Modeling fractures as interfaces for flow and transport in porous media

Clarisse Alboin [*] Jérôme Jaffré [†] Jean E. Roberts [‡] Christophe Serres [§]

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Abstract

Fractures in a porous medium are considered individually and are supposed to be a porous medium of higher permeability than in the surrounding rock. Since their thickness is supposed to be small with respect to the dimension of the domain of calculation they are modelled as interfaces. We formulate the flow and the transport in the medium, taking into account interaction between the fracture and the surrounding rock. We proved existence and uniqueness of the flow problem and give numerical experiments illustrating the model. The case of intersecting fractures is also considered.

1 Introduction

We are concerned with flow and transport in a fractured porous medium at a scale where the fractures can be modelled individually. The fractures themselves are porous media with large permeability in comparison with that in the surrounding rock. For a general description of fracture modeling we refer to [1] and for examples of numerical experiments with fractures see for instance [5, 6, 8]. Contrarily to many studies in which the contrast in permeabilities is of such an order that the flow outside of the fracture is neglected, the purpose of this work is to consider the case where the exchange between the fractures and the surrounding rock is significant. Then it is necessary to take into account this interaction because it has a profound effect on the flow and the transport of a solute.

The main idea for this work is to treat fractures as interfaces. Then it will not be necessary to use mesh refinements around the fractures, which is an important drawback of most models. Treating fractures as interfaces leads to nonoverlapping domain decomposition methods, using the natural domain decomposition suggested by the fracture network.

This paper is organized as follows. In Section 2, we present the model, and in Section 3, we show that the corresponding problem has a unique solution. In Section 4, we reduce the approximate problem to a problem with unknowns on the interface. Numerical results are given in Section 5 for the simple case of a domain divided into two subdomains by one fracture. In Section 6 we extend the formulation to the case of intersecting fractures and in Section 7 to that of a solute transport.

[*]Clarisse Alboin, Thales, Elancourt, France, Clarisse.Alboin@thales.fr

[†]Jérôme Jaffré, Inria-Rocquencourt, BP 105, 78153 Le Chesnay Cedex, France, Jerome.Jaffre@inria.fr

[‡]Jean E. Roberts, Inria-Rocquencourt, BP 105, 78153 Le Chesnay Cedex, France, Jean.Roberts@inria.fr

[§]Christophe Serres, IPSN, Fontenay aux Roses, France, Christophe.Serres@ipsn.fr

2 The flow model

Let Ω, be a domain in $\mathbb{R}^n$, $n = 2$ or 3, with boundary Γ, divided into two subdomains $\Omega_i, i = 1,2$ by a fracture γ as in Figure 1. γ is a surface of dimension $n-1$. We denote by $\Gamma_i = \partial\Omega \cap \partial\Omega_i$ the part of the boundary of the subdomains which coïncides with that of the whole domain. Also $\mathbf{n}_i$ denotes the unit outside normal to Γ_i, $\mathbf{n}$ that to γ pointing toward Ω_2, and $\mathbf{n}_\gamma$ that to $\partial\gamma$.

The domain Ω represents a porous medium. The fracture itself is supposed to be a porous medium with high permeability K_γ compared to that in the subdomains K_i. Its thickness d is supposed to be small with respect to the size of the domain Ω, so it is represented actually by an interface of dimension $\mathbb{R}^{n-1}$.

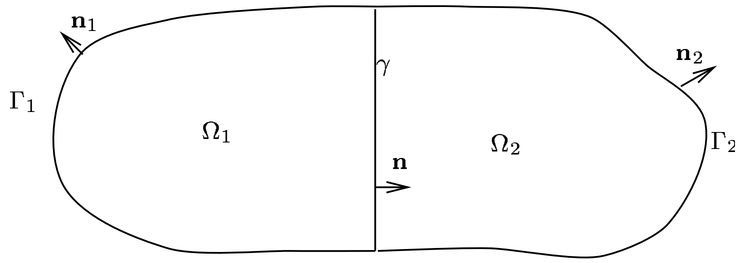

Figure 1: The domain Ω with the fracture γ as an interface

In the subdomains Ω_i as well as in the fracture γ the flow is assumed to be incompressible and to satisfy Darcy's law. We denote by $p_i, \mathbf{u}_i, p_\gamma, \mathbf{u}_\gamma$ the pressures and the Darcy velocities in the subdomains and in the fracture. Then the flow can be modelled by the following system of equations, in the subdomains,

$$\begin{aligned} \mathrm{div}_i\,\mathbf{u}_i &= q_i && \text{on } \Omega_i, \\ \mathbf{u}_i &= -K_i \boldsymbol{\nabla}\, p_i && \text{on } \Omega_i, \\ p_i &= p_{di} && \text{on } \Gamma_i, \\ p_i &= p_\gamma && \text{on } \gamma, \quad i = 1,2, \end{aligned} \tag{1}$$

and in the fracture

$$\begin{aligned} \mathrm{div}_\gamma\,\mathbf{u}_\gamma &= q_\gamma + (\mathbf{u}_1 \cdot \mathbf{n} - \mathbf{u}_2 \cdot \mathbf{n}) && \text{on } \gamma, \\ \mathbf{u}_\gamma &= -d\,K_\gamma \boldsymbol{\nabla}_\gamma\, p_\gamma && \text{on } \gamma, \\ p_\gamma &= p_{d\gamma} && \text{on } \partial\gamma, \end{aligned} \tag{2}$$

where the divergence and gradient operator along the fracture are defined as

$$\mathrm{div}_\gamma \mathbf{v} \;=\; \mathrm{div}\,\mathbf{v} - \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{n})\cdot\mathbf{n} \quad \text{and} \quad \boldsymbol{\nabla}_\gamma r = \boldsymbol{\nabla} r - \boldsymbol{\nabla} r \cdot \mathbf{n}.$$

