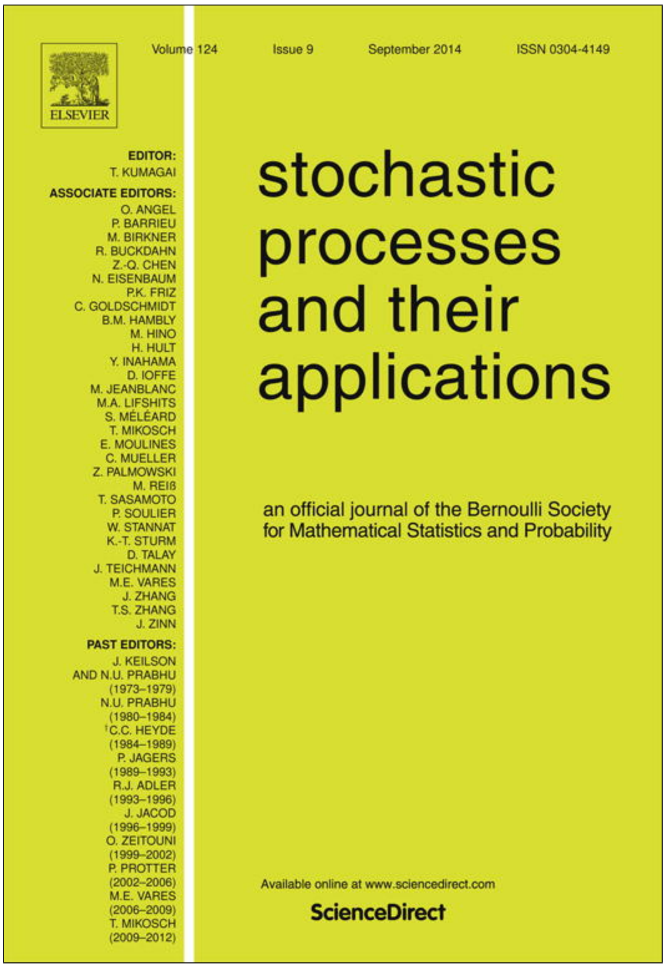
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Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps

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Abstract

We study the optimal stopping problem for dynamic risk measures represented by Backward Stochastic Differential Equations (BSDEs) with jumps and its relation with reflected BSDEs (RBSDEs). The financial position is given by an RCLL adapted process. We first state some properties of RBSDEs with jumps when the obstacle process is RCLL only. We then prove that the value function of the optimal stopping problem is characterized as the solution of an RBSDE. The existence of optimal stopping times is obtained when the obstacle is left-upper semi-continuous along stopping times. Finally, we investigate robust optimal stopping problems related to the case with model ambiguity and their links with mixed control/optimal stopping game problems. We prove that, under some hypothesis, the value function is equal to the solution of an RBSDE. We then study the existence of saddle points when the obstacle is left-upper semi-continuous along stopping times.

MSC: 93E20; 60J60; 47N10

Keywords: Backward stochastic differential equations; Reflected backward stochastic equations; g-conditional expectation; Jump processes; Optimal stopping; Dynamic risk measures; Game problems

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1. Introduction

In the last years, there has been a lot of studies on risk measures and related issues, see e.g. Föllmer and Schied [13], Fritelli and Rosazza-Gianin [14], Bion-Nadal [4], Delbaen [6], Klöppel and Schweizer [19]. In this paper we study optimal stopping problems for a dynamic risk measure (ρ_t) represented by a Backward Stochastic Differential Equation (BSDE) with jumps. Properties of such a risk measure have been studied recently in [23]. Related studies can be found in [11] in the Brownian case and in [3,2].

In the first part of the paper, we study the following optimal stopping problem: given a dynamic financial position (ξ_t), represented by an RCLL adapted process, we want to determine a stopping time τ which minimizes the risk of the position ξ_τ, and compute the corresponding value. To this purpose, we study the links between this optimal stopping problem and reflected BSDEs (RBSDEs) with jumps. These equations have been introduced by N. El Karoui et al. [10] in the case of a Brownian filtration and a continuous obstacle. We state some preliminary results on RBSDEs with jumps when the obstacle is RCLL only, such as existence, uniqueness, comparison and strict comparison theorems, which complete some results in Hamadène, Ouknine, Issaky, Crépey and Matoussi [15,16,12,5,21]. We then prove that the value function of our optimal stopping problem is the solution of an RBSDE with obstacle given by the dynamic position (ξ_t). We provide a sufficient condition and a necessary condition of optimality. When the obstacle is left-upper semicontinuous (l.u.s.c.) along stopping times, we show the existence of optimal stopping times. If the obstacle is only RCLL, we prove the existence of ε-optimal stopping times.

