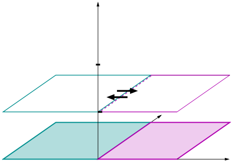

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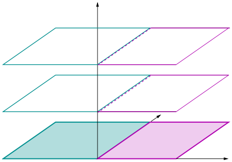

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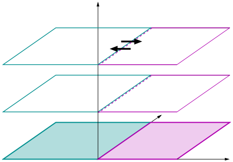

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- ✗ Same time step on the whole domain.

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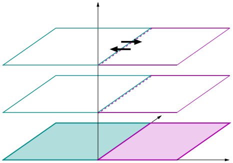

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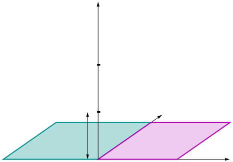
Space-time domain decomposition



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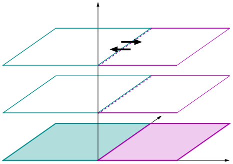

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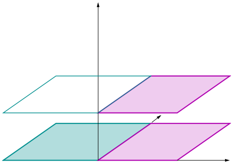
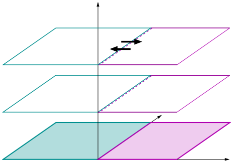


- Solve time-dependent problems in the subdomains

Domain decomposition in space



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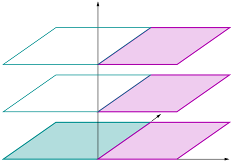

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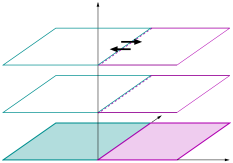

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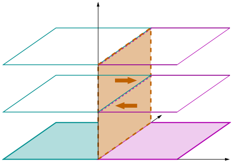
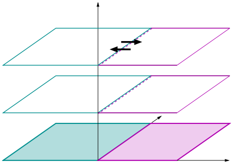


- Solve time-dependent problems in the subdomains
- Exchange information through the space-time interface · · · Following [Halpern-Nataf-Gander (03), Martin (05)]

Domain decomposition in space



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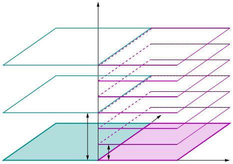


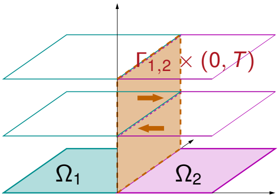
- Solve time-dependent problems in the subdomains
- Exchange information through the space-time interface · · · Following [Halpern-Nataf-Gander (03), Martin (05)]
- ✓ Different time steps can be used in each subdomain according to its physical properties.
  · · · Following [Halpern-J.-Szeftel (12), Hoang-Japhet-Jaffré-K.-Roberts (13)]

Equivalent space-time DD formulation

Solve the transmission problem, with $i = 1, 2$

$$
\begin{aligned}
\mathbf{u}_i &= -\boldsymbol{S}\nabla p_i && \text{in} \quad \Omega_i \times (0,T),\\
\phi_i \frac{\partial p_i}{\partial t} + \nabla\cdot\mathbf{u}_i &= f && \text{in} \quad \Omega_i \times (0,T),\\
p_i &= g_{\mathrm{D}} && \text{on} \quad (\Gamma_{\mathrm{D}} \cap \partial\Omega_i) \times (0,T),\\
-\mathbf{u}_i \cdot \boldsymbol{n}_i &= g_{\mathrm{N}} && \text{on} \quad (\Gamma_{\mathrm{N}} \cap \partial\Omega_i) \times (0,T),\\
p_i(\cdot,0) &= p_0 && \text{in} \quad \Omega_i,\\
p_1 &= p_2 && \text{on} \quad \Gamma_{1,2},\\
\mathbf{u}_1 \cdot \boldsymbol{n}_1 &= \mathbf{u}_2 \cdot \boldsymbol{n}_1 && \text{on} \quad \Gamma_{1,2},
\end{aligned}
$$

- with physical transmission conditions



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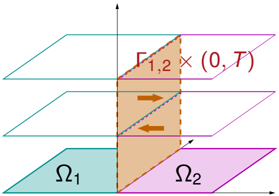
p_i(\cdot, 0) &= p_0 && \text{in} \quad \Omega_i,\\
-\beta_{1,2}\mathbf{u}_1 \cdot \boldsymbol{n}_1 + p_1 &= -\beta_{1,2}\mathbf{u}_2 \cdot \boldsymbol{n}_1 + p_2 && \text{on} \quad \Gamma_{1,2},\\
-\beta_{2,1}\mathbf{u}_2 \cdot \boldsymbol{n}_2 + p_2 &= -\beta_{2,1}\mathbf{u}_1 \cdot \boldsymbol{n}_2 + p_1 && \text{on} \quad \Gamma_{1,2},
\end{aligned}
$$

- with Robin transmission conditions
  ··· Following [P.-L. Lions (88)]
- Equivalent to original problem

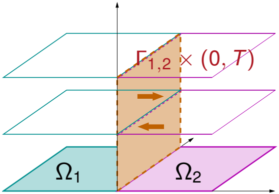


Optimized Schwarz waveform relaxation algorithm

For $k \geq 0$, at step k, solve **in parallel** the **space-time** Robin subdomain problems $(i = 1, 2)$:

$$
\begin{aligned}
\mathbf{u}_i^{k+1} &= -\boldsymbol{S}\nabla p_i^{k+1} && \text{in} \quad \Omega_i \times (0, T),\\
\phi_i \frac{\partial p_i^{k+1}}{\partial t} + \nabla\cdot\mathbf{u}_i^{k+1} &= f && \text{in} \quad \Omega_i \times (0, T),\\
p_i^{k+1} &= g_{\mathrm{D}} && \text{on} \quad (\Gamma_{\mathrm{D}} \cap \partial\Omega_i) \times (0, T),\\
-\mathbf{u}_i^{k+1} \cdot \boldsymbol{n}_i &= g_{\mathrm{N}} && \text{on} \quad (\Gamma_{\mathrm{N}} \cap \partial\Omega_i) \times (0, T),\\
p_i^{k+1}(\cdot, 0) &= p_0 && \text{in} \quad \Omega_i,\\
-\beta_{1,2}\mathbf{u}_1^{k+1} \cdot \boldsymbol{n}_1 + p_1^{k+1} &= -\beta_{1,2}\mathbf{u}_2^{k} \cdot \boldsymbol{n}_1 + p_2^{k} && \text{on} \quad \Gamma_{1,2} \times (0, T),\\
-\beta_{2,1}\mathbf{u}_2^{k+1} \cdot \boldsymbol{n}_2 + p_2^{k+1} &= -\beta_{2,1}\mathbf{u}_1^{k} \cdot \boldsymbol{n}_2 + p_1^{k} && \text{on} \quad \Gamma_{1,2} \times (0, T),
\end{aligned}
$$