The first equations (1),(2) represent volume conservation in the subdomains and in the fracture (volume conservation is equivalent to mass conservation since we assumed the incompressibility of the flow). $q_i, i = 1,2$ and q_γ are given source terms in the subdomains and in the fracture. One should notice that in the first equation (2) an extra source term $\mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n}$ appears in the righthand side. This source term represents the contribution of the subdomain flows to the fracture flow.

The second equations (1),(2) are Darcy's laws for the subdomains and the fracture. Note that $\mathbf{u}_\gamma$ gives the flow rate through the cross section of the fracture, which is the reason for the presence of the width d of the fracture in the righthand side of the second equation (2).

The third equations are Dirichlet boundary conditions on $\partial\Omega$ and on $\partial\gamma$. Of course we could also replace the Dirichlet conditions by Neumann conditions (given flow rate through the boundary) on part of these boundaries.

Finally the fourth equation (1) represents continuity of the pressure across the fracture γ. This continuity condition is physically valid when the permeability in the fracture is much larger than that in the subdomains.

3 Existence and uniqueness of a solution

To obtain a weak formulation of equations (1),(2), we introduce the Hilbert spaces

$$\begin{aligned}\mathbf{W} &= \{\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_\gamma) \in L^2(\Omega_1)^n \times L^2(\Omega_2)^n \times L^2(\gamma)^{n-1} : \\ &\qquad \text{div } \mathbf{u}_i \in L^2(\Omega_i), i = 1, 2,\ \text{div}_\gamma\, \mathbf{u}_\gamma - (\mathbf{u}_1 \cdot \mathbf{n} - \mathbf{u}_2 \cdot \mathbf{n}) \in L^2(\gamma)\} \\ M &= \{p = (p_1, p_2, p_\gamma) \in L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\gamma)\},\end{aligned}$$

and their norms

$$\|\mathbf{u}\|^2_{\mathbf{W}} = \sum_{i=1}^{2} \left(\|\mathbf{u}_i\|^2_{0,\Omega_i} + \|\text{div } \mathbf{u}_i\|^2_{0,\Omega_i}\right) + \|\mathbf{u}_\gamma\|^2_{0,\gamma} + \|\text{div}_\gamma \mathbf{u}_\gamma - (\mathbf{u}_1 \cdot \mathbf{n} - \mathbf{u}_2 \cdot \mathbf{n})\|^2_{0,\gamma}$$

$$\|p\|^2_M = \sum_{i=1}^{2} \|p_i\|^2_{0,\Omega_i} + \|p_\gamma\|^2_{0,\gamma}.$$

We also need the bilinear forms

$$\alpha : \mathbf{W} \times \mathbf{W} \to \mathbb{R} \qquad \text{and} \qquad \beta : \mathbf{W} \times M \to \mathbb{R}$$

defined by

$$\begin{aligned}\alpha(\mathbf{u}, \mathbf{v}) &= \sum_{i=1}^{2} \int_{\Omega_i} K_i^{-1} \mathbf{u}_i \cdot \mathbf{v}_i + \int_\gamma (dK_\gamma)^{-1} \mathbf{u}_\gamma \cdot \mathbf{v}_\gamma \\ \beta(\mathbf{u}, r) &= \sum_{i=1}^{2} \int_{\Omega_i} \text{div } \mathbf{u}_i\, r_i + \int_\gamma (\text{div}_\gamma \mathbf{u}_\gamma - (\mathbf{u}_1 \cdot \mathbf{n} - \mathbf{u}_2 \cdot \mathbf{n}))\, r_\gamma\end{aligned}$$

and the linear form $L : M \to \mathbb{R}$

$$L(r) = \sum_{i=1}^{2} \int_{\Omega_i} q_i\, r_i + \int_\gamma q_\gamma\, r_\gamma.$$

Assuming that the Dirichlet data are null, $p_{di} = p_{d\gamma} = 0, i = 1, 2$, the mixed formulation of equations (1),(2)) is

$$\begin{array}{lll}\text{Find } \mathbf{u} \in \mathbf{W}, p \in M \text{ such that} & & \\ \alpha(\mathbf{u}, \mathbf{v}) - \beta(\mathbf{v}, p) = 0 & & \forall \mathbf{v} \in \mathbf{W} \\ \beta(\mathbf{u}, r) = L(r) & & \forall r \in M.\end{array} \tag{3}$$

Theorem 1 *Assume that $0 < K_{\min} \le K_i, K_\gamma \le K_{\max}$. Then problem (3) has a unique solution.*