In a second part, we address the optimal stopping problem when there is ambiguity on the risk measure represented by parameters α belonging to some non empty set $\mathcal{A}$. To this purpose, we first study the following optimal control problem for RBSDEs: Let $\{f^\alpha, \alpha \in \mathcal{A}\}$ be a family of Lipschitz drivers and let $\{Y^\alpha, \alpha \in \mathcal{A}\}$ be the solutions of the RBSDEs associated with drivers $\{f^\alpha\}$ and RCLL obstacle (ξ_t). The problem is to minimize Y^α over α. Under appropriate hypotheses, the value function is characterized as the solution of an RBSDE. We then focus on the robust optimal stopping problem for dynamic risk measures: let $\{\rho_t^\alpha, \alpha \in \mathcal{A}\}$ be the family of dynamic risk measures induced by the BSDEs with jumps associated with drivers $\{f^\alpha, \alpha \in \mathcal{A}\}$. Consider the risk measure defined as the supremum of the risk measures ρ^α over all ambiguity parameters α. Given the dynamic position (ξ_t), the problem is to determine a stopping time τ^* which minimizes the risk of the position ξ_τ over all stopping times τ. This leads to a mixed control/optimal stopping game problem. When the financial position (ξ_t) is RCLL only, we prove that, under some additional hypothesis, there exists a value function for the game problem, which corresponds to the minimal risk measure. Using the previous results on optimization problems for RBSDEs, we show that this value function is equal to the solution of an RBSDE. When the obstacle is l.u.s.c. along stopping times and under additional assumptions, we prove the existence of a saddle point (τ^*, α^*). The stopping time τ^* is optimal for the robust risk minimization problem and α^* corresponds to a worst case scenario. Hence, our robust optimal stopping problem reduces to a classical optimal stopping problem associated with a worst case scenario among all the possible ambiguity parameters $\alpha \in \mathcal{A}$.

Links with partial integro-differential equations in the Markovian framework are studied in Dumitrescu, Quenez and Sulem [8].

The paper is organized as follows. In Section 2, we state the notation and give some preliminary results on BSDEs and reflected BSDEs with jumps and irregular obstacle. Relations between optimal stopping problems for dynamic risk measures induced by BSDEs with jumps and RBSDEs are given in Section 3. In Section 4, we provide comparison theorems for RBSDEs with jumps and optimization principles. The robust optimal stopping problem for risk measures when there is ambiguity on the risk measure is addressed in Section 5. An application to a case of multiple priors is presented in Section 6.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion. Let $(U, \mathcal{U})$ be a measurable space equipped with a σ-finite positive measure ν. Let $N(dt, du)$ be a Poisson random measure with compensator $\nu(du)dt$ (see e.g. [17]). Let $\tilde{N}(dt, du)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N.

Notation. Let $\mathcal{P}$ be the predictable σ-algebra on $[0, T] \times \Omega$.

For each $T > 0$ and $p > 1$, we use the following notation:

- $L^p(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and p-integrable.
- $\mathbb{H}^{p,T}$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^p_{\mathbb{H}^{p,T}} := E\left[(\int_0^T \phi_t^2 dt)^{\frac{p}{2}}\right] < \infty$.
- L^p_ν is the set of measurable functions $\ell : U \to \mathbf{R}$ such that $\int_U |\ell(u)|^p \nu(du) < +\infty$. The set L^2_ν is a Hilbert space equipped with the scalar product $\langle \delta, \ell \rangle_\nu := \int_U \delta(u)\ell(u)\nu(du)$ for all $\delta, \ell \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_U |\ell(u)|^2 \nu(du) < +\infty$.
- $\mathbb{H}^{p,T}_\nu$ is the set of processes l which are *predictable*, that is, measurable $l : ([0, T] \times \Omega \times U, \mathcal{P} \otimes \mathcal{U}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$; $(\omega, t, u) \mapsto l_t(\omega, u)$ with
$$\|l\|^p_{\mathbb{H}^{p,T}_\nu} := E\left[\left(\int_0^T \|l_t\|^2_\nu \, dt\right)^{\frac{p}{2}}\right] < \infty.$$
- $\mathcal{S}^{p,T}$ is the set of real-valued RCLL adapted processes ϕ with $\|\phi\|^p_{\mathcal{S}^{p,T}} := E(\sup_{0 \leq t \leq T} |\phi_t|^p) < \infty$.

