# Results on hypergraph planarity

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**Abstract.** Using the notion of planarity and drawing for hypergraphs introduced respectively by Johnson and Pollak [9] and Mäkinen [14], we show in this paper that any hypergraph having less than nine hyperedges is vertex-planar and can be drawn in the edge standard and in the subset standard without edge crossing.

**Key words:** hypergraphs, planarity, vertex-planarity, drawing in the edge standard, drawing in the subset standard, Euler diagrams

## 1 Introduction

Hypergraphs can be viewed as generalizations of graphs: a hypergraph is an ordered pair $(V, \mathcal{E})$ where $V$ is a set of vertices and $\mathcal{E}$ is a set of hyperedges, each hyperedge being a subset of $V$.
Drawings of hypergraphs are used in various contexts as, for example, vlsi design [1], databases [6, 4, 3] and information visualization [8]. Obtaining "good looking" layouts of hypergraphs becomes an important point in information visualization where readable cartographies may constitute an useful help to access huge sets of documents. In this context, we want to create Euler-like diagrams to represent the interconnections of a collection of semantic fields (cf. [19, 17] for a more detailed description of our purpose). As the number of hyperedges corresponds to the number of semantic fields, we need to study the existence of planar hypergraphs layouts with respect to the number of hyperedges.
In this paper, we study the following problem:
*what is the maximum number of hyperedges a hypergraph can contain to ensure that it has a planar representation ?*
After having described in section 1.1 the existing graphical representations for hypergraphs, we show in section 1.2 the equivalence between the notions of vertex-planarity and planar drawing in edge standard for hypergraphs. Finally, we prove in section 2 that any hypergraph having less than nine hyperedges is vertex planar.

### 1.1 Graphical representation of hypergraphs

Several graphical representations have been introduced for hypergraphs (see figure 1 for an example):

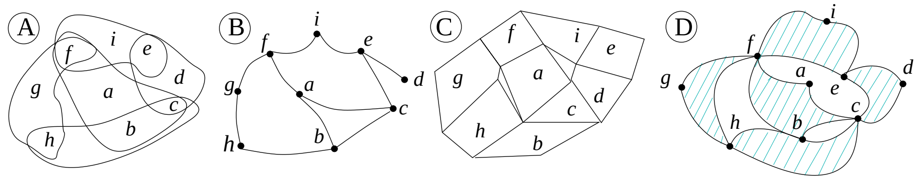


**Fig. 1.** The hypergraph $H = (V, \mathcal{E})$ with $V = \{a, b, c, d, e, f, g, h, i\}$ and $\mathcal{E} = \{\{a, b, c, f\}, \{c, d, e\}, \{e, f, i\}, \{f, g, h\}, \{b, c, h\}\}$ is drawn in the subset standard (A) and in the edge standard (B). A *vertex-based Venn diagram* representing $H$ is drawn in (C). The hypergraph $H$ is Zykov-planar and its representation is drawn in (D).

- Zykov, as described in [9], considers that a hypergraph is planar when it can be represented by the faces of a planar map, the vertices belonging to the boundaries of the faces (cf. [7] for a study on the minimal non planar hypergraphs using Zykov's definition of planarity).
- Johnson and Pollak [9] introduce two notions of planarity for hypergraphs, based on dual generalizations of Venn diagrams: the edge-planarity and the vertex-planarity. They show that the general problem of determining whether a hypergraph is (vertex- / edge-) planar is NP-complete (other complexity results related to vertex-planarity and hypergraphs can be found in [1]). Let us recall the definition of vertex-planarity [9]:
  "Given a hypergraph $H = (V, \mathcal{E})$, a *vertex-based Venn diagram representing* $H$ consists of a planar graph $G$, an embedding $M$ of $G$ into the plane, and a one-to-one map from the set $V$ of vertices of $H$ to the set of faces of $M$, such that for each hyperedge $e \in \mathcal{E}$, the union of the faces corresponding to vertices in $e$ comprises a region of the plane whose interior is connected. A hypergraph is said *vertex-planar* if there is a *vertex-based Venn diagram* that represents it."
  As noticed in [15, 12, 16, 5], the name *Venn* corresponds to diagrams containing $2^n$ regions when $H$ has $n$ hyperedges. The vertex-based Venn diagrams defined are similar to the *extended Euler diagrams* introduced in [18, 19].
- Mäkinen [14] introduces two types of drawings for hypergraphs: for "the edge standard", the vertices belonging to a same hyperedge are connected together (in [10] an implementation of a edge standard drawing method is described); for the "subset standard", hyperedges are represented by planar regions bounded by curves (see [2] for a description of a drawing system based on this definition).

In fact, the notion of Zykov-planarity and vertex-planarity can be considered as a generalization of the planarity's notion for graphs because we have [9]: a graph $G$ is planar in the ordinary sense if and only if it is vertex-planar (resp. Zykov-planar) when viewed as a hypergraph. Then, as the graph $K_{3,3}$ is the non planar graph having the smallest number of edges, which is equal to nine, we know that there is at least one non vertex-planar hypergraph having nine hyperedges (we use the notations of [13] for $K_{3,3}$ and $K_5$ and Kuratowski's theorem [11]).
Thus, as we want to know the lower bound $n$ on the number of hyperedges such that the following assertion is true:

"all the hypergraphs having at most $n$ hyperedges are vertex-planar".

We already know that $n$ is lower than nine.
We show in this paper that $n$ is equal to eight, i.e. that all the hypergraphs having at most

eight hyperedges are vertex-planar. This is made by a constructive proof in section 2. Let us first of all compare the notions of vertex-planarity and planar drawing in the edge standard for hypergraphs.

### 1.2 Vertex-planarity and drawing in the edge standard

Let $H = (V, \mathcal{E})$ be a hypergraph, with the set of vertices $V$ and the set of hyperedges $\mathcal{E}$. Following Mäkinen, we define an equivalence relation $r$ on $V$ by:
$vrv'$ if and only if for each edge $e$ in $\mathcal{E}$, $v \in e$ if and only if $v' \in e$.
Then the *condensation* of $H$ is the hypergraph $H' = (V', \mathcal{E}')$ in which $V'$ contains a vertex $v'$ for each equivalence class of $V$ w.r.t. $r$ and $\mathcal{E}'$ has an edge $e'$ with vertex $v' \in V'$ if and only if the corresponding edge $e$ in $\mathcal{E}$ contains $v'$ (cf. figure 2).