Proof: Let us introduce the subspace $\tilde{\mathbf{W}} = \{\mathbf{v} \in \mathbf{W} : \beta(\mathbf{v}, r) = 0 \quad \forall r \in M\}$. To show existence and uniqueness of the solution of (3), it is sufficient to show that α is $\tilde{\mathbf{W}}$-elliptic and that β satisfies the inf-sup condition (see [7, 2]), that is there exist constants C_α and C_β such that

$$\inf_{\mathbf{v}\in\tilde{\mathbf{W}}} \frac{\alpha(\mathbf{v},\mathbf{v})}{\|\mathbf{v}\|^2_{\mathbf{W}}} \ge C_\alpha, \qquad \inf_{r\in M} \sup_{\mathbf{v}\in\mathbf{W}} \frac{\beta(\mathbf{v},r)}{\|r\|_M \, \|\mathbf{v}\|_{\mathbf{W}}} \ge C_\beta.$$

To check that α is $\tilde{\mathbf{W}}$-elliptic, we notice that $\|\mathbf{u}\|^2_W = \sum_{i=1}^{2} \|\mathbf{u}_i\|^2_{0,\Omega_i} + \|\mathbf{u}_\gamma\|^2_{0,\gamma}$ for $\mathbf{u} \in \tilde{\mathbf{W}}$, so

$$\begin{aligned}
\alpha(\mathbf{u},\mathbf{u}) &= \sum_{i=1}^{2} \int_{\Omega_i} K_i^{-1}\mathbf{u}_i \cdot \mathbf{u}_i + \int_\gamma (dK_\gamma^{-1})\mathbf{u}_\gamma \cdot \mathbf{u}_\gamma \\
&\ge K_{\max}^{-1} \left(\sum_{i=1}^{2} \|\mathbf{u}_i\|^2_{0,\Omega_i} + \|\mathbf{u}_\gamma\|^2_{0,\gamma} \right) \\
&= K_{\max}^{-1} \|\mathbf{u}\|^2_W .
\end{aligned}$$

To see that β satisfies the inf-sup condition, given $r \in M$, using the adjoint equation we construct a $\mathbf{v} \in \mathbf{W}$ such that $\beta(\mathbf{v}, r) = \|r\|^2_M$ and $\|\mathbf{v}\|_{\mathbf{W}} \le C\|r\|_M$, where C is the constant of elliptic regularity for the adjoint problem.

For $r = (r_1, r_2, r_\gamma) \in M$, let $(\varphi_1, \varphi_2, \varphi_\gamma) \in H^2(\Omega_1) \times H^2(\Omega_2) \times H^2(\gamma)$ be the solution of

$$\begin{aligned}
-\triangle\varphi &= \tilde{r} \quad \text{on } \Omega \\
\varphi &= 0 \quad \text{on } \Gamma,
\end{aligned}$$

where $\tilde{r} \in L^2(\Omega)$ is given by $\tilde{r}_{|\Omega_i} = r_i$ and

$$\begin{aligned}
-\triangle_\gamma\varphi_\gamma &= r_\gamma \quad \text{on } \gamma \\
\varphi_\gamma &= 0 \quad \text{on } \partial\gamma.
\end{aligned}$$

Pose $\mathbf{v}_i = -\boldsymbol{\nabla}\varphi_{|\Omega_i}, i = 1, 2$, and $\mathbf{v}_\gamma = -\boldsymbol{\nabla}_\gamma \varphi_\gamma$ and note that $\operatorname{div} \mathbf{v}_i = r_i \in L^2(\Omega_i), i = 1, 2$, $\operatorname{div}_\gamma \mathbf{v}_\gamma = r_\gamma \in L^2(\gamma)$ and $\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n} = 0$. Thus $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_\gamma) \in \mathbf{W}$ and it is easy to check that $\mathbf{v}$ has the desired properties.

4 Formulation of the interface problem

In this section we formulate our problem as an interface problem where the unknowns are defined on the interface between subdomains. Such a formulation is useful for defining a domain decomposition method with nonoverlapping subdomains. Our formulation is an extension to the case with fractures of the formulation given in [4] for a standard elliptic equation. To approximate the problem we too use mixed finite element methods [7, 2].

Introduce a quasi regular triangulation $\mathcal{T}_h$ (of triangles and/or rectangles) of Ω compatible with the decomposition of Ω into the subdomains $\Omega_i, \; i = 1, 2$. Note that in this case a triangulation is induced on the interface γ. We let $M_h = M_{h,1} \times M_{h,2} \times M_{h,\gamma}$ and $\mathbf{W}_h = \mathbf{W}_{h,1} \times \mathbf{W}_{h,2} \times \mathbf{W}_{h,\gamma}$ be finite dimensional subspaces of $L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\gamma)$ and $H(\operatorname{div}; \Omega_1) \times$

$H(\mathrm{div};\Omega_2) \times H(\mathrm{div}_\gamma;\gamma)$ respectively, such that the pair $(M_{h,i},\, \mathbf{W}_{h,i})$ is a Raviart-Thomas space of order k for Ω_i, $i=1,2$, subordinate to the triangulation $\mathcal{T}_{h,i}$ determined by $\mathcal{T}_h$, and the pair $(M_{h,\gamma},\, \mathbf{W}_{h,\gamma})$ is a Raviart-Thomas space of order k for γ associated with the triangulation $\mathcal{T}_{h,\gamma}$ on γ induced by $\mathcal{T}_h$. Denote by $TM_{h,i}$ and $TM_{h,\gamma}$ the subspaces of traces of functions of $M_{h,i}$ and $M_{h,\gamma}$ where the boundary data p_{di} and $p_{d\gamma}$ lie.