When T is fixed and there is no ambiguity, we denote $\mathbb{H}^p$ instead of $\mathbb{H}^{p,T}$, $\mathbb{H}^p_\nu$ instead of $\mathbb{H}^{p,T}_\nu$, $\mathcal{S}^p$ instead of $\mathcal{S}^{p,T}$.

- $\mathcal{T}_0$ denotes the set of stopping times τ such that $\tau \in [0, T]$ a.s.
- For S in $\mathcal{T}_0$, let $\mathcal{T}_S$ be the set of stopping times τ such that $\tau \in [S, T]$ a.s.

Definition 2.1. A progressive process (ϕ_t) is said to be *left-upper semicontinuous (l.u.s.c.) along stopping times* if for all $\tau \in \mathcal{T}_0$ and for each nondecreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s.,

$$\phi_\tau \geq \limsup_{n \to \infty} \phi_{\tau_n} \quad \text{a.s.} \tag{2.1}$$

Remark 2.2. Note that in this definition, no condition is required at a totally unaccessible stopping time. Since there the filtration is generated by W and N, this means that no condition is required at the jump times of N. For example, a jump–diffusion process is l.u.s.c. along stopping times. When the process (ϕ_t) is left-limited, (ϕ_t) is l.u.s.c. along stopping times if and only if for each predictable stopping time $\tau \in \mathcal{T}_0$, $\phi_{\tau^-} \leq \phi_\tau$ a.s.

Definition 2.3 (*Driver, Lipschitz Driver*)**.** A function f is said to be a *driver* if

- $f : [0, T] \times \Omega \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$ $(\omega, t, x, \pi, \ell(\cdot)) \mapsto f(\omega, t, x, \pi, \ell(\cdot))$ is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$measurable,
- $f(., 0, 0, 0) \in \mathbb{H}^2$.

A driver f is called a *Lipschitz driver* if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s., for each (x_1, π_1, ℓ_1), (x_2, π_2, ℓ_2),

$$|f(\omega, t, x_1, \pi_1, \ell_1) - f(\omega, t, x_2, \pi_2, \ell_2)| \leq C(|x_1 - x_2| + |\pi_1 - \pi_2| + \|\ell_1 - \ell_2\|_\nu).$$

Existence and uniqueness result for BSDEs with jumps. By the martingale representation theorem for diffusion processes with jumps (see [18,25]), we have the following result [1]: Let $T > 0$. For each Lipschitz driver f, and each terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(X, \pi, l) \in \mathcal{S}^{2,T} \times \mathbb{H}^{2,T} \times \mathbb{H}^{2,T}_\nu$ satisfying

$$-dX_t = f(t, X_{t^-}, \pi_t, l_t(\cdot))\, dt - \pi_t\, dW_t - \int_U l_t(u)\tilde{N}(dt, du); \qquad X_T = \xi. \tag{2.2}$$

This solution is denoted by $(X(\xi, T), \pi(\xi, T), l(\xi, T))$.

This result can be extended to the case when the terminal time is a stopping time $\tau \in \mathcal{T}_0$ and the terminal condition is given by a random variable ξ in $L^2(\mathcal{F}_\tau)$. In this case, $(X(\xi, \tau), \pi(\xi, \tau), l(\xi, \tau))$ is defined as the unique solution of the BSDE with driver $f(t, x, \pi, l)\mathbf{1}_{\{t \leq \tau\}}$ and terminal conditions (T, ξ). Note that $X_t(\xi, \tau) = \xi$, $\pi_t(\xi, \tau) = 0$, $l_t(\xi, \tau) = 0$ for $t \geq \tau$. The process $X_t(\xi, \tau)$ corresponds to the f-conditional expectation $\mathcal{E}^f_{t,\tau}(\xi)$ (see [22]).

Reflected BSDEs with RCLL obstacle. Let $T > 0$ be a fixed terminal time and f be a Lipschitz driver. Let $\xi_. = (\xi_t)$ be an RCLL adapted process in $\mathcal{S}^2$, that is, such that $E(\sup_{0 \leq t \leq T} |\xi_t|^2) < \infty$.