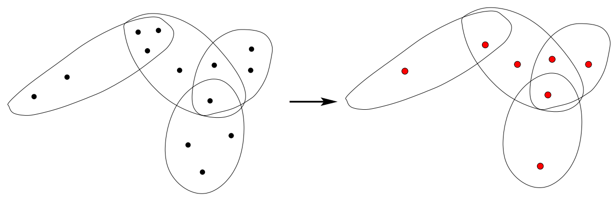

**Fig. 2.** A hypergraph and its condensation.

One can notice that the condensation of $H$ is vertex-planar if and only if $H$ is vertex-planar. Then, we will work with the condensation of $H$ instead of hypergraphs $H$ in the rest of the paper.

*Remark 1.* According to Mäkinen's definition, a drawing of $H$ in the edge standard is strictly constrained by the fact that a hyperedge is represented by a path of edges connecting all its vertices together. Then, when $H$ is vertex-planar, the adjacency graph of the regions forming *the vertex-based Venn diagram representing* $H$ is a drawing of $H$ in the edge standard. Conversely, when $H$ has a planar drawing in the edge standard, a vertex based Venn diagram representing $H$ can be build (cf. figure 3 (B) and (C) for an example). Thus, the two notions: "$H$ is vertex-planar" and "$H$ has a planar drawing in edge standard" are equivalent.

More formally, we say that a graph $G = (V, E)$ is *a representation of* $H = (V, \mathcal{E})$ *in the edge standard* when it satisfies: $\forall e \in \mathcal{E}$, the subgraph of $G$ induced by $e$ is connected. By extension, a graph $G' = (V', E')$, where $V' \subset V$, is *a representation of* $H$ *restricted to* $V'$ *in the edge standard* if $\forall e \in \mathcal{E}$, the subgraph of $G$ induced by $e \cap V'$ is connected. Then we can say that "*$H$ has a planar drawing in edge standard*" or "*$H$ is vertex-planar*" when there exists a planar graph $G = (V, E)$ which is *a representation of* $H = (V, \mathcal{E})$ *in the edge standard*.

## 2 The constructive proof

To show that a hypergraph $H = (V, \mathcal{E})$ having less than nine hyperedges is vertex planar, we will show how to build a planar representation of $H$ in the edge standard.
Let $H = (V, \mathcal{E})$, $v$ be a vertex of $V$ and $W$ a subset of $V$. We note: $e(v)$ the subset of $\mathcal{E}$

formed by the hyperedges containing $v$; $n_e(v)$ the number of hyperedges of $\mathcal{E}$ containing $v$; $e(W)$ the set of hyperedges of $\mathcal{E}$ containing at least a vertex of $W$; $e_u(v, W)$ the subset of $e(v)$ for which $v$ is the unique vertex in $W$, and $e_u(W) = \bigcup_{v \in W} e_u(v, W)$.

Given $H = (V, \mathcal{E})$, the condensation of a hypergraph having $n < 9$ hyperedges, to show how to build a planar graph representation of $H$ in the edge standard, we proceed as follows:

1. we first choose a subset $V_0$ of $V$ such that:
   (i) any hyperedge of $\mathcal{E}$ has a vertex in $V_0$
   (ii) $V_0$ has a minimal number of vertices
   (iii) $V_0$ is maximal w.r.t. the relation $\succeq$ : $W \succeq W'$ if the ordered list $(n_e(v))_{v \in W}$ is greater than the ordered list $(n_e(v'))_{v' \in W'}$ for the lexicographic order.
   (cf. example 1 for an illustration of those properties)
2. a planar *representation of $H$ restricted to $V_0$ in the edge standard*, $G_0 = (V_0, E_0)$, is built.
   The properties of $V_0$ and the planarity of $G_0$ are studied in section 2.1.
3. $V \backslash V_0$ is partitioned into $k$ sets of vertices $V_1$,...,$V_k$ such that $v \in V_i$ when $i$ is the minimum number of edges to be inserted in $G_0 = (V_0, E_0)$ to obtain a representation of $H$ restricted to $V_0 \cup \{v\}$ in the edge standard.
   The construction of the partition of $V \backslash V_0$ is described in section 2.2.
4. a planar *representation of $H$ in the edge standard* is built by inserting progressively the vertices of $V_k$,$V_{k-1}$, ..., $V_1$ in $G_0$.
   This is the subject of section 2.3.

### 2.1 Properties of $V_0$

In the following, $H = (V, \mathcal{E})$ is the condensation of a hypergraph having less than nine hyperedges and $V_0$ is a subset of $V$ satisfying (i),(ii) and (iii).
As $V_0$ has a minimal number of vertices, we have:

**Lemma 1.** $e_u(V_0)$ *contains at least* $card(V_0)$ *elements. and* $card(V_0) \leq card(\mathcal{E})$.

**Proposition 1.** *When* $card(\mathcal{E}) < 9$*, there is a planar* representation of $H$ restricted to $V_0$ in the edge standard*;*

*Proof.* The complete graph $K_{card(V_0)}$ on $V_0$ is a representation of $H$ restricted to $V_0$ in the edge standard. Let $G_0 = (V_0, E_0)$ be a subgraph of $K_{card(V_0)}$, with minimal number of edges, and being a representation of $H$ restricted to $V_0$ in the edge standard. We have the following cases to consider:
$-$ $\boldsymbol{card(V_0) \leq 4}$. $G_0 = (V_0, E_0)$ and is planar because we need at least 5 vertices to build a non planar graph.
$-$ $\boldsymbol{card(V_0) \geq 5}$. By definition, we have $card(e_u(V_0)) \geq 5$ then at most 3 hyperedges have to be represented by a path in $G_0$. As $G_0$ is minimal in number of edges among the representations of $H$ restricted to $V_0$ in the edge standard, it cannot contains any subdivision of $K_5$. When $card(V_0) \geq 6$, only 2 hyperedges have to be represented by a path in $G_0$, thus $G_0$ cannot contain a subdivision of $K_{3,3}$. □

In the rest of the paper, $G_0 = (V_0, E_0)$ denotes a planar representation of $H$ restricted to $V_0$ in the edge standard.