Introduce the bilinear forms

$$\begin{array}{lcll}
\alpha_i(\mathbf{u},\mathbf{v}) & = & \displaystyle\int_{\Omega_i} K_i^{-1}\,\mathbf{u}\cdot\mathbf{v}, & \mathbf{u},\mathbf{v}\in\mathbf{W}_{h,i},\\
\beta_i(\mathbf{v},r) & = & \displaystyle\int_{\Omega_i} \mathrm{div}\,\mathbf{v}\,r, & \mathbf{v}\in\mathbf{W}_{h,i},\ r\in M_{h,i},\\
L_{\gamma i}(r,\mathbf{v}) & = & \displaystyle\int_{\gamma} r\,\mathbf{v}\cdot\mathbf{n}, & r\in M_{h,\gamma},\ \mathbf{v}\in\mathbf{W}_{h,i}\quad i=1,2,\\
\alpha_\gamma(\mathbf{u},\mathbf{v}) & = & \displaystyle\int_{\gamma} (d\,K_\gamma)^{-1}\,\mathbf{u}\cdot\mathbf{v}, & \mathbf{u},\mathbf{v}\in\mathbf{W}_{h,\gamma},\\
\beta_\gamma(\mathbf{v},r) & = & \displaystyle\int_{\gamma} \mathrm{div}_\gamma\mathbf{v}\,r, & \mathbf{v}\in\mathbf{W}_{h,\gamma},\ r\in M_{h,\gamma}.
\end{array}$$

For $q_i\in M_{h,i}, p_{di}\in TM_{h,i}, q_\gamma\in M_{h,\gamma}, p_{d\gamma}\in TM_{h,\gamma}$, we define the linear forms

$$\begin{array}{lcll}
L_{\sigma i}(q_i;r) & = & \displaystyle\int_{\Omega_i} q_i\,r, & r\in M_{h,i},\\
L_{\Gamma i}(p_{di};\mathbf{v}) & = & \displaystyle\int_{\Gamma_i} p_{di}\,\mathbf{v}\cdot\mathbf{n}_i, & \mathbf{v}\in\mathbf{W}_{h,i},\\
L_{\sigma\gamma}(q_\gamma;r) & = & \displaystyle\int_{\gamma} q_\gamma\,r, & r\in M_{h,\gamma},\\
L_{\partial\gamma}(p_{d\gamma};\mathbf{v}) & = & \displaystyle\int_{\partial\gamma} p_{d\gamma}\mathbf{v}\cdot\mathbf{n}_\gamma, & \mathbf{v}\in\mathbf{W}_{h,\gamma}.
\end{array}$$

Then we may write the approximate weak formulation of equations (1),(2):

$$\begin{array}{lcll}
\multicolumn{4}{l}{\mathbf{u}_i\in\mathbf{W}_{h,i},\ p_i\in M_{h,i},\ \mathbf{u}_\gamma\in\mathbf{W}_{h,\gamma},\ p_\gamma\in M_{h,\gamma},}\\
\alpha_i(\mathbf{u}_i,\mathbf{v})-\beta_i(\mathbf{v},p_i) & = & (-1)^i L_{\gamma i}(p_\gamma;\mathbf{v})+L_{\Gamma i}(p_{di},v) & \forall\mathbf{v}\in\mathbf{W}_{h,i},\\
\beta_i(\mathbf{u}_i,r) & = & L_{\sigma i}(q_i;r) & \forall r\in M_{h,i},\\
\alpha_\gamma(\mathbf{u}_\gamma,\mathbf{v})-\beta_\gamma(\mathbf{v},p_\gamma) & = & L_{\partial\gamma}(p_{d\gamma};\mathbf{v}) & \forall\mathbf{v}\in\mathbf{W}_{h,\gamma},\\
\beta_\gamma(\mathbf{u}_\gamma,r) & = & L_{\sigma\gamma}(q_\gamma;r)+ & \\
 & & L_{\gamma1}(r,\mathbf{u}_1)-L_{\gamma2}(r,\mathbf{u}_2) & \forall r\in M_{h,\gamma}.
\end{array}\tag{4}$$

Following [4, 3] we split $\mathbf{u}_i, p_i, \mathbf{u}_\gamma, p_\gamma$ into two parts

$$\mathbf{u}_i=\mathbf{u}_i^0+\mathbf{u}_i^1,\quad p_i=p_i^0+p_i^1,\quad \mathbf{u}_\gamma=\mathbf{u}_\gamma^0+\mathbf{u}_\gamma^1,\quad p_\gamma=p_\gamma^0+p_\gamma^1$$