- where $-\beta_{i,j}\mathbf{u}_j^0 \cdot \boldsymbol{n}_i + p_j^0 = g_i^0$, with g_i^0 a given function, $i = 1, 2$.
  ··· Following [Halpern-Nataf-Gander (03), Martin (05)]



The semi-discrete in time subdomain problem

(DG0 time stepping)

- $\{t^n\}_{0 \le n \le N}$ discrete times: $t^0 = 0 < t^1 < \cdots < t^n < \cdots < t^N = T$.
- $\mathcal{T}_\tau$ the partition of $(0, T)$ into sub-intervals $I_n := (t^{n-1}, t^n]$, and $\tau^n := t^n - t^{n-1}$, $1 \le n \le N$
- $P^0_{\mathcal{T}_\tau}(E) := \{v_\tau : (0, T) \to E;$ where v_τ is constant on $I_n, \ 1 \le n \le N\}$.
- $v^n := v_\tau|_{I_n}$ and $\tilde{f}^n := \dfrac{1}{\tau^n} \displaystyle\int_{I_n} f(\cdot, t)dt$.

The semi-discrete in time subdomain problem is:

Find $(p_{\tau,i}, \mathbf{u}_{\tau,i}) \in P^0_\tau(L^2(\Omega_i)) \times P^0_\tau(\mathbf{H}(\mathrm{div}, \Omega_i))$ solution of the following problem, for $n = 1, \ldots, N$:

$$\begin{aligned}
\mathbf{u}_i^n &= -\boldsymbol{S}\nabla p_i^n && \text{in} \quad \Omega_i, \\
\frac{p_i^n - p_i^{n-1}}{\tau^n} + \nabla\cdot\mathbf{u}_i^n &= \tilde{f}^n && \text{in} \quad \Omega_i, \\
-\beta_{i,j}\mathbf{u}_i^n \cdot \boldsymbol{n}_i + p_i^n &= \xi_{i,j}^n && \text{on} \quad \Gamma_{i,j}, \ \forall j \in B^i, \\
p_i^0 &= p_0 && \text{in} \quad \Omega_i.
\end{aligned}$$

- Later, for the a posteriori estimates, we also define:
  $P^1_{\mathcal{T}_\tau}(E) := \{v_\tau : (0, T) \to E; \ v_\tau \in C^0(0, T), \ v_\tau$ is affine on $I_n, \ 1 \le n \le N\}$.

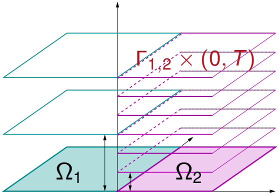
Semi-discrete in time interface problem

- Robin to Robin operators, for $i = 1, 2,\ j = 3 - i$:

$$\mathcal{S}_i^{\text{RtR}} : \left(\xi_{\tau,i}, \tilde{f}, p_0\right) \to \left(-\mathbf{u}_{\tau,i} \cdot \boldsymbol{n}_j + \beta_{j,i} p_{\tau,i}\right)_{|\Gamma_{i,j}}$$

where $(p_{\tau,i}, \mathbf{u}_{\tau,i})$ $(i = 1, 2)$ solves, for $n = 1, \ldots, N$:

$$\begin{aligned} \mathbf{u}_i^n &= -\boldsymbol{S}\nabla p_i^n && \text{in} \quad \Omega_i, \\ \frac{p_i^n - p_i^{n-1}}{\tau^n} + \nabla\cdot\mathbf{u}_i^n &= \tilde{f}^n && \text{in} \quad \Omega_i, \\ -\beta_{i,j}\mathbf{u}_i^n \cdot \boldsymbol{n}_i + p_i^n &= \xi_{i,j}^n && \text{on} \quad \Gamma_{i,j}, \ \forall j \in B^i, \\ p_i^0 &= p_0 && \text{in} \quad \Omega_i. \end{aligned}$$



- Space-time interface problem

$$\begin{aligned} \xi_{1,2} &= S_1^{\text{RtR}}(\xi_{2,1}, \tilde{f}, p_0) \\ \xi_{2,1} &= S_2^{\text{RtR}}(\xi_{1,2}, \tilde{f}, p_0) \end{aligned} \quad \text{on } \Gamma \times (0,T) \quad \text{or } S_R \begin{pmatrix} \xi_{1,2} \\ \xi_{2,1} \end{pmatrix} = \chi$$

- Solve with block-Jacobi (OSWR algorithm) or GMRES

Semi-discrete in time interface problem

- Robin to Robin operators, for $i = 1, 2,\ j = 3 - i$:

$$\mathcal{S}_i^{\text{RtR}} : \left(\xi_{\tau,i}, \tilde{f}, p_0\right) \to (-\mathbf{u}_{\tau,i} \cdot \boldsymbol{n}_j + \beta_{j,i} p_{\tau,i})_{|\Gamma_{i,j}}$$

where $(p_{\tau,i}, \mathbf{u}_{\tau,i})$ $(i = 1, 2)$ solves, for $n = 1, \ldots, N$:

$$\begin{aligned}
\mathbf{u}_i^n &= -\boldsymbol{S}\nabla p_i^n && \text{in} \quad \Omega_i, \\
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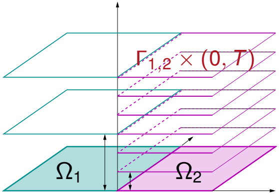

- Space-time interface problem

$$\begin{aligned}
\xi_{1,2} &= \Pi_{i,j} S_1^{\text{RtR}}(\xi_{2,1}, \tilde{f}, p_0) \\
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\end{aligned} \quad \text{on } \Gamma \times (0,T) \quad \text{or } S_R \begin{pmatrix} \xi_{1,2} \\ \xi_{2,1} \end{pmatrix} = \chi$$

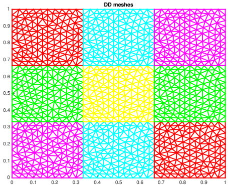
- Solve with block-Jacobi (OSWR algorithm) or GMRES
- L^2 projection operator $\Pi_{i,j}$ from $P^0_{\mathcal{T}_{\tau,j}}(L^2(\Gamma_{i,j}))$ onto $P^0_{\mathcal{T}_{\tau,i}}(L^2(\Gamma_{i,j}))$,

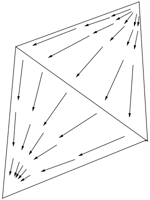
DD using the lowest-order Raviart–Thomas (RT0)