## 2.2 The partition of $V \backslash V_0$

Once a subset $V_0$ of $V$ satisfying (i), (ii) and (iii) is chosen, a partition of $V \backslash V_0$ is made, classifying the vertices $v$ of $V \backslash V_0$ with respect to the number of edges necessary to extend $V_0$ to $V_0 \cup \{v\}$ while keeping the property of being a representation of $H$ restricted to $V_0 \cup \{v\}$ in the edge standard. More precisely:
$V$ is partitioned into $k+1$ sets of vertices $V_0$,...,$V_k$. Each $V_i$, with $i > 0$ is such that $v \in V_i$ if and only if $i$ is the minimum number of edges to be inserted in $G_0 = (V_0, E_0)$ to obtain a representation of $H$ restricted to $V_0 \cup \{v\}$ in the edge standard. These edges connect $v$ and vertices of $V_0$. In the following, $W_i(v) = \{w_1, ..., w_t\}$ denotes a minimum subset of $V_0$ such that its vertices can be connected to $v$ to form a representation of $H$ restricted to $V_0 \cup \{v\}$ in the edge standard. We have: $card(W_i(v)) = i$.

**Example 1** *Consider the two hypergraphs (cf. figure 3):*
• *For the hypergraph of figure 1, we have* $\mathcal{E} = \{e_1, ..., e_5\}$ *with* $e_1 = \{a, b, c, f\}$, $e_2 = \{c, d, e\}$, $e_3 = \{e, f, i\}$, $e_4 = \{f, g, h\}$ *and* $e_5 = \{b, c, h\}$. *Then* $e(a) = \{e_1\}$, $e(b) = \{e_1, e_5\}$, $e(c) = \{e_1, e_2, e_5\}$, $e(d) = \{e_2\}$, $e(e) = \{e_2, e_3\}$, $e(f) = \{e_1, e_3, e_4\}$, $e(g) = \{e_4\}$ *and* $e(h) = \{e_4, e_5\}$. *The set of vertices $V$ is partitioned into three sets:* $V_0 = \{c, f\}$, $V_1 = \{a, b, d, g, i\}$ *and* $V_2 = \{e, h\}$ *(case A of figure 3).*
• *Suppose* $H = (V, \mathcal{E})$ *with* $V = \{a, b, c, d, e, f, g, h\}$ *and* $\mathcal{E} = \{e_1, ..., e_8\}$ *with* $e_1 = \{a, f, i\}$, $e_2 = \{b, g\}$, $e_3 = \{c, h\}$, $e_4 = \{d, i\}$, $e_5 = \{a, b, e\}$, $e_6 = \{b, c, f, h\}$, $e_7 = \{c, d, e, g\}$ *and* $e_8 = \{a, g\}$. *We have* $e(a) = \{e_1, e_5, e_8\}$, $e(b) = \{e_2, e_5, e_6\}$, $e(c) = \{e_3, e_6, e_7\}$, $e(d) = \{e_4, e_7\}$, $e(e) = \{e_7, e_5\}$, $e(f) = \{e_1, e_6\}$, $e(g) = \{e_2, e_5, e_8\}$, $e(h) = \{e_3, e_6\}$ *and* $e(i) = \{e_1, e_4\}$. *The sets of vertices* $W = \{a, b, c, d\}$ *and* $W' = \{a, g, c, i\}$ *both satisfy (i), (ii) and (iii): the ordered list of the* $(n_e(v))_{v \in W}$ *and the ordered list of the* $(n_e(v'))_{v' \in W'}$ *are both equal to* $(3, 3, 3, 2)$. *The set* $W'' = \{a, g, h, i\}$ *satisfies (i) and (ii) but not (iii) because the ordered list of the* $(n_e(v''))_{v'' \in W''}$ *is equal to* $(3, 3, 2, 2)$ *which is lower than the others for the lexicographic order. Let us take* $V_0 = \{a, b, c, d\}$. *Then* $V_1 = \{h\}$, $V_2 = \{e, f, i\}$ *and* $V_3 = \{g\}$. *Two different subsets of* $V_0$ *can be connected to* $f$ *with a minimal number of vertices:* $W_2(f)$ *can be equal to* $\{a, b\}$ *or to* $\{a, c\}$.

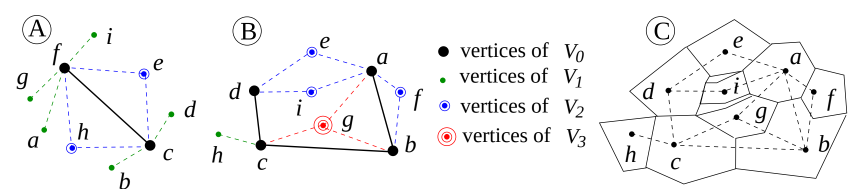


**Fig. 3.** The graph $G_0$ and the partition of $V$ into $V_0$, ..., $V_k$ for the two hypergraphs of example 1. The dashed lines connect vertices of $V_i$, $i > 0$, with vertices of $V_0$ according to $\mathcal{E}$. The solid lines represent the graph $G_0$. in (A), $V_0$ contains two vertices and $G_0$ contains one edge connecting them. In (B), three edges connect the four vertices of $V_0$. In (C), a vertex-based Venn diagram representing the second hypergraph is drawn.

Because of the minimality of $V_0$ and the definition of the $V_i$, we have:

**Lemma 2.** *(1) if $v$ is a vertex of $V_i$, $i > 0$, $e(v)$ contains at least $i$ hyperedges of* $e_u(W_i(v))$.

*(2) $V_i$ is empty when $i > card(V_0)$*
*(3) $V_i$ is empty when $i > card(\mathcal{E}) - card(V_0)$*

### 2.3 Extension of $G_0$

*Remark 2.* Suppose that $G'$ is a planar representation of $H$ restricted to $V' = V_0 \cup W$ in the edge standard, where $W$ is a subset of $V \backslash V_0$.

- a vertex $v$ of $V_1$ is inserted in $G'$ by adding an edge to obtain a graph $G''$, a representation of $H$ restricted to $V' \cup \{v\}$ in the edge standard. As $v$ is connected to only one vertex and $G'$ is planar, $G''$ is also planar. Thus any insertion of a vertex of $V_1$ can be made without breaking the planarity.

Without loss of generality, we will suppose in the rest of the paper that any vertex of $V$ is included in at least two hyperedges of $\mathcal{E}$. As a consequence, vertices of $V_0$ are included in at least two hyperedges of $\mathcal{E}$.

- a vertex $v$ of of $V_2$ is inserted in $G'$ by adding two edges joining $v$ with two vertices $w_1$ and $w_2$ of $V_0$. If $(w_1, w_2)$ is an edge of $G'$, this insertion leads to a planar representation of $H$ restricted to $V' \cup \{v\}$ in the edge standard. In the other cases, we must show that the edge $(w_1, w_2)$ does not break the planarity while being inserted into $G'$.