so equations (4) can be rewritten as

$$\begin{array}{lcll}
\multicolumn{4}{l}{\mathbf{u}_i^0\in\mathbf{W}_{h,i},\ p_i^0\in M_{h,i},\ \mathbf{u}_\gamma^0\in\mathbf{W}_{h,\gamma},\ p_\gamma^0\in M_{h,\gamma},}\\
\alpha_i(\mathbf{u}_i^0,\mathbf{v})-\beta_i(\mathbf{v},p_i^0) & = & (-1)^i L_{\gamma i}(p_\gamma^0;\mathbf{v}) & \forall\mathbf{v}\in\mathbf{W}_{h,i},\\
\beta_i(\mathbf{u}_i^0,r) & = & 0 & \forall r\in M_{h,i},\\
\alpha_\gamma(\mathbf{u}_\gamma^0,\mathbf{v})-\beta_\gamma(\mathbf{v},p_\gamma^0) & = & 0 & \forall\mathbf{v}\in\mathbf{W}_{h,\gamma},\\
\beta_\gamma(\mathbf{u}_\gamma^0,r) & = & L_{\gamma1}(r,\mathbf{u}_1^0)-L_{\gamma2}(r,\mathbf{u}_2^0)+ & \\
 & & L_{\gamma1}(r,\mathbf{u}_1^1)-L_{\gamma2}(r,\mathbf{u}_2^1) & \forall r\in M_{h,\gamma}.
\end{array}\tag{5}$$

$$
\begin{array}{lcll}
\mathbf{u}_i^1 \in \mathbf{W}_{h,i},\ p_i^1 \in M_{h,i}, & & & \\
\alpha_i(\mathbf{u}_i^1, \mathbf{v}) - \beta_i(\mathbf{v}, p_i^1) & = & (-1)^i L_{\gamma i}(p_\gamma^1, \mathbf{v}) + L_{\Gamma i}(p_{di}; \mathbf{v}) & \forall \mathbf{v} \in \mathbf{W}_{h,i}, \\
\beta_i(\mathbf{u}_i^1, r) & = & L_{\sigma i}(q_i; r) & \forall r \in M_{h,i},
\end{array} \tag{6}
$$

$$
\begin{array}{lcll}
\mathbf{u}_\gamma^1 \in \mathbf{W}_{h,\gamma},\ p_\gamma^1 \in M_{h,\gamma}, & & & \\
\alpha_\gamma(\mathbf{u}_\gamma^1, \mathbf{v}) - \beta_\gamma(\mathbf{v}, p_\gamma^1) & = & L_{\partial\gamma}(p_{d\gamma}; \mathbf{v}) & \forall \mathbf{v} \in \mathbf{W}_{h,\gamma}, \\
\beta_\gamma(\mathbf{u}_\gamma^1, r) & = & L_{\sigma\gamma}(q_\gamma; r) & \forall r \in M_{h,\gamma}.
\end{array} \tag{7}
$$

We remark that equations (6),(7) can be solved separately in the subdomains and in the fracture, starting with that in the fracture (7), and that these equations can be solved prior to equations (5) where $\mathbf{u}_i^1, p_i^1,\ \mathbf{u}_\gamma^1, p_\gamma^1$ can then be considered as data calculated beforehand. .

We denote by $(\mathbf{u}_i^0(s_\gamma), p_i^0(s_\gamma), (\mathbf{u}_\gamma^0(s_\gamma), p_\gamma^0(s_\gamma))$ the solution of (5) for a given $p_\gamma^0 = s_\gamma$ in the righthand side of the first equation, and we define, for $i = 1, 2$, the following bilinear form on $M_{h,\gamma}$:

$$
\mathcal{A}_i(s_\gamma, r_\gamma) = \alpha_i(\mathbf{u}_i^0(s_\gamma), \mathbf{u}_i^0(r_\gamma)), \quad s_\gamma, r_\gamma \in M_{h,\gamma}.
$$

Using the symmetry of the forms α_i, the first two equations (5), and the decomposition of $\mathbf{u}_i$ into $\mathbf{u}_i^0$ and $\mathbf{u}_i^1$ we have,

$$
\begin{aligned}
\sum_{i=1}^{2} \mathcal{A}_i(p_\gamma^0, r_\gamma) &= \sum_{i=1}^{2} \alpha_i(\mathbf{u}_i^0(r_\gamma), \mathbf{u}_i^0(p_\gamma^0)) \\
&= \sum_{i=1}^{2} (-1)^i\, L_{\gamma i}(r_\gamma, \mathbf{u}_i^0(p_\gamma^0)) \\
&= \sum_{i=1}^{2} (-1)^i\, L_{\gamma i}(r_\gamma, \mathbf{u}_i(p_\gamma^0)) \; - \; \sum_{i=1}^{2} (-1)^i\, L_{\gamma i}(r_\gamma, \mathbf{u}_i^1).
\end{aligned}
$$

From the last two equations (5) we obtain

$$
\begin{aligned}
\sum_{i=1}^{2} (-1)^i\, L_{\gamma i}(r_\gamma, \mathbf{u}_i(p_\gamma^0)) &= -\,\beta_\gamma(\mathbf{u}_\gamma^0(p_\gamma^0), r_\gamma) \\
&= -\,\delta_\gamma(p_\gamma^0, r_\gamma),
\end{aligned}
$$

where the bilinear form $\delta_\gamma(.,.)$ is defined as

$$
\delta_\gamma(s_\gamma, r_\gamma) = < B_\gamma A_\gamma^{-1} B_\gamma^T\, s_\gamma, r_\gamma >, \quad s_\gamma, r_\gamma \in M_{h,\gamma}
$$

with A_γ and B_γ being the linear mappings respectively associated to the bilinear forms $\alpha_\gamma(.,.)$ and $\beta_\gamma(.,.)$.