Let $\mathcal{T}_h$ be a matching mesh of Ω, with $\mathcal{T}_{h,i} := \mathcal{T}_h|_{\Omega_i}$, $i = 1, .., \mathcal{N}$. Discrete spaces:

$$M_{h,i} = \{q_{h,i} \in L^2(\Omega_i), q_{h,i}{}_{|K} \in \mathcal{P}^0(K), \forall K \in \mathcal{T}_{hi}\}$$
$$\mathbf{W}_{h,i} = \{\mathbf{v}_{h,i} \in \mathbf{H}(\text{div}\,, \Omega_i), \mathbf{v}_{h,i}{}_{|K} \in \mathbf{RT}_0(K), \forall K \in \mathcal{T}_{hi}\}.$$

Find the discrete solutions $p^{k+1}_{h\tau,i} \in P^0_{\mathcal{T}_\tau}(M_{h,i})$ and $\mathbf{u}^{k+1}_{h\tau,i} \in P^0_{\mathcal{T}_\tau}(\mathbf{W}_{h,i})$





- The energy norm on $H^1_{\Gamma_D}(\Omega)$ is $|||v|||^2 := \|\boldsymbol{S}^{\frac{1}{2}} \nabla v\|^2$
- The energy norm for vectors on $\mathbf{L}^2(\Omega)$ is defined by: $|||\mathbf{v}|||^2_\star := \|\boldsymbol{S}^{-\frac{1}{2}} \mathbf{v}\|^2$

Outline

A posteriori estimates: overview

- $\underbrace{|||p - p_h^{k+1}|||}_{\textbf{unknown}}$

A posteriori estimates: overview

- $\underbrace{|||p - p_h^{k+1}|||}_{\textbf{unknown}} \leq \underbrace{\textbf{Fully computable estimators}}_{\text{depend on } \mathbf{H}(\mathrm{div},\Omega) \text{ flux and } H^1(\Omega) \text{ potential reconstructions}}$

A posteriori estimates: overview

- $\underbrace{|||p - \tilde{p}_{h\tau}^{k+1}|||}_{\textbf{unknown}} \leq \underbrace{\textbf{Fully computable estimators}}$

  depend on $\mathbf{H}(\mathrm{div},\Omega)$ flux and $H^1(\Omega)$ potential reconstructions

- Post-processing $\tilde{p}_{h\tau,i}^{k}$ of the pressure $p_{h\tau,i}^{k}$, at each time step n, $n = 0, ..., N$:

  $\nabla p_h^{k,n} = 0$, as $p_h^{k,n} \in \mathcal{P}_0(\mathcal{T}_{h,i})$ in the MFE, so that $||\boldsymbol{S}^{\frac{1}{2}} \nabla (p - p_h^{k,n})||^2 = ||\boldsymbol{S}^{\frac{1}{2}} \nabla p||^2$ not suitable.

A posteriori estimates: overview

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A posteriori estimates: overview

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- ✖ MFE method gives $\tilde{p}_h^{k,n} \notin H^1(\Omega_i)$
- ✖ Robin DD method gives $\mathbf{u}_h^{k,n} \notin \mathbf{H}(\mathrm{div}\,,\Omega)$ and $\tilde{p}_h^{k,n}$ jumps across Γ_{ij}

A posteriori estimates: overview

- $\underbrace{|||p - \tilde{p}_{h\tau}^{k+1}|||}_{\textbf{unknown}} \leq \underbrace{\textbf{Fully computable estimators}}$
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  $\nabla p_h^{k,n} = 0$, as $p_h^{k,n} \in \mathcal{P}_0(\mathcal{T}_{h,i})$ in the MFE, so that $||\boldsymbol{S}^{\frac{1}{2}} \nabla(p - p_h^{k,n})||^2 = ||\boldsymbol{S}^{\frac{1}{2}} \nabla p||^2$ not suitable.
- ✗ MFE method gives $\tilde{p}_h^{k,n} \notin H^1(\Omega_i)$
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Pressure and flux reconstructions:

- $\overline{s}_{h\tau,i}^{k+1}$: $H^1(\Omega_i)$-conforming but not continuous over the DD interfaces **(New strategy)**, continuous and piecewise affine in time
- $s_{h\tau}^{k+1}$: $H^1(\Omega)$-conforming, continuous and piecewise affine in time
- $\boldsymbol{\sigma}_h^{k+1}$: $\mathbf{H}(\mathrm{div}, \Omega)$-conforming and local conservative in each element, piecewise constant in time

[Vohralík (10), Pencheva-Vohralík-Wheeler-Wildey (13), Ern-Vohralík (10), Ern-Smears-Vohralík (16)]

Extension to Robin DD in this work

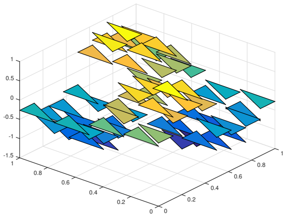


Figure: p_h^{k+1}

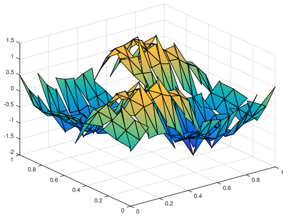


Figure: $\tilde{p}_h^{k+1}$

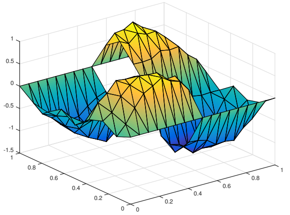


Figure: $\overline{s}_{h\tau,i}^{k+1}$

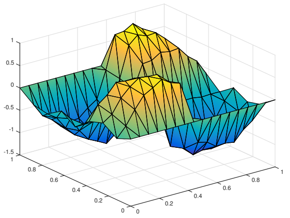


Figure: $s_{h\tau}^{k+1}$

Following [Ern-Vohralík (10), Ern-Smears-Vohralík (16)]

$$X := L^2(0,T;H_0^1(\Omega)),\ |||q|||_X^2 := \sum_{n=1}^{N}\int_{I_n}\sum_{K\in\mathcal{T}_h}\|\mathbf{S}^{\frac{1}{2}}\nabla q(\cdot,t)\|_K^2 dt,$$

$$X' := L^2(0,T;H^{-1}(\Omega)),$$

$$Y := \{q\in X;\ \partial_t q\in X'\},\ |||q|||_Y^2 := |||q|||_X^2 + \|\partial_t q\|_{X'}^2 + \|q(\cdot,T)\|^2.$$

A posteriori error estimate distinguishing error components

At each iteration $k+1$ of the DD method :

$$|||p-\tilde{p}_{h\tau}^{k+1}|||_Y \le \boldsymbol{\eta}_{\mathbf{sp}}^{\mathbf{k+1}} + \boldsymbol{\eta}_{\mathbf{DD}}^{\mathbf{k+1}} + \boldsymbol{\eta}_{\mathbf{tm}}^{\mathbf{k+1}} + \eta_{\mathrm{IC}}^{k+1} + ||f-\tilde{f}||_{X'} + ||s_h^{k+1,N} - \tilde{p}_h^{k+1,N}||,$$

where :