Considering the cardinalities of $\mathcal{E}$ and of $V_0$, we have the following cases to consider to show that the insertion of vertices of $V \setminus V_0$ leads to a planar representation of $H$ in the edge standard:

| $card(V_0)$ | $V \setminus V_0$ | | | | | |
|---|---|---|---|---|---|---|
| | $card(\mathcal{E}) = 8$ | $card(\mathcal{E}) = 7$ | $card(\mathcal{E}) = 6$ | $card(\mathcal{E}) = 5$ | $card(\mathcal{E}) = 4$ | $card(\mathcal{E}) = 3$ |
| 1 | $V_1$ | $V_1$ | $V_1$ | $V_1$ | $V_1$ | $V_1$ |
| 2 | $V_1 \cup V_2$ | $V_1 \cup V_2$ | $V_1 \cup V_2$ | $V_1 \cup V_2$ | $V_1 \cup V_2$ | $V_1$ |
| 3 | $\boldsymbol{V_1 \cup ... V_3}$ | $\boldsymbol{V_1 \cup ... V_3}$ | $\boldsymbol{V_1 \cup .. V_3}$ | $V_1 \cup V_2$ | $V_1$ | $\emptyset$ |
| 4 | $\boldsymbol{V_1 \cup ... V_4}$ | $\boldsymbol{V_1 \cup ... V_3}$ | $V_1 \cup V_2$ | $V_1$ | $\emptyset$ | $\emptyset$ |
| 5 | $\boldsymbol{V_1 \cup ... V_3}$ | $V_1 \cup V_2$ | $V_1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 6 | $V_1 \cup V_2$ | $V_1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 7 | $V_1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

*Remark 3.* Only the cases written in bold letters have to be examined precisely. The other cases are solved as follows:
- When $card(V_0) = 2$, all the vertices $v$ of $V_2$ are inserted in $G_0$ by adding two edges joining $v$ with the two vertices of $V_0$ without breaking the planarity of the resulting graph.
- When $card(\mathcal{E}) - card(V_0) = 2$, two hyperedges $e'$ and $e''$ of $\mathcal{E}$ contain several vertices of $V_0$ and $card(V_0)$ hyperedges of $\mathcal{E}$ contain only one vertex of $V_0$. Then $G_0$ is composed of at most two paths connecting the vertices of $V_0$ belonging to $e'$ (resp. $e''$). As $V_0$ has a minimal number of vertices, a vertex of $V_2$ cannot belong to more than one hyperedge of $e_u(V_0)$ and then it must belong to either $e'$ or $e''$. Thus the insertions of vertices of $V_2$ correspond to insertions of edges between vertices of $e'$ and vertices of $e''$ in $G_0$. These insertions can be made without breaking the planarity of $G_0$ (cf. figure 4).

Let us now study how vertices of $V_3$ can be inserted in $G_0$. Using the fact that $card(\mathcal{E}) < 9$ and that a vertex of $V_i$ is included in at least $i$ hyperedges of $\mathcal{E}$, we have:

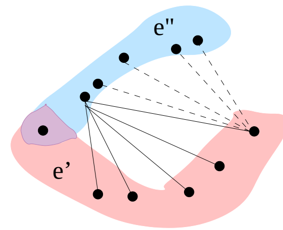


**Fig. 4.** When $card(\mathcal{E}) - card(V_0) = 2$, edges corresponding to insertion of vertices of $V_2$ included in $e'$ (resp. $e''$) are drawn in dashed (resp. solid) lines.

**Lemma 3.** *$V_3$ contains at most two vertices $v$ and $v'$ which are not both included in a same hyperedge of $\mathcal{E}$.*

*Remark 4.* When two vertices $v$ and $v'$ of $V_3$ are both linked to $W_3(v) = \{w_1, w_2, w_3\}$ and when the face defined by $W_3(v)$ do not contain any edge in $G_0$, if there exists $i$ in $\{1, 2, 3\}$ with $e(v) \cap e_u(w_i, W_3(v)) = e(v') \cap e_u(w_i, W_3(v))$ then these vertices can be inserted inside the face defined by $W_3(v)$ without edge crossing (cf figure 5 (A)).
- $v$ is inserted inside the face $(w_1, w_2, w_3)$ by adding three edges.
- by adding the edges $(v, v')$, $(w_2, v')$ and $(w_3, v')$, we obtain a planar representation of $H$ restricted to $V_0 \cup \{v, v'\}$ in the edge standard.

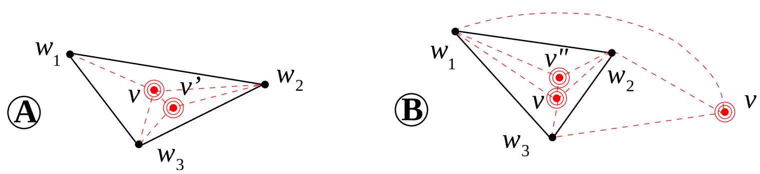


**Fig. 5.** The insertion is made by adding the dashed edges. A: insertion of $v$ and $v'$ inside the face $(w_1, w_2, w_3)$. B: when $V_0 = \{w_1, w_2, w_3\}$, the insertion of $v$, $v'$ and $v''$ can be made either inside or outside the face $(w_1, w_2, w_3)$.

**Proposition 2.** *When $card(V_0) = 3$ and $card(\mathcal{E}) < 9$, the insertions of vertices of $V \backslash V_0$ in $G_0$ lead to a planar representation of $H$ in the edge standard.*

*Proof.* Vertices of $V_3$ are inserted as follows (cf figure 5 (B)):
- a vertex $v$ of $V_3$ is inserted inside the face $(w_1, w_2, w_3)$ by adding three edges.
- if a vertex $v'$ of $V_3$, is such that for any $i = 1, 2, 3$, $e(v) \cap e_u(w_i, W_3(v)) \neq e(v') \cap e_u(w_i, W_3(v))$, then $v'$ is inserted outside the face defined by $V_0$ by adding three edges.
- if $v$ has been inserted inside $(w_1, w_2, w_3)$ and $v'$ inserted outside $(w_1, w_2, w_3)$, then, as $V_0$ has a minimum number of vertices and $card(\mathcal{E}) < 9$, any other vertex $v''$ of $V_3$ must satisfy : there exists $w_i$ in $V_0$ such that either $e_u(w_i, V_0) \cap e(v) = e_u(w_i, V_0) \cap e(v'')$ or $e_u(w_i, V_0) \cap e(v') = e_u(w_i, V_0) \cap e(v'')$. Then $v$" is inserted as described in remark 4.
We proceed similarly for the other vertices of $V_3$ and obtain a planar representation of $H$ restricted to $V_0 \cup V_3$ in the edge standard.
Vertices of $V_2$ are inserted by adding paths along the edges of the triangle defined by the three vertices of $V_0$. The resulting graph is a planar representation of $H$ in the edge standard. □

Now, to prove that any hypergraph having less than 9 hyperedges has a planar representation in the edge standard, we have the following cases to consider:

- $card(V_0) = 4$ and $card(\mathcal{E}) = 7$ or 8.
- $card(V_0) = 5$ and $card(\mathcal{E}) = 8$.