Combining the last two equations we obtain our interface problem:

$$
\begin{array}{lcll}
p_\gamma^0 \in M_{h,\gamma} & & & \\
\sum_{i=1}^{2} \mathcal{A}_i(p_\gamma^0, r_\gamma) \; + \; \delta_\gamma(p_\gamma^0, r_\gamma) & = & \sum_{i=1}^{2} (-1)^{i+1} L_{\gamma i}(r_\gamma, \mathbf{u}_i^1) & \forall r_\gamma \in M_{h,\gamma}.
\end{array} \tag{8}
$$

The lefthand side of the above equation determines a symmetric, positive definite form on $M_{h,\gamma}$. Thus there exists a unique solution $p_\gamma^0 \in M_{h,\gamma}$ to which we need to add p_γ^1 given by equations (7). We remark that in the absense of a fracture, i.e. when the flux is continuous across the interface γ, $\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n}$, the second term in the righthand side of (8) vanishes (p_γ^1 is of course in this case equal to 0) and we obtain the standard interface problem given in [4, 3] for the case of two subdomains.

5 First numerical results

To illustrate the model we consider an ideal dimensionless problem. The domain is an horizontal rectangular slice of porous medium, of dimensions 2×1, with a given pressure on the left and right boundaries and no flow conditions on the top and bottom boundaries. In the domain the permeability is equal to one. The domain is divided into two equally large sub-domains by a linear fracture parallel to the x_2 axis. The permeability in the fracture $\times$ the width of the fracture is equal to 2. For example the fracture could be of width 0.1 and could have a permeability equal to 20. Flow in the fracture is driven by a pressure drop of 10 between the two extremities of the fracture for the first example and a pressure drop of 5 for the second example.

Two cases are considered. A symmetric case where pressures on the left and on the right boundaries of the domain are equal. So the flow is driven only by the fracture and is symmetric. In the other case there is a pressure drop from the right boundary to the left boundary. Then the flow is a combination of the flow in the fracture and that going from left to right in the rest of the porous medium.

Numerical results are shown on figure 2. Arrows represent the flow field with length proportional to the magnitude of the velocity. The gray scale represents also the magnitude of the velocity with the lightest color corresponding to the largest velocity. We see that there is actual flow interaction between the fracture and the rest of the porous medium. In particular one can observe that some fluid is coming out of the fracture and then is coming back into it. In the nonsymmetric case we notice also that even though most of the flow is attracted into the fracture, there is still some flow on the left part of the domain pointing toward the left.

Figure 2: Calculated Darcy's velocity for a symmetric and a nonsymmetric flow pattern

6 Intersecting fractures

We know extend the previous formulation to the case of intersecting fractures as in Figure 3. We denote by γ_{ij} the fracture separating the subdomains Ω_i and Ω_j, and by x_{ijk} the intersection of the fractures γ_{ij}, $\gamma_{jk}$and γ_{ik}.

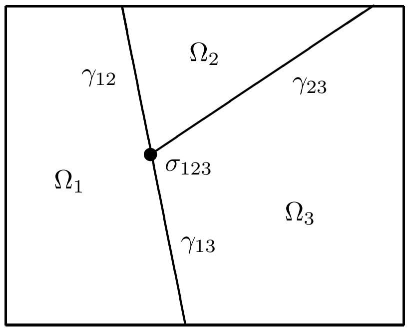

Figure 3: Intersecting fractures

Equations in the subdomains and in the fractures are as in equations (1),(2). However we need to add extra equations on σ_{ijk}, one is conservation of mass, the other being continuity of the pressure:

$$\mathbf{u}_{\gamma ij} + \mathbf{u}_{\gamma jk} + \mathbf{u}_{\gamma ik} = 0, \quad p_{\gamma ij} = p_{\gamma jk} = p_{\gamma ik}.$$

These equations are reasonable physically, continuity of pressure being justified when the permeabilities in the fractures are higher than in the subdomains.

We present two academic numerical examples demonstrating flow interaction between the subdomains and the intersecting fractures. A rectangular domain is divided into four subdomains by four fractures intersecting at the center of the rectangle. The domain has the same size as in Section 5 and its boundary supports the same boundary conditions: north and south boundaries are closed, except for the vertical fracture for which a pressure drop of 1 is imposed from the bottom end to the top end. In the subdomains $K_i = 1, i = 1, \ldots, 4$ while in the fractures $dK_\gamma = 2$. Results are shown in Figure 4. The left picture corresponds to the case where a pressure drop of 1 is imposed from left to right, while the right picture corresponds to the case where a pressure drop of 10 is imposed from left to right. Notice that in this case, due to this higher pressure drop, the direction of the flow in the lower vertical fracture is opposite to that corresponding to a lower pressure drop.