$\boldsymbol{\eta}_{\mathbf{sp}}^{\mathbf{k+1}}$:= *subdomain discretization estimator*,

$\boldsymbol{\eta}_{\mathbf{DD}}^{\mathbf{k+1}}$:= *domain decomposition estimator*,

$\eta_{\mathrm{IC}}^{k+1} := \|s_h^{k+1,0} - p_0\|$ *initial condition estimator*,

$$\boldsymbol{\eta}_{\mathbf{tm}}^{\mathbf{k+1}} := \left\{\sum_{n=1}^{N}\sum_{K\in\mathcal{T}_h}\frac{1}{3}\tau^n|||s_h^{k+1,n} - s_h^{k+1,n-1}|||_K^2\right\}^{\frac{1}{2}}$$ *time discretization estimator*,

$\cdots\quad \tilde{p}_{h\tau,i}^{k+1},\ s_{h\tau}^{k+1},\ \overline{s}_{h\tau,i}^{k+1}$, and $\boldsymbol{\sigma}_{h\tau}^{k+1}$.

$$\eta_{\mathrm{sp}}^{k+1} := \left\{ \sum_{n=1}^{N} \tau^n \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{osc},K}^{k+1,n} + \eta_{\mathrm{DF},K}^{k+1,n})^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{i=1}^{\mathcal{N}} \sum_{K \in \mathcal{T}_{h,i}} (\eta_{\mathrm{NCP},1,K}^{k+1}(t))^2 \mathrm{d}t \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{N} \tau^n \sum_{i=1}^{\mathcal{N}} \sum_{K \in \mathcal{T}_{h,i}} (\eta_{\mathrm{NCP},2,K}^{k+1,n})^2 \right\}^{\frac{1}{2}}$$

$$\eta_{\mathrm{osc},K}^{k+1,n} := \frac{h_K}{\pi} c_{\mathbf{S},K}^{-\frac{1}{2}} \|\tilde{f}^n - \partial_t s_h^{k+1,n} - \nabla \cdot \boldsymbol{\sigma}_h^{k+1,n}\|_K \quad \text{“data oscillation”},$$

$$\eta_{\mathrm{DF},K}^{k+1,n} := |||\mathbf{S}\nabla \overline{s}_h^{k+1,n} + \mathbf{u}_h^{k+1,n}|||_{\star,K}, \quad \text{“constitutive relation”},$$

$$\eta_{\mathrm{NCP},1,K}^{k+1}(t) := |||(\tilde{p}_{h\tau,i}^{k+1} - \overline{s}_{h\tau,i}^{k+1})(t)|||_K, \quad t \in I_n \quad \text{“scheme potential nonconformity”},$$

$$\eta_{\mathrm{NCP},2,K}^{k+1,n} := \frac{h_K}{\pi} c_{\mathbf{S},K}^{-\frac{1}{2}} ||\partial_t(\tilde{p}_{h,i}^{k+1,n} - \overline{s}_{h,i}^{k+1,n})||_K, \quad \text{“scheme potential nonconformity”},$$

$$\boldsymbol{\eta}_{\mathbf{sp}}^{\boldsymbol{k+1}} := \left\{ \sum_{n=1}^{N} \tau^n \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{osc},K}^{k+1,n} + \eta_{\mathrm{DF},K}^{k+1,n})^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{i=1}^{\mathcal{N}} \sum_{K \in \mathcal{T}_{h,i}} (\eta_{\mathrm{NCP},1,K}^{k+1}(t))^2 \mathrm{d}t \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{N} \tau^n \sum_{i=1}^{\mathcal{N}} \sum_{K \in \mathcal{T}_{h,i}} (\eta_{\mathrm{NCP},2,K}^{k+1,n})^2 \right\}^{\frac{1}{2}}$$

$$\eta_{\mathrm{osc},K}^{k+1,n} := \frac{h_K}{\pi} c_{\boldsymbol{S},K}^{-\frac{1}{2}} \| \tilde{f}^n - \partial_t s_h^{k+1,n} - \nabla \cdot \boldsymbol{\sigma}_h^{k+1,n} \|_K$$ "data oscillation",

$$\eta_{\mathrm{DF},K}^{k+1,n} := ||| \boldsymbol{S} \nabla \overline{s}_h^{k+1,n} + \mathbf{u}_h^{k+1,n} |||_{\star,K},$$ "constitutive relation",

$$\eta_{\mathrm{NCP},1,K}^{k+1}(t) := ||| (\tilde{p}_{h\tau,i}^{k+1} - \overline{s}_{h\tau,i}^{k+1})(t) |||_K, \quad t \in I_n$$ "scheme potential nonconformity",

$$\eta_{\mathrm{NCP},2,K}^{k+1,n} := \frac{h_K}{\pi} c_{\boldsymbol{S},K}^{-\frac{1}{2}} || \partial_t (\tilde{p}_{h,i}^{k+1,n} - \overline{s}_{h,i}^{k+1,n}) ||_K,$$ "scheme potential nonconformity",

$$\boldsymbol{\eta}_{\mathbf{DD}}^{\boldsymbol{k+1}} := \left\{ \sum_{n=1}^{N} \tau^n \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{DDF},K}^{k+1,n} + \eta_{\mathrm{DDP},1,K}^{k+1}(t^n))^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{i=1}^{\mathcal{N}} \sum_{K \in \mathcal{T}_{h,i}} (\eta_{\mathrm{DDP},1,K}^{k+1}(t))^2 \mathrm{d}t \right\}^{\frac{1}{2}} + \left\{ \sum_{n=1}^{N} \tau^n \sum_{i=1}^{\mathcal{N}} \sum_{K \in \mathcal{T}_{h,i}} (\eta_{\mathrm{DDP},2,K}^{k+1,n})^2 \right\}^{\frac{1}{2}}$$

$$\eta_{\mathrm{DDF},K}^{k+1,n} := ||| \mathbf{u}_h^{k+1,n} - \boldsymbol{\sigma}_h^{k+1,n} |||_{\star,K},$$ "DD flux nonconformity",

$$\eta_{\mathrm{DDP},1,K}^{k+1}(t) := ||| (\overline{s}_{h\tau,i}^{k+1} - s_{h\tau,i}^{k+1})(t) |||_K, \quad t \in I_n$$ "DD potential nonconformity",

$$\eta_{\mathrm{DDP},2,K}^{k+1,n} := \frac{h_K}{\pi} c_{\boldsymbol{S},K}^{-\frac{1}{2}} || \partial_t (\overline{s}_{h,i}^{k+1,n} - s_{h,i}^{k+1,n}) ||_K,$$ "DD potential nonconformity".