**Proposition 3.** *When $card(V_0) = 4$ and $card(\mathcal{E}) < 9$, the insertions of vertices of $V \backslash V_0$ in $G_0$ lead to a planar representation of $H$ in the edge standard.*

*Proof.* Let us consider the two cases separately:
**-A-** When $\boldsymbol{card(\mathcal{E}) = 7}$, we first prove that $V_4$ is empty and that the vertices of $V_3$ can always be inserted inside the faces defined by $W_3(v)$. Then, as $K_4$ is planar, the resulting graph will be a planar representation of $H$ in the edge standard.
$V_4$ is empty: suppose that $v$ is a vertex of $V_4$, then $v$ belongs to at least four hyperedges of $e_u(V_0)$. As $V_0$ satisfies (iii), $V_0$ contains a vertex $v_0$ which belongs to at least four hyperedges. There is only one hyperedge of $\mathcal{E}$ which contains both $v$ and $v_0$. Then the set $\{v, v_0\}$ satisfies (i) and contains only two vertices which contradicts the hypothesis on $V_0$.
If $V_3$ contains two vertices $v$ and $v'$ associated to $W_3(v) = \{w_1, w_2, w_3\}$ and such that for any $i = 1, 2, 3$, $e(v) \cap e_u(w_i, W_3(v)) \neq e(v') \cap e_u(w_i, W_3(v))$, then at most one hyperedge $e$ of $\mathcal{E}$ does not contain $v$ or $v'$. Let $v_0$ be a vertex of $V_0$ belonging to $e$. then the set $\{v, v', v_0\}$ satisfies (i) and contains only three vertices which contradicts the hypothesis on $V_0$.
**-B-** When $\boldsymbol{card(\mathcal{E}) = 8}$, $V_4$ may not be empty.
**-B-1.** Suppose $\boldsymbol{V_4}$ **contains a vertex** $\boldsymbol{v_4}$.
We will show that the insertion of $v_4$ in $G_0$ leads to a planar representation of $H$ restricted to $V_0 \cup \{v_4\}$ in the edge standard and that the insertion of vertices of $V \backslash (V_0 \cup \{v_4\})$ can be made by adding at most two edges to this graph.
When $v_4$ is a vertex of $V_4$, four hyperedges of $e_u(V_0)$ contains $v$. Suppose that there is a vertex $v_0$ in $V_0$ such that $card(e_u(v_0, V_0)) = 1$. If $v_0$ belongs to 4 hyperedges, then there is $v_0'$ in $V_0$ such that $\{v_4, v_0, v_0'\}$ satisfies (i) and contains only three vertices which contradicts the hypothesis on $V_0$.
Otherwise, $V' = \{v_4\} \cup V_0 \backslash \{v_0\}$ satisfies (i), (ii) and is such that $V' \succeq V_0$ which contradicts the hypothesis on $V_0$.
Thus when $V_4$ contains a vertex $v_4$, $card(e_u(v, V_0)) = 2$ for any vertex $v$ of $V_0$ and $G_0$ does not contain any edge. $v_4$ is inserted in $G_0$ by adding four edges and the resulting graph is planar.
Let $v$ be a vertex of $V \backslash (V_0 \cup \{v_4\})$. Suppose that there are two hyperedges $e$ and $e'$ of $\mathcal{E}$ which contain $v$ but not $v_4$. Let $v_1$ and $v_2$ be two vertices of $V_0$ included neither in $e$ nor in $e'$. Then $V' = \{v_1, v_2, v, v_4\}$ satisfies (i), (ii) and is such that $V' \succeq V_0$ which contradicts the hypothesis on $V_0$.
Thus the vertices of $V \backslash (V_0 \cup \{v_4\})$ can be inserted by adding two edges connecting $v_4$ and a vertex of $V_0$ which leads to a planar graph.

**-B-2.** Suppose $\boldsymbol{V_4}$ **is empty**.
Let us examine the insertion of vertices of $V_3$:
• If all the vertices of $V_3$ can be inserted inside the faces defined by $K_4$ without edge crossing then, as $K_4$ is planar, the resulting graph will be planar.
• Otherwise, suppose that the two vertices $v$ and $v'$ of $V_3$ are both associated to

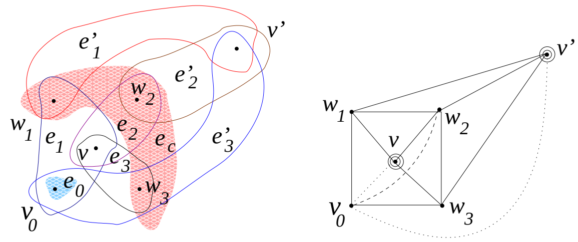


**Fig. 6.** When $card(V_0) = 4$, $card(\mathcal{E}) = 8$ and vertices $v$ and $v'$ are both associated to $\{w_1, w_2, w_3\}$. The path $v_0, w_2$ is replaced by one of the two paths $v_0, v'$ or $v_0, v$.

$W_3(v) = \{w_1, w_2, w_3\}$ and are such that $\forall i = 1, ..., 3$, $e_u(w_i, W_3(v)) \cap e(v) \neq e_u(w_i, W_3(v)) \cap e(v')$. Then, six hyperedges of $\mathcal{E}$ contain either $v$ or $v'$. Thus using the fact that $card(\mathcal{E}) < 9$, that $v$ and $v'$ belong to $V_3$ and the hypothesis satisfied by $V_0$, we have (cf. figure 6 for an example):

(a) $\forall i = 1, 2, 3$, $e_u(w_i, W_3(v)) = \{e_i, e'_i\}$, with $v$ included in $e_i$ and $v'$ in $e'_i$, and there is a hyperedge $e_c$ in $\mathcal{E}$ which contains the three vertices $w_1$, $w_2$ and $w_3$.