Figure 4: Calculated Darcy's velocity for intersecting fractures

7 The transport model

We now consider a solute which is transported by diffusion and by a flow calculated as in the previous section. We denote by $c_i, \boldsymbol{\varphi}_i, c_\gamma, \boldsymbol{\varphi}_\gamma$ the concentrations and the transport velocities of the solute in the subdomains and in the fracture. Then the solute transport can be modelled by the following set of equations, in the subdomains,

$$
\begin{array}{rcll}
\phi_i \dfrac{\partial c_i}{\partial t} + \operatorname{div} \boldsymbol{\varphi}_i & = & q_{c,i} & \text{on } \Omega_i, \\
\boldsymbol{\varphi}_i & = & -D_i \boldsymbol{\nabla}\, c_i + \mathbf{u}_i\, c_i & \text{on } \Omega_i, \\
c_i & = & c_{di} & \text{on } \Gamma_i, \\
c_i & = & c_\gamma & \text{on } \gamma, \\
c_i(x,0) & = & c_{0i} & \text{on } \Omega_i, \quad i = 1,2,
\end{array}
\tag{9}
$$

and in the fracture

$$
\begin{array}{rcll}
\phi_\gamma \dfrac{\partial c_\gamma}{\partial t} + \operatorname{div}_\gamma \boldsymbol{\varphi}_\gamma & = & q_{c\gamma} + (\boldsymbol{\varphi}_1 \cdot \mathbf{n} - \boldsymbol{\varphi}_2 \cdot \mathbf{n}) & \text{on } \gamma, \\
\boldsymbol{\varphi}_\gamma & = & -d\, D_\gamma \boldsymbol{\nabla}_\gamma\, c_\gamma + \mathbf{u}_\gamma\, c_\gamma & \text{on } \gamma, \\
c_\gamma & = & c_{d\gamma} & \text{on } \partial\gamma, \\
c_\gamma(x,0) & = & c_{0\gamma} & \text{on } \gamma,
\end{array}
\tag{10}
$$

The first equations (9),(10) represent volume conservation in the subdomains and in the fracture. $q_{ci}, i = 1,2$ and $q_{c\gamma}$ are given source terms in the subdomains and in the fracture, and $\phi_i, i = 1,2, \phi_\gamma$ the porosity in the subdomains and in the fracture. In the righthand side of the first equation (10) the extra source term $\boldsymbol{\varphi}_1 \cdot \mathbf{n} - \boldsymbol{\varphi}_2 \cdot \mathbf{n}$ represents the contribution of the subdomain to the solute transport in the fracture.

The second equations (9),(10) say that the solute is transported by diffusion and convection. $D_i, i = 1,2, D_\gamma$ are diffusion coefficients in the subdomains and in the fracture and the Darcy velocities $\mathbf{u}_i, i = 1,2, \mathbf{u}_\gamma$ are solution of equations (1),(2).

The third equations (9),(10) are Dirichlet boundary conditions on $\partial\Omega$ and on $\partial\gamma$, and the fifth equation (9) and the fourth equation (10) are initial conditions with $c_{di}, c_{0i}, i = 1,2, c_{d\gamma}, c_{0\gamma}$ being given concentrations. Finally the fourth equation (9) represents continuity of the concentration across the fracture γ.

We illustrate this model by a calculation for a situation represented in Figure 5. A

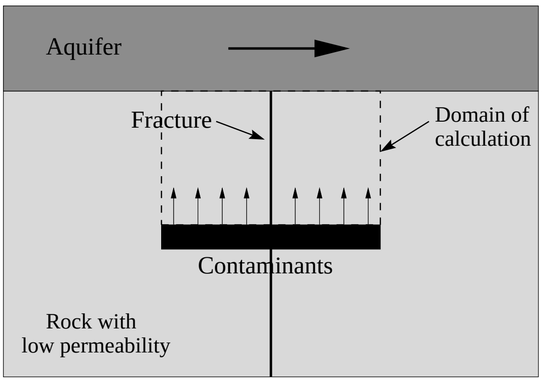

Figure 5: A contaminant storage crossed by a fracture

contaminant repository, located in a rock with low permeability, is leaking. The repository is crossed by a fracture and transported mostly upward. The rock is covered by an aquifer and the contaminant is assumed to be moved away instantly at the top boundary of the domain of calculation so the boundary condition there is a vanishing concentration. The actual physical parameters are given in table 1. The diffusion coefficient is of the form

$$D = d_m I + \alpha_L P_L(\mathbf{u}) + \alpha_T (I - P_L)(\mathbf{u}),$$

where d_m is the molecular diffusion and α_L, α_T are the longitudinal and transversal coefficients, I being the identity matrix in $\mathbb{R}^n$, and $P_L(\mathbf{u})$ being the projection onto the direction of the velocity $\mathbf{u}$.

Parameters	*Subdomains*	*Fracture*
Hydraulic conductivity ($m\, an^{-1}$)	$3.15\, 10^{-8}$	10^{-7}
Transversal dispersion (m)	1	-
Longitudinal dispersion (m)	0.1	10
Molecular diffusion ($m^2\, an^{-1}$)	10^{-5}	$3.15\, 10^{-4}$
Porosity	0.05	0.1
Subdomains dimensions (m)	10×10	-
Fracture width (m)	-	1

Table 1: Physical parameters for experience shown in Fig. 5

Boundary conditions are as follows. For velocity we assume that there is no horizontal flow on the lateral sides of the domain while a presure drop constant in time is given between the top and bottom boundaries. At the top the pressure is constant in space while at the bottom it is increasing slightly from the fracture toward the lateral sides. For concentration, it is given, constant, at the top and bottom boundaries, vanishing at the top. On the lateral sides we assume that there is no exchange with the outside.