Postprocessing $\tilde{p}_{h\tau,i}^{k+1}$ of $p_{h\tau,i}^{k+1}$

$\tilde{p}_{h,i}^{k+1,n} \in \mathcal{P}_2(\mathcal{T}_{h,i})$ at each iteration $k+1$ and at each time step n, $n = 0, ..., N$, is constructed as:

$$-\boldsymbol{S}_K \nabla \tilde{p}_{h,i}^{k+1,n}|_K = \mathbf{u}_{h,i}^{k+1,n}|_K, \qquad \forall K \in \mathcal{T}_{h,i},$$
$$\pi_0(\tilde{p}_{h,i}^{k+1,n}|_K) = p_{h,i}^{k+1,n}|_K, \qquad \forall K \in \mathcal{T}_{h,i}.$$

⚠ $\tilde{p}_{h,i}^{k+1} \notin H^1(\Omega_i)$,

- $\tilde{p}_{h,i}^{k+1} \in W_0(\mathcal{T}_{h,i}) := \{\varphi \in H^1(\mathcal{T}_{h,i}); \langle [\![\varphi]\!], 1\rangle_e = 0, \quad \forall e \in \mathcal{E}_{h,i}^{\mathrm{int}}\} \cdots$ weak continuity.

Potential reconstruction $s_{h\tau}^{k+1}$

Captures

- scheme potential nonconformity
- DD potential nonconformity

Potential reconstruction $s_{h\tau}^{k+1}$

Captures

- scheme potential nonconformity
- DD potential nonconformity

$s_{h\tau}^{k+1}$ is $H^1(\Omega)$-conforming in space and piecewise affine continuous in time:

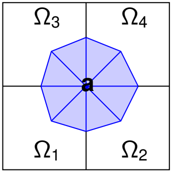
$$s_{h\tau}^{k+1} \in P_\tau^1(H^1(\Omega) \cap C^0(\overline{\Omega})),$$
$$(s_h^{k+1,n}, 1)_K = (\tilde{p}_{h,i}^{k+1,n}, 1)_K, \quad \forall K \in \mathcal{T}_h, \quad 0 \leq n \leq N.$$

$$s_h^{k+1,n} := \mathcal{I}_{\mathrm{av}}(\tilde{p}_h^{k+1,n}) + \sum_{K \in \mathcal{T}_h} \alpha_K^{k+1,n} b_K,$$

where $\mathcal{I}_{\mathrm{av}}(\tilde{p}_h^{k+1,n})(\mathbf{a}) = \frac{1}{|\mathcal{T}_\mathbf{a}|} \sum_{K \in \mathcal{T}_\mathbf{a}} \tilde{p}_h^{k+1,n}|_K(\mathbf{a}),$

b_K is a bubble function on K, and

$$\alpha_K^{k+1,n} := \frac{1}{(b_K, 1)_K} (\tilde{p}_h^{k+1,n} - \mathcal{I}_{\mathrm{av}}(\tilde{p}_h^{k+1,n}), 1)_K$$

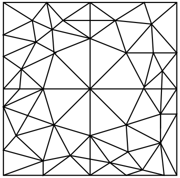


$|\mathcal{T}_\mathbf{a}| = 8$

Subdomain potential reconstruction $\overline{s}_{h\tau,i}^{k+1}$

Captures the scheme potential nonconformity in each subdomain

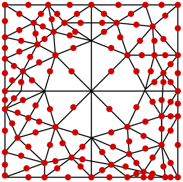
- $\overline{s}_{h,i}^{k+1,n} \in H^1(\Omega_i) \qquad \forall i \in [\![1, \mathcal{N}]\!]$,



Subdomain potential reconstruction $\overline{s}_{h\tau,i}^{k+1}$

Captures the scheme potential nonconformity in each subdomain

- $\overline{s}_{h,i}^{k+1,n} \in H^1(\Omega_i) \qquad \forall i \in [\![1, \mathcal{N}]\!]$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) = s_{h,i}^{k+1,n}(\mathbf{a})$,



Subdomain potential reconstruction $\overline{s}_{h\tau,i}^{k+1}$

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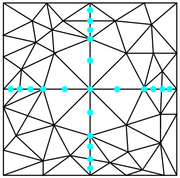
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) =$
$w_{i,\mathbf{a}}^{k+1,n} \sum_{K\in\mathcal{T}_{\mathbf{a}}^i} \tilde{p}_{h,i}^{k+1,n}|_K(\mathbf{a}) + w_{i,\mathbf{a}}^{k+1,n}(1-\overline{w}_{\mathbf{a}}^{k+1,n}) \sum_{j\in\tilde{B}^i} \sum_{K\in\mathcal{T}_{\mathbf{a}}^j} \tilde{p}_{h,j}^{k+1,n}|_K(\mathbf{a}), \quad \mathbf{a} \subset \Gamma_i.$

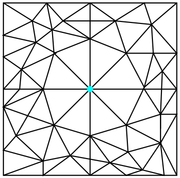


Subdomain potential reconstruction $\overline{s}_{h\tau,i}^{k+1}$

Captures the scheme potential nonconformity in each subdomain

- $\overline{s}_{h,i}^{k+1,n} \in H^1(\Omega_i) \qquad \forall i \in [\![1, \mathcal{N}]\!]$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) = s_{h,i}^{k+1,n}(\mathbf{a})$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) =$
  $$w_{i,\mathbf{a}}^{k+1,n} \sum_{K \in \mathcal{T}_{\mathbf{a}}^i} \tilde{p}_{h,i}^{k+1,n}|_K(\mathbf{a}) + w_{i,\mathbf{a}}^{k+1,n}(1 - \overline{w}_{\mathbf{a}}^{k+1,n}) \sum_{j \in \tilde{B}^i} \sum_{K \in \mathcal{T}_{\mathbf{a}}^j} \tilde{p}_{h,j}^{k+1,n}|_K(\mathbf{a}), \quad \mathbf{a} \subset \Gamma_i.$$

☞ Redistribute nonuniform weights **that depend on the mean jump of $\tilde{p}_{h,i}^{k+1,n}$**.