(b) the vertex $v_0$ of $V_0 \backslash W_3(v)$ has only one hyperedge in $e_u(v_0, V_0)$; $v_0$ is not included in $e_c$ and cannot be included in both $e_i$ and $e'_i$ for $i = 1, 2, 3$.

We insert $v$ inside the face defined by $W_3(v)$ and $v'$ outside this face. This leads to a planar graph $G'$. Then one of the three edges joining $v_0$ with the $w_i$, for example the edge $(w_2, v_0)$ cannot be drawn without edge crossing in $G'$, as in figure 6. Thus to insert the other vertices of $V_3$ and the vertices of $V_2$ in $G'$ without edge crossing, we replace the paths joining $v_0$ to $w_2$ by either a path joining $v_0$ to $v$ or a path joining $v_0$ to $v'$ (this can always be done because $e_2$ and $e'_2$ cannot both contain $v_0$). These insertions lead to a planar representation of $H$ in the edge standard. □

The following remark will be used in the proof of proposition 4.

*Remark 5.* Let $v$ be a vertex of $V \backslash V_0$ included in $e_1 \cap e_2$ where $\{e_1\} = e_u(v_1, V_0)$ $\{e_2\} = e_u(v_2, V_0)$ with $v_1$ and $v_2$ two distinct vertices of $V_0$. Then we have:

1. $\mathcal{E}$ contains a hyperedge $e_{1,2}$ of such that $e_{1,2} \cap V_0 = \{v_1, v_2\}$, otherwise $V_0$ could be replaced by $(V_0 \backslash \{v_1, v_2\}) \cup \{v\}$.
2. Consequently, the edge $(v_1, v_2)$ belongs to $G_0$. Then, if $v$ belongs to $V_2$, it can be inserted in $G_0$ without breaking the planarity.
3. Moreover, when $v$ belongs to $V_3$ and $card(V_0) = 5$, we have:
   (a) $v_1$ and $v_2$ must be included in more than two hyperedges of $\mathcal{E}$, otherwise $V_0$ would not be maximal w.r.t. the relation $\succeq$.
   (b) Consequently, if $v$ is included in a hyperedge $e$ of $\mathcal{E} \backslash e_u(V_0)$, then $\mathcal{E} \backslash e_u(V_0)$ must contain at least three hyperedges.

Because of (a), at least two hyperedges of $\mathcal{E}\backslash e_u(V_0)$ are used for $v_1$ and $v_2$ and they are distinct from $e$.

(c) if $v$ is included in a hyperedge of $e_u(v_0, V_0)$ with $v_0 \in V_0\backslash\{v_1, v_2\}$, then $v_0$ must be such that $card(e_u(v_0, V_0)) > 1$.
Otherwise, using point 1 of this remark, six hyperedges are necessary for $v_1, v_2$ and $v_0$ ($e_i$ for each $e_u(v_i, V_0)$ and $e_{i,j}$ for each couple $(v_i, v_j)$). Then when $card(V_0) = 5$, three other hyperedges are needed for the two vertices of $V_0\backslash\{v_0, v_1, v_2\}$, which is impossible when $card(\mathcal{E}) < 9$.

**Proposition 4.** *When $card(V_0) = 5$ and $card(\mathcal{E}) = 8$ the insertions of vertices of $V\backslash V_0$ in $G_0$ lead to a planar representation of $H$ in the edge standard.*

*Proof.* We have: $V\backslash V_0 = V_1 \cup V_2 \cup V_3$. We will prove that the insertion of vertices of $V\backslash V_0$ leads to a planar representation of $H$ in the edge standard considering the number $N$ of hyperedges of $\mathcal{E}\backslash e_u(V_0)$. Because of the cardinality of $\mathcal{E}$ and $V_0$, we have $3 \geq N > 0$.

**- case A-** Only one hyperedge $e$ of $\mathcal{E}$ contains several vertices of $V_0$.
There are at least one vertex $v_0$ such that $card(e_u(v_0, V_0)) > 1$, and three vertices of $V_0$, $v_1$, $v_2$ and $v_3$, such that $card(e_u(v_i, V_0)) = 1$ and $v_i$ belongs to $e$ for $i = 1, 2, 3$.

- Suppose $v_0$ is the unique vertex of $V_0$ satisfying $card(e_u(v_0, V_0)) > 1$. Then $V_3$ is empty, otherwise this would contradict the fact that $V_0$ has a minimum number of vertices (cf. figure 7(A)).

- Suppose $V_0$ contains another vertex $v'_0$ such that $card(e_u(v'_0, V_0)) > 1$. Then, to satisfy the minimality condition of $V_0$, the vertices of $V_3$ must be included in $e$, in one hyperedge of $e_u(v_0, V_0)$ and in one hyperedge of $e_u(v'_0, V_0)$. These vertices can be inserted inside the face defined by $(v_0, v'_0, v_1)$ as done in remark 4 (cf. figure 7(B)).

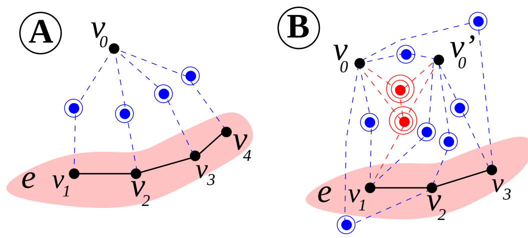


**Fig. 7.** When $card(V_0) = 5$, $card(\mathcal{E}) = 8$ and only one hyperedge $e$ of $\mathcal{E}$ contains several vertices of $V_0$. The paths corresponding to potential insertions of elements of $V_2$ and $V_3$ are drawn in dashed lines. In this figure, the positions of the vertices of $V_2$ and $V_3$ w.r.t. $e$ are not significant.

Consider now the subgraph $G'_0$ of $G_0$ restricted to the set of vertices $\{v_1, v_2, v_3\}$. As $G_0$ is minimal in number of edges and $e$ is the unique hyperedge containing $v_1$, $v_2$ and $v_3$, $G'_0$ does not contain a $K_3$. Using remark 5, a vertex $v$ of $V_2$ cannot be such that $W_2(v) = \{v_i, v_j\}$ with $i, j \in \{1, 2, 3\}$. As the elements of $V_3$, if they exist, are inserted inside the face defined by $(v_0, v'_0, v_1)$ , the insertion of vertices of $V\backslash V_0$ in $G'_0$ cannot create a subdivision of $K_3$. Thus, the insertions of vertices of $V\backslash V_0$ in $G_0$ do not create a subdivision of $K_5$, as illustrated in figure 7 (A) et (B).