The calculated velocity is shown in Figure 6 and the calculated concentration at different times is shown in Figure 7. In Figure 6 the grey shades represent pressure values while in Figure 7 they represent concentration values.

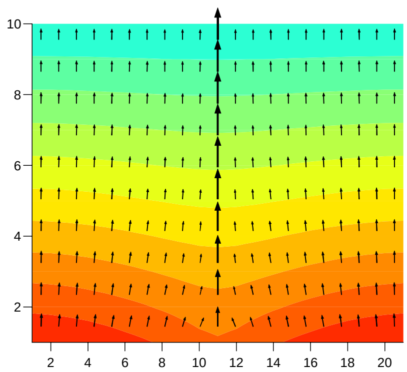

Figure 6: Calculated Darcy velocity

The contaminant are quickly transported upward by the fracture, but there is also a slow spreading in the subdomains due to subdomain-fracture interaction. Remember also that the concentration is always vanishing at the top because of the presence of the aquifer.

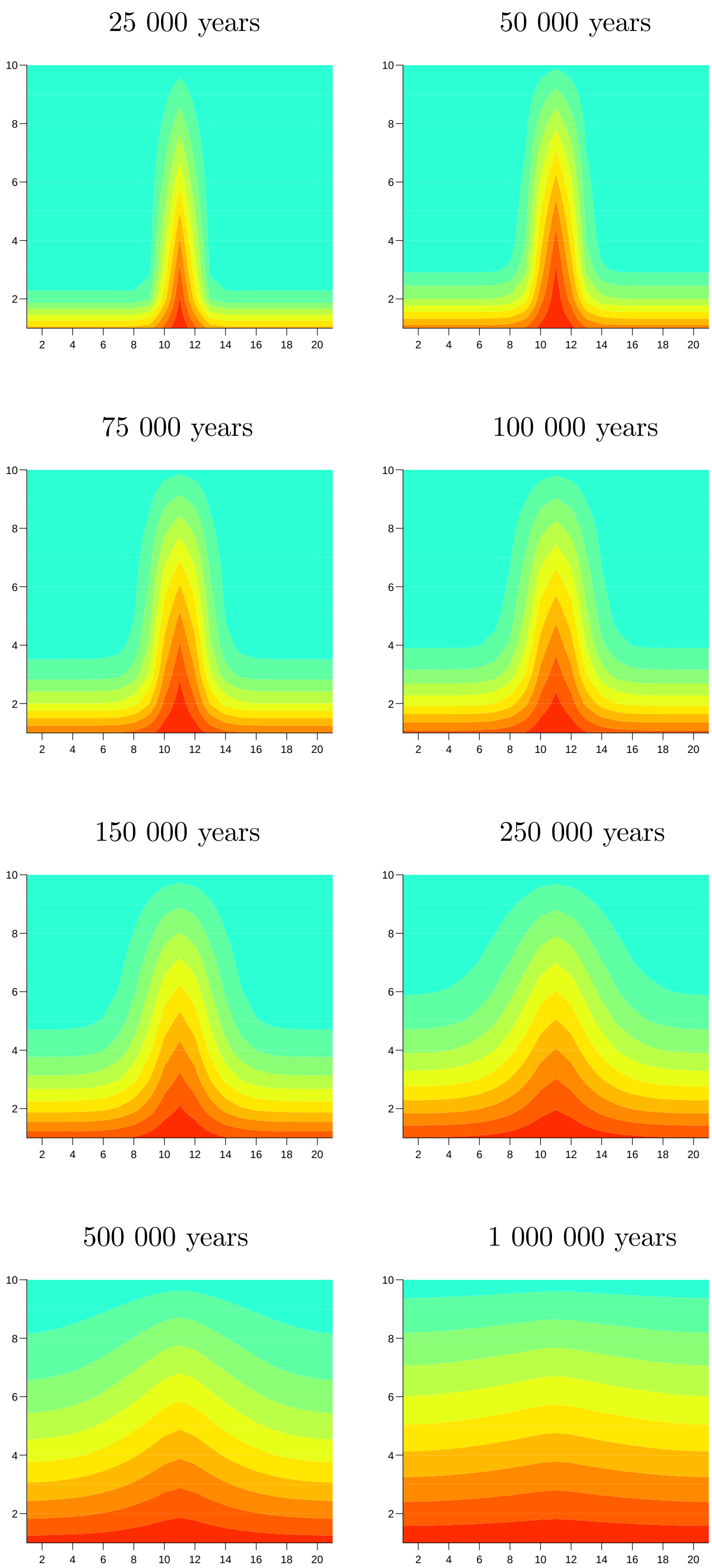

Figure 7: Calculated concentration in the subdomains and in the fracture

8 Conclusion

A model for flow interaction between a fracture and the rest of the porous medium has been presented. In this model the fracture is an interface dividing the domain of calculation into subdomains. Existence and uniqueness of the solution has been shown and the model has been reformulated as an interface problem. Extensions to the case of intersecting fractures and to solute transport have been presented. Simple numerical experiments illustrated actual flow and transport interactions between the fracture and the rest of the porous medium.

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