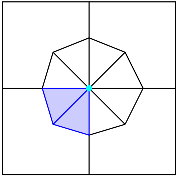


Subdomain potential reconstruction $\overline{s}^{k+1}_{h\tau,i}$

Captures the scheme potential nonconformity in each subdomain

- $\overline{s}^{k+1,n}_{h,i} \in H^1(\Omega_i) \qquad \forall i \in [\![1, \mathcal{N}]\!]$,
- $\overline{s}^{k+1,n}_{h,i}(\mathbf{a}) = s^{k+1,n}_{h,i}(\mathbf{a})$,
- $\overline{s}^{k+1,n}_{h,i}(\mathbf{a}) =$
$$\underbrace{w^{k+1,n}_{i,\mathbf{a}}}_{\frac{1}{|\mathcal{T}^i_{\mathbf{a}}|}} \sum_{K \in \mathcal{T}^i_{\mathbf{a}}} \tilde{p}^{k+1,n}_{h,i}|_K(\mathbf{a}) + \underbrace{w^{k+1,n}_{i,\mathbf{a}}(1 - \overline{w}^{k+1,n}_{\mathbf{a}})}_{0} \sum_{j \in B^i} \sum_{K \in \mathcal{T}^j_{\mathbf{a}}} \tilde{p}^{k+1,n}_{h,j}|_K(\mathbf{a}), \quad \mathbf{a} \subset \Gamma_i$$

at the beginning of the DD algorithm (k=0)



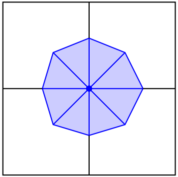
Subdomain potential reconstruction $\overline{s}_{h\tau,i}^{k+1}$

Captures the scheme potential nonconformity in each subdomain

- $\overline{s}_{h,i}^{k+1,n} \in H^1(\Omega_i) \qquad \forall i \in [\![1, \mathcal{N}]\!]$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) = s_{h,i}^{k+1,n}(\mathbf{a})$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) =$

  $$\underbrace{w_{i,\mathbf{a}}^{k+1,n}}_{\frac{1}{|\mathcal{T}_\mathbf{a}|}} \sum_{K \in \mathcal{T}_\mathbf{a}^i} \tilde{p}_{h,i}^{k+1,n}|_K(\mathbf{a}) + \underbrace{w_{i,\mathbf{a}}^{k+1,n}(1 - \overline{w}_\mathbf{a}^{k+1,n})}_{\frac{1}{|\mathcal{T}_\mathbf{a}|}} \sum_{j \in B^i} \sum_{K \in \mathcal{T}_\mathbf{a}^j} \tilde{p}_{h,j}^{k+1,n}|_K(\mathbf{a}), \quad \mathbf{a} \subset \Gamma_i$$

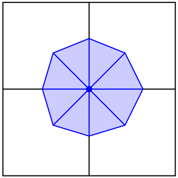
  at convergence of the DD algorithm $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) = s_{h,i}^{k+1,n}(\mathbf{a}) \cdots \eta_{\mathrm{DDP},K}$ disappears.



Subdomain potential reconstruction $\overline{s}_{h\tau,i}^{k+1}$

Captures the scheme potential nonconformity in each subdomain

- $\overline{s}_{h,i}^{k+1,n} \in H^1(\Omega_i) \qquad \forall i \in [\![1,\mathcal{N}]\!]$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) = s_{h,i}^{k+1,n}(\mathbf{a})$,
- $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) =$
  $$\underbrace{w_{i,\mathbf{a}}^{k+1,n}}_{\frac{1}{|\mathcal{T}_\mathbf{a}|}} \sum_{K\in\mathcal{T}_\mathbf{a}^i} \tilde{p}_{h,i}^{k+1,n}|_K(\mathbf{a}) + \underbrace{w_{i,\mathbf{a}}^{k+1,n}(1-\overline{w}_\mathbf{a}^{k+1,n})}_{\frac{1}{|\mathcal{T}_\mathbf{a}|}} \sum_{j\in B^i} \sum_{K\in\mathcal{T}_\mathbf{a}^j} \tilde{p}_{h,j}^{k+1,n}|_K(\mathbf{a}), \quad \mathbf{a}\subset\Gamma_i$$
  at convergence of the DD algorithm $\overline{s}_{h,i}^{k+1,n}(\mathbf{a}) = s_{h,i}^{k+1,n}(\mathbf{a}) \ \cdots \ \eta_{\mathrm{DDP},K}$ disappears.



Add bubble function to ensure $(\overline{s}^{k+1,n}, 1)_K = (\tilde{p}_{h,i}^{k+1,n}, 1)_K, \quad \forall K \in \mathcal{T}_h, \quad 0 \le n \le N$.

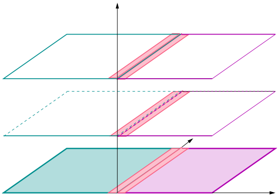
Equilibrated flux reconstruction $\boldsymbol{\sigma}_{h\tau}^{k+1}$

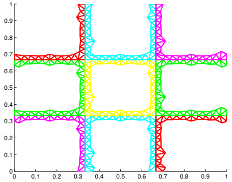
$$\begin{aligned}\boldsymbol{\sigma}_{h\tau}^{k+1} &\in P_\tau^0(\mathbf{H}(\mathrm{div},\Omega)),\\ (\nabla\cdot\boldsymbol{\sigma}_h^{k+1,n},1)_K &= (\tilde{f}^n - \partial_t \tilde{p}_h^{k+1,n},1)_K, \qquad \forall K \in \mathcal{T}_h.\end{aligned}$$

1. Solve the following system for $\mathcal{N}$ balancing conditions at each time step n:

$$\sum_{\substack{b=1,2/\\ |\partial\Omega_i\cap\partial\Omega|>0}} c_{\Gamma_i^b}^{k+1,n} + \sum_{j\in B^i} (\boldsymbol{n}_{\Gamma_{i,j}}\cdot\boldsymbol{n}_{\partial\Omega_i^{\mathrm{ext}}}) c_{\Gamma_{i,j}}^{k+1,n} = (\tilde{f}^n - \partial_t\tilde{p}_h^{k+1,n},1)_{\Omega_i^{\mathrm{ext}}} - \langle \{\!\{\mathbf{u}_h^{k+1,n}\cdot\boldsymbol{n}_{\partial\Omega_i^{\mathrm{ext}}}\}\!\},1\rangle_{\partial\Omega_i^{\mathrm{ext}}}$$

2. Then solve local Neumann problems in bands near the interface with the corrections on the interfaces in order to obtain the flux reconstruction in the bands.

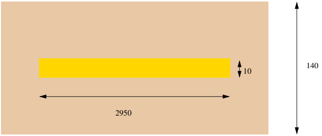


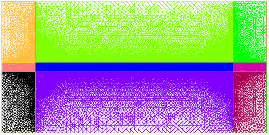


ANDRA test case

$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \boldsymbol{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m}/s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m}/s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5} \text{ years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$



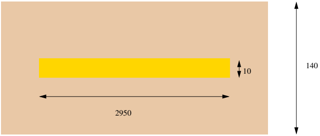


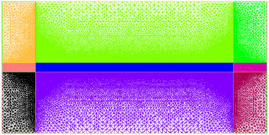
Mesh $|\mathcal{T}_h| = 106638$

ANDRA test case

$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \boldsymbol{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m/}s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m/}s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5} \text{ years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





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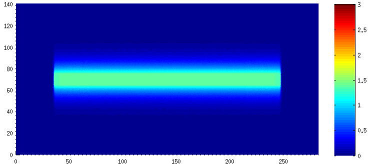
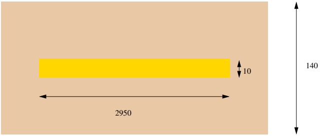