**- case B -** Two hyperedges $e$ and $e'$ of $\mathcal{E}$ contain several vertices of $V_0$.
Then exactly one vertex $v_0$ of $V_0$ is such that $e_u(v_0, V_0) = \{e_0, e'_0\}$.
Let us examine the vertices of $V_3$.

- If a vertex $v$ of $V_3$ is included in the hyperedge $e_1$ s.t. $\{e_1\} = e_u(v_1, V_0)$. As $V_0$ must be maximal w.r.t. the relation $\succeq$, $v_1$ must be included in $e$ and in $e'$. Then as $v$ belongs to $V_3$, $v$ must be included in $e_2$, $\{e_2\} = e_u(v_2, V_0)$ with $v_2 \neq v_1$ and in one hyperedge of $e_u(v_0, V_0)$, because of point c of remark 5.3. Thus we have the following configuration: $e(v_0) = \{e_0, e'_0\}$ or $e(v_0) = \{e_0, e'_0, e'\}$, $e(v_i) = \{e_i, e, e'\}$ for $i = 1, 2$ and $e(v_i) = \{e_i, e'\}$ for $i = 3, 4$, and $v$ is inserted inside the face $(v_0v_1v_2)$, as illustrated in figure 8 (A).

- Otherwise, $V_3$ may contain vertices that must be included in $e$, in $e'$ and in one of the hyperedges of $e_u(v_0, V_0)$. They can be inserted in a unique face, following remark 4 and we have the configuration of figure 8 (B).

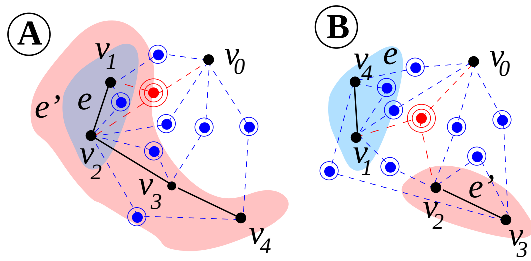


**Fig. 8.** When $card(V_0) = 5$, $card(\mathcal{E}) = 8$ and two hyperedges, $e$ and $e'$, of $\mathcal{E}$ contains several vertices of $V_0$. The paths corresponding to potential insertions of elements of $V_2$ and $V_3$ are drawn in dashed lines. In this fi gure, the positions of the vertices of $V_2$ and $V_3$ w.r.t. $e$ and $e'$ are not signifi cant.

Consider now the subgraph $G'_0$ of $G_0$ restricted to $V_0\backslash\{v_0\}$. As $G_0$ is minimal in number of edges and there are only two hyperedges $e$ and $e'$ containing several vertices of $V_0\backslash\{v_0\}$, $G'_0$ does not contain a $K_4$.
As noticed in remark 5 .2, a vertex of $V_2$ belonging to two hyperedges of $e_u(V_0)\backslash e_u(v_0, V_0)$ can be inserted in $G_0$ along an existing edge of $G_0$. The other vertices of $V_2$ associated to vertices of $V_0\backslash\{v_0\}$ belong to at least one hyperedge of $\{e, e'\}$ and can be inserted as described in figure 4 without creating a subdivision of $K_4$. Thus the insertion of the vertices of $V\backslash V_0$ can be done without creating a subdivision of $K_5$ (as illustrated in figure 8 (A) and (B)). Please, notice that to simplify the drawing, multiple insertions between two vertices have been represented only once in those figures.
**- case C-** Three hyperedges $e$, $e'$ and $e''$ of $\mathcal{E}$ contain several vertices of $V_0$.
As $card(V_0) = 5$, we can remark that:

- any vertex of $V_0$ belongs to exactly one hyperedge of $e_u(V_0)$. Then using point c of remark 5.3, a vertex $v$ of $V_3$ cannot contain three hyperedges of $e_u(V_0)$.

- there is at least one vertex of $V_0$ included in at least two hyperedges of $\{e, e', e''\}$. Then $v$ must be included in at least one hyperedge $e_0$ with $\{e_0\} = e_u(v_0, V_0)$.
As $V_0$ is maximal w.r.t. the relation $\succeq$, $v_0$ must be included in two hyperedges $e$ and $e'$ of $\mathcal{E}\backslash e_u(V_0)$. Then $v$ necessarily belongs to $e''$, $e_0$ and $e_1$ with $e \cap V_0 = \{v_0, v_1\}$. Thus $V_0$ must be such that: $e(v_0) = \{e_0, e, e'\}$, $e(v_1) = \{e_1, e, e'\}$, $\{e_2, e''\} \in e(v_2)$ and

$e \cap V_0 = \{v_0, v_1\}$.
Suppose that $e''$ contains exactly two vertices $v_2$ and $v_3$ of $V_0$ distinct from $v_0$ and $v_1$. $V_3$ could contain a second vertex $v'$ included in $e_2, e_3$ and $e$ but, in this case, $V_0$ would not be minimal because $e(\{v, v', v_4\}) = e(V_0)$.
Consequently, $V_3$ contains at most one vertex $v$ that is inserted inside the face $(v_0v_1v_2)$ (cf. figure 9(A)).

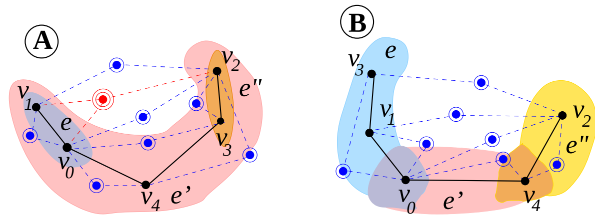


**Fig. 9.** When $card(V_0) = 5$, $card(\mathcal{E}) = 8$ and three hyperedges $e$, $e'$ and $e''$ of $\mathcal{E}$ contains several vertices of $V_0$. The paths corresponding to potential insertions of elements of $V_2$ and $V_3$ are drawn in dashed lines. In this figure, the positions of the vertices of $V_2$ and $V_3$ w.r.t. $e$, $e'$ and $e''$ are not significant.

Let us now examine the vertices of $V_2$. As previously, we suppose that $v_0$ and $v_2$ are vertices of $V_0$ respectively included in $\{e, e'\}$ and in $e''$. A vertex $v$ of $V_2$ is:

- either included in two hyperedges of $e_u(V_0)$. Following remark 5.2, $v$ is inserted along an existing edge of $G_0$.