Definition 2.4. A process $(Y, Z, k(.), A)$ is said to be a solution of the reflected BSDE associated with driver f and obstacle $\xi_.$ if

$$(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$$

$$-dY_t = f(t, Y_t, Z_t, k_t(\cdot))dt + dA_t - Z_t dW_t - \int_U k_t(u)\tilde{N}(dt, du); \qquad Y_T = \xi_T, \tag{2.3}$$

$$Y_t \geq \xi_t, \quad 0 \leq t \leq T \text{ a.s.,}$$

A is a nondecreasing RCLL predictable process with $A_0 = 0$ and such that

$$\int_0^T (Y_t - \xi_t)dA^c_t = 0 \quad \text{a.s.} \quad \text{and} \quad \Delta A^d_t = -\Delta Y_t\, \mathbf{1}_{\{Y_{t^-} = \xi_{t^-}\}} \quad \text{a.s.}$$

Here A^c denotes the continuous part of A and A^d its discontinuous part.

The following lemma provides a first link between RBSDEs and optimal stopping problems.

Lemma 2.5. *Suppose that f does not depend on y, z, k, that is $f(\omega, t, y, z, k(\cdot)) = f(\omega, t)$, where f is a process in $\mathbb{H}^2 := \mathbb{H}^{2,T}$. Let (ξ_t) be an RCLL adapted process in $\mathcal{S}^2$. Then, RBSDE (2.3) admits a unique solution $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$, and for each $S \in \mathcal{T}_0$, we have*

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} E\left[\xi_\tau + \int_S^\tau f(t)dt \mid \mathcal{F}_S\right] \quad a.s. \tag{2.4}$$

If (ξ_t) is l.u.s.c. *along stopping times, then (A_t) is continuous.*

Moreover, for each $\varepsilon > 0$ and each $S \in \mathcal{T}_0$, the stopping time τ^ε_S defined by

$$\tau^\varepsilon_S := \inf\{t \geq S, Y_t \leq \xi_t + \varepsilon\}$$

is ε-optimal for (2.4), that is

$$Y_S \le E\left[\xi_{\tau_S^\varepsilon} + \int_S^{\tau^\varepsilon} f(s)ds \mid \mathcal{F}_S\right] + \varepsilon \quad a.s.$$

In the Appendix, we give a short and direct proof of this lemma, and provide some estimates which imply the following existence and uniqueness theorem.

Theorem 2.6. *Let $\xi_.$ be an RCLL adapted process in $\mathcal{S}^2$ and let f be a Lipschitz driver. The RBSDE (2.3) admits a unique solution $(Y, Z, k(.), A) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2$. If (ξ_t) is* l.u.s.c. *over stopping times, then (A_t) is continuous.*

3. Optimal stopping for dynamic risk measures

3.1. Formulation of the problem

Let $T > 0$ be a time horizon and f be a Lipschitz driver. For each $T' \in [0, T]$ and $\eta \in L^2(\mathcal{F}_{T'})$, set

$$\rho_t^f(\eta, T') = \rho_t(\eta, T') := -X_t(\eta, T'), \quad 0 \le t \le T', \tag{3.5}$$

where $X_t(\eta, T')$ denotes the solution of the BSDE (2.2) with driver f and terminal conditions (T', η). If T' represents a given maturity and η a financial position at time T', then $\rho_t(\eta, T')$ is interpreted as the risk of η at time t. The functional $\rho : (\eta, T') \mapsto \rho_.(\eta, T')$ thus represents a *dynamic risk measure* induced by the BSDE with driver f. Properties of these dynamic risk measures, such as monotonicity, translation invariance, convexity are satisfied under appropriate hypotheses on the driver (see Section 5 in [23] for details). Contrary to the Brownian case, the *monotonicity* property of ρ, that is, the non increasing property with respect to financial position, which is naturally required for risk measures, is not automatically satisfied. We thus assume from now on that the driver f satisfies the following assumption, which ensures the monotonicity property of ρ by the comparison theorem for BSDEs with jumps (see [23], Th 4.2).

Assumption 3.1. Assume that $dP \otimes dt$-a.s for each $(x, \pi, \ell_1, \ell_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$f(t, x, \pi, \ell_1) - f(t, x, \pi, \ell_2) \ge \langle \theta_t^{x,\pi,\ell_1,\ell_2}, \ell_1 - \ell_2 \rangle_\nu,$$

with

$$\theta : [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu; \qquad (\omega, t, x, \pi, \ell_1, \ell_2) \mapsto \theta_t^{x,\pi,l_1,l_2}(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s., for each $(x, \pi, \ell_1, \ell_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta_t^{x,\pi,\ell_1,\ell_2}(u) \ge -1 \quad \text{and} \quad |\theta_t^{x,\pi,\ell_1,\ell_2}(u)| \le \psi(u), \tag{3.6}$$

where $\psi \in L^2_\nu$.