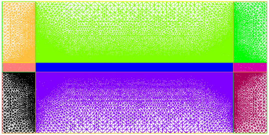
t=20000

ANDRA test case

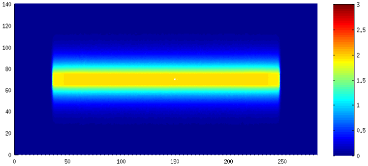
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \boldsymbol{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m/}s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m/}s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5} \text{ years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





Mesh $|\mathcal{T}_h| = 106638$

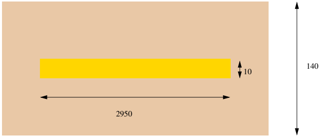


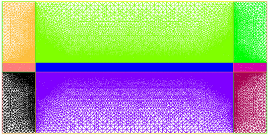
t=60000

ANDRA test case

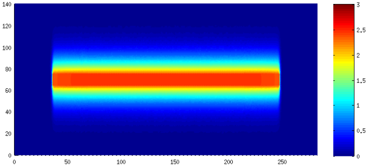
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \boldsymbol{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m/}s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m/}s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5}\ \text{years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





Mesh $|\mathcal{T}_h| = 106638$

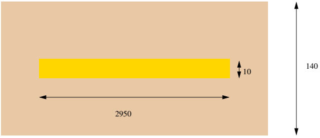


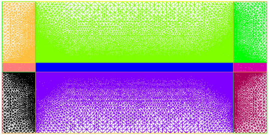
t=80000

ANDRA test case

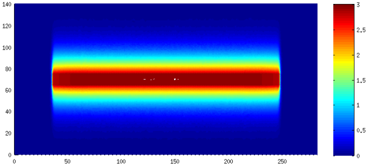
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \mathbf{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m/}s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m/}s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5}\ \text{years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





Mesh $|\mathcal{T}_h| = 106638$

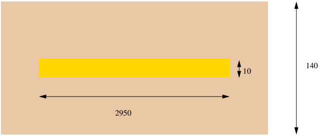


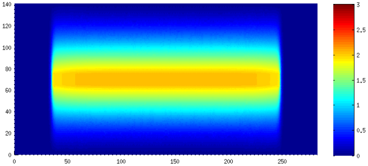
t=100000

ANDRA test case

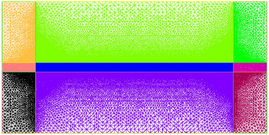
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \mathbf{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m}/s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m}/s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5} \text{ years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





t=200000

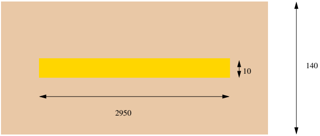


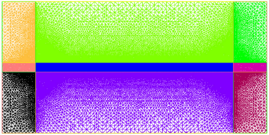
Mesh $|\mathcal{T}_h| = 106638$

ANDRA test case

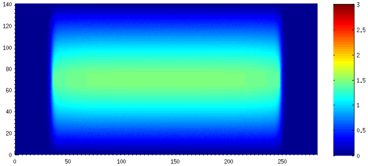
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \mathbf{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m}/s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m}/s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5}\ \text{years}^{-1} & \text{if } t \le 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





Mesh $|\mathcal{T}_h| = 106638$

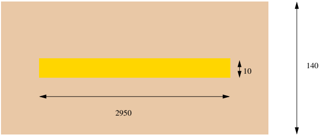


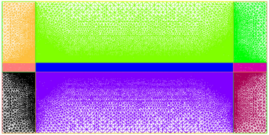
t=400000

ANDRA test case

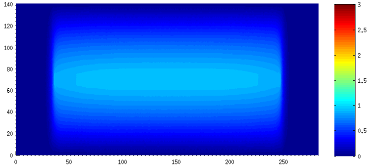
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \boldsymbol{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m/}s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m/}s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5} \text{ years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





Mesh $|\mathcal{T}_h| = 106638$

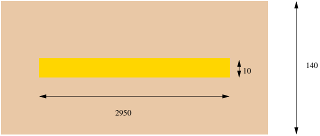


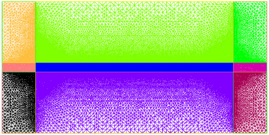
t=800000

ANDRA test case

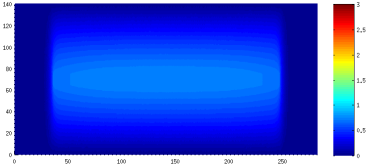
$T = 10^6$ years, $\mathcal{N} = 9$ domains, the repository Ω_r is the yellow one

$$\phi = \begin{cases} 0.2 & \text{in } \Omega_r \\ 0.05 & \text{else} \end{cases}, \quad \boldsymbol{S} = \begin{cases} 2\,10^{-9}\mathbf{I} & \text{m/}s^2 \text{ in } \Omega_r \\ 5\,10^{-12}\mathbf{I} & \text{m/}s^2 \text{ else} \end{cases}, \quad f = \begin{cases} 10^{-5} \text{ years}^{-1} & \text{if } t \leq 10^5 \text{ years} \\ 0 & \text{else} \end{cases}$$





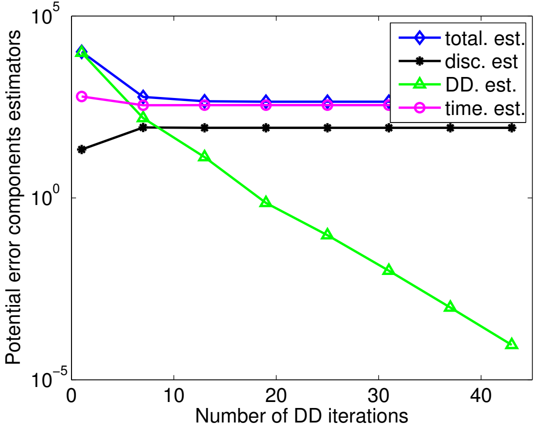
Mesh $|\mathcal{T}_h| = 106638$



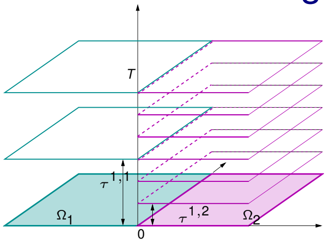
t=1000000

Stopping criterion

- $\tau = 4000$ years
- Relative DD stopping criterion : 10^{-6} $\Rightarrow$ iterations: 44 (GMRES)
- **A posteriori stopping criterion:** $\eta_{DD} \leq 0.1 \max(\eta_{tm}, \eta_{sp})$ $\Rightarrow$ iterations: 11
- **Iterations saved**: $\approx$ 75% (GMRES)