- either included in only one hyperedge of $e_u(V_0)$. Suppose that $v$ is included in $e_u(v_1, V_0)$. Then $v$ will be inserted by adding two edges between either $v_1$ and $v_0$ or between $v_1$ and $v_2$.

- or included only in hyperedges of $\mathcal{E} \backslash e_u(V_0)$. Then $v$ will be inserted by adding two edges between $v_2$ and $v_0$.

The paths created by these insertions join $v_0$ or $v_2$ with the other vertices of $V_0$ or are located along existing edges of $G_0$, as shown in figure 9 (A) and (B). Then, the resulting graph does not contain a subdivision of a $K_5$. Finally, we obtain a planar representation of $H$ in the edge standard. □

Then using remark 3, proposition 2, 3 and 4, we have:

**Theorem 1.** *Any hypergraph having at most eight hyperedges is vertex-planar and has a planar representation in the edge standard.*

## 3 Conclusion

We have shown by a constructive proof that any hypergraph having less than nine hyperedges is vertex-planar and has a planar drawing in the edge standard.

The vertex-based Venn diagram representing a hypergraph $H$ is an extended Euler diagram (cf. [19]). The only difference with a planar drawing of $H$ in the subset standard is that a hyperedge is represented by several closed curves: one of the curves is the external contour and the others are the internal contours. The internal contours are included in the planar region bounded by the external contour and represent holes in this region. The planar region defined by this set of curves is connected. Then, by adding curves

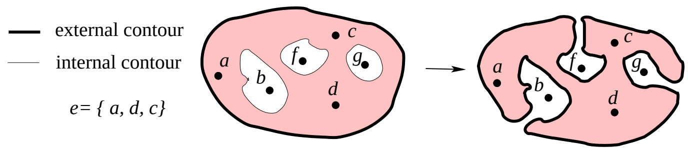


**Fig. 10.** The region corresponding to the hyperedge $e$ in a vertex-based Venn diagram is transformed into a region bounded by a curve.

connecting the external curve and the internal curves and opening the internal curves as in figure 10, the vertex-based Venn diagram representing $H$ can be easily transformed into a planar drawing of $H$ in the subset standard. Thus our method can also be used to compute a planar drawing of any hypergraph having less than nine hyperedges in the subset standard (cf. figure 11 for an example).

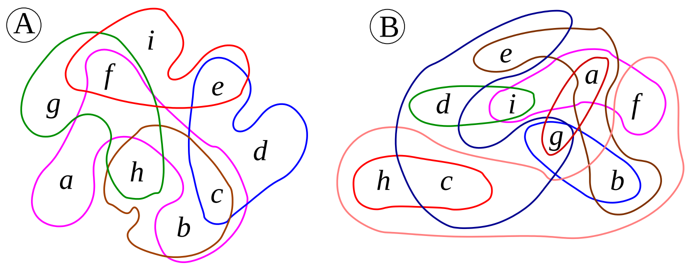


**Fig. 11.** A drawing in the subset standard of the two hypergraphs of example 1, according to the representation in the edge standard built with our method.

We are currently implementing a system computing a planar drawing in the edge standard, given a hypergraph having less than nine hyperedges. This work will be integrated in a graphical user interface for digital library access [17].

## References


1. Circuit verification via hypergraph realization. *Simulation standard, Silvaco international*, 9(3):7–9, March 1998.
2. F. Bertault and P. Eades. Drawing hypergraphs in the subset standard. In *Graph Drawing 2000*, LNCS 1984, pages 164–169. Springer Verlag, 2000.
3. T. Catarci, M.F. Costabile, S. Levialdi, and C.Batini. Visual query systems for databases: a survey. *Journal of Visual Languages and Computing*, 8:215–260, 1997.
4. T. Catarci and L. Tarantino. A hypergraph-based framework for visual interaction with databases. *Journal of Visual Languages and Computing*, 6(2):135–166, 1995.
5. S. Chow and F. Ruskey. Drawing area proportional Venn and Euler diagrams. In *Graph Drawing 2003*, number 2912 Springer 2004 in Lecture Notes in Computer Science, pages 466–477, Perugia, Italy, September 2003.
6. M. Consens. *Creating and filtering structural data visualizations using Hygraph patterns*. Tech. Rep. CSRI– 302, Toronto, Canada, February 1994.

7. M. Debowsky. Results on planar hypergraphs and on cycle decompositions. Master's thesis, University of Vermont, Burlington, April 2002.
8. D. Harel. On visual formalisms. *Communications of the ACM*, 31(5):514–530, 1988.
9. D.S. Johnson and H. O. Pollak. Hypergraph planarity and the complexity of drawing Venn diagrams. *Journal of Graph Theory*, 11(3):309–325, 1987.
10. H. Klemetti, I. Lapinleimu, E. Mäkinen, and M. Sieranta. A programming project: trimming the spring algorithm for drawing hypergraphs. *SIGCSE Bulletin*, 27(3):34–38, 1995.
11. K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15:271–283, 1930.
12. O. Lemon and I. Pratt. Spatial logic and the complexity of diagrammatic reasoning. *Machine Graphics and Vision (Special Issue on Diagrammatic Reasoning)*, 6(1):89–109, 1997.
13. A. Liebers. Planarizing graphs a survey and annotated bibliography. *Journal of Graph Algorithms and Applications*, 5(1):1–74, 2001.
14. E. Mäkinen. How to draw a hypergraph. *International Journal of Computer Mathematics*, 34:177–185, 1990.
15. F. Ruskey. A survey of Venn diagrams. *The electronic journal of combinatorics*, March 2001.
16. S-J. Shin and O. Lemon. Diagrams. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. 2003. http://plato.stanford.edu/entries/diagrams/.
17. J. Thièvre, M.L. Viaud, and A. Verroust-Blondet. Using Euler diagrams in traditional library environment. In *Euler Diagrams 2004*, Brighton, U.K., September 2004.
18. A. Verroust and M.L. Viaud. Ensuring the drawability of extended Euler diagrams for up to 8 sets. Technical Report 4973, INRIA, Rocquencourt, France, October 2003.
19. A. Verroust and M.L. Viaud. Ensuring the drawability of extended Euler diagrams for up to 8 sets. In *Diagrammatic Representation and Inference, Diagrams 2004*, number 2980 in Lecture Notes in Artifi cial Intelligence, pages 128–141, Cambridge, U.K., March 2004.