Assumption 3.1 is satisfied if for example f is continuously differentiable with respect to ℓ on L^2_ν and such that $dP \otimes dt \otimes d\nu$-a.s., for each (x, π, ℓ), $\nabla_\ell f(t, x, \pi, \ell)(.) \ge -1$, $|\nabla_\ell f(t, x, \pi, \ell)(.)| \le \psi(.)$, where $\psi \in L^2_\nu$.

If in (3.6), $\theta_t^{x,\pi,\ell_1,\ell_2} > -1$, then ρ is strictly monotonous by the strict comparison theorem for BSDEs with jumps (see [23], Th 4.4).

Remark 3.2. If Assumption 3.1 is not satisfied, then the risk measure may fail to be monotonous; for example, the risk measure associated with the BSDE driven by a Poisson process with parameter 1 (i.e. $\nu = \delta_1$) and driver $f(\ell) := \langle \theta, \ell \rangle_\nu = \theta \ell(1)$ with constant $\theta < -1$ is not monotonous (see [23] Example 3.1). If $\theta = -1$, then it is monotonous but not strictly.

We can also ask ourselves if Assumption 3.1 is necessary to ensure the monotonicity property. A result given in [24] (or [23] Prop. 5.1) provides that, under a technical assumption, a strict monotonous and translation invariant dynamic risk measure induced by a BSDE with jumps is associated with a driver satisfying Assumption 3.1.

Let us now formulate our optimal stopping problem. Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL adapted process in $\mathcal{S}^2$, representing a dynamic financial position. Let $S \in \mathcal{T}_0$. The problem is to minimize the risk measure at time S. Let $v(S)$ be the associated value function, equal to the $\mathcal{F}_S$-measurable random variable (unique for the equality in the almost sure sense) defined by

$$v(S) := \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_S} \rho_S(\xi_\tau, \tau). \tag{3.7}$$

This random variable $v(S)$ corresponds to the minimal risk measure at time S.

Since by definition $\rho_S(\xi_\tau, \tau) = -X_S(\xi_\tau, \tau)$, we have, for each stopping time $S \in \mathcal{T}_0$,

$$v(S) = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_S} -X_S(\xi_\tau, \tau) = -\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau). \tag{3.8}$$

The aim is to characterize $v(S)$ for each $S \in \mathcal{T}_0$, and to study the existence of an S-optimal stopping time $\tau^* \in \mathcal{T}_S$, that is such that $v(S) = \rho_S(\xi_{\tau^*}, \tau^*)$ a.s.

3.2. Characterization of the value function as the solution of an RBSDE

We prove that, under Assumption 3.1, the minimal risk measure v defined by (3.8) coincides with $-Y$, where Y is the solution of the reflected BSDE associated with driver f and obstacle ξ, and we show the existence of ε-optimal stopping times.

Theorem 3.3. *Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 3.1. Suppose that $(Y, Z, k(\cdot), A)$ is the solution of the reflected BSDE (2.3).*

- *For each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) \quad a.s. \tag{3.9}$$

where for $\tau \in \mathcal{T}_S$, $X_\cdot(\xi_\tau, \tau)$ is the solution of the BSDE associated with terminal time τ, terminal condition ξ_τ, and driver f.

- *For each $S \in \mathcal{T}_0$ and each $\varepsilon > 0$, let τ_S^ε be the stopping time defined by*

$$\tau_S^\varepsilon = \inf\{t \geq S, Y_t \leq \xi_t + \varepsilon\}. \tag{3.10}$$

We have

$$Y_S \leq X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \quad a.s., \tag{3.11}$$

*where $K = K(T, C)$ is a constant which only depends on T and the Lipschitz constant C of f. In other words, τ_S^ε is a $(K\varepsilon)$-*optimal stopping time *for (3.9).*

Proof. Let $\tau \in \mathcal{T}_S$. Note that the process $(Y_s, Z_s, k_s; 0 \le s \le \tau)$ is the solution of the BSDE associated with terminal time τ, terminal condition Y_τ, and (generalized) driver

$$f(s, y, z, k)ds + dA_s.$$

We have $f(s, y, z