Global-in-time DD using nonconforming time grids

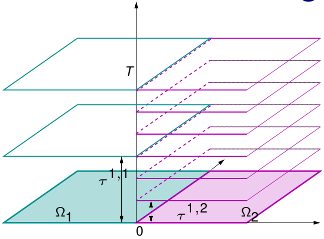


- $\{t^{n,i}\}_{0 \le n \le N_i}$ discrete times of Ω_i, $i \in [\![1, \mathcal{N}]\!]$:

$$t^{0,i} = 0 < t^{1,i} < \cdots < t^{N_i,i} = T.$$

- $\{t^{n,i}\}_{0 \le n \le N_i} \neq \{t^{n,j}\}_{0 \le n \le N_j}, \quad j \in B^i.$
- Information on one time grid at the interface is passed to the other time grid using L^2-projections.
  Following
  [Hoang-Jaffré-Japhet-Kern-Roberts (13)]

Global-in-time DD using nonconforming time grids



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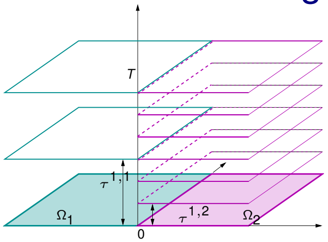
A posteriori estimates for NC time grids

- $\{t^n_{\mathbf{new}}\}_{0\le n\le N_{\mathbf{new}}}$ discrete times: $\forall \Omega_i$ $i \in [\![1, \mathcal{N}]\!]$:

$$t^0_{\mathbf{new}} = 0 < t^1_{\mathbf{new}} < \cdots < t^{N_{\mathbf{new}}}_{\mathbf{new}} = T.$$

- This new sequence is defined as:
  $\{t^n_{\mathbf{new}}\}_{0\le n\le N_{\mathbf{new}}} = \bigcup_{i=1}^{\mathcal{N}} \{t^{n,i}\}_{0\le n\le N_i}.$

Global-in-time DD using nonconforming time grids



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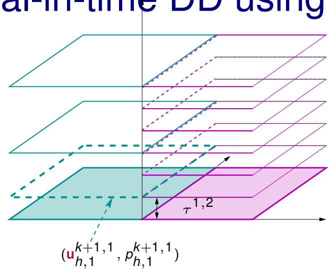
- Do a linear interpolation

Global-in-time DD using nonconforming time grids



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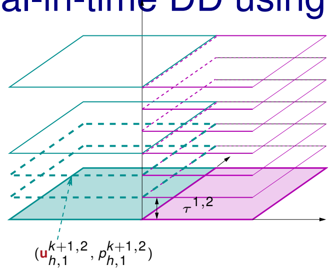
Global-in-time DD using nonconforming time grids



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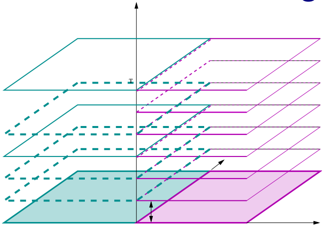
Global-in-time DD using nonconforming time grids



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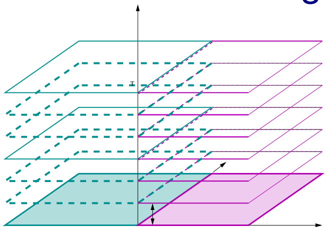
Global-in-time DD using nonconforming time grids



- $\{t^{n,i}\}_{0 \le n \le N_i}$ discrete times of Ω_i, $i \in [\![1, \mathcal{N}]\!]$:

$$t^{0,i} = 0 < t^{1,i} < \cdots < t^{N_i,i} = T.$$

- $\{t^{n,i}\}_{0 \le n \le N_i} \neq \{t^{n,j}\}_{0 \le n \le N_j}, \quad j \in B^i.$
- Information on one time grid at the interface is passed to the other time grid using L^2-projections.
  Following
  [Hoang-Jaffré-Japhet-Kern-Roberts (13)]

A posteriori estimates for NC time grids

- $\{t^n_{\mathbf{new}}\}_{0 \le n \le N_{\mathbf{new}}}$ discrete times: $\forall \Omega_i$ $i \in [\![1, \mathcal{N}]\!]$:

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- This new sequence is defined as:
  $\{t^n_{\mathbf{new}}\}_{0 \le n \le N_{\mathbf{new}}} = \bigcup_{i=1}^{\mathcal{N}} \{t^{n,i}\}_{0 \le n \le N_i}.$
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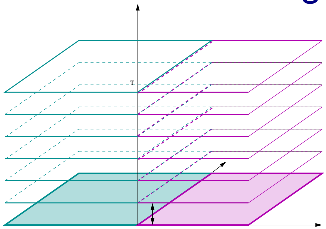

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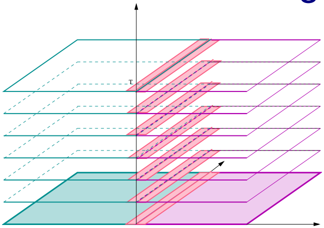

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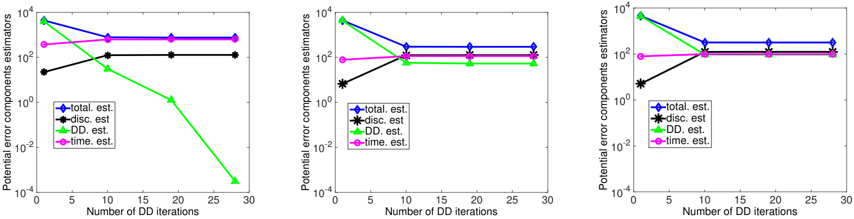
- $|||p-\tilde{p}_{h\tau}^{k+1}|||_Y \leq \eta_{\mathrm{sp}}^{k+1}+\eta_{\mathrm{tm}}^{k+1}+\eta_{\mathrm{DD},\mathbf{NC_{tm}}}^{k+1}+\eta_{\mathrm{IC}}^{k+1}+||f-\tilde{f}||_{X'}+||s_h^{k+1,N_{\mathbf{new}}}-\tilde{p}_h^{k+1,N_{\mathbf{new}}}||$
- $|||\mathbf{u}_h^{k+1,n}-\boldsymbol{\sigma}_h^{k+1,n}|||_{\star,K}$ is the source of this new **NC** discretization error in time.



Error component estimates for the Andra example with the GMRES solver for different ratios of discretization in time $\frac{N_5}{N_i}$, for $i \neq 5$: 1, 5, 10.

Conclusions

- The quality of the result is assured by controlling the error between the approximate solution and the exact solution at each iteration of the DD algorithm.
- Different components of the error have been distinguished.
- An efficient stopping criterion for the domain decomposition iterations has been established.
- Many of the domain decomposition iterations usually performed can be saved.

Future work

- Assess how much computing time can be saved
- Extend to advection-diffusion
- Study the local efficiency
- Develop an a posteriori coarse-grid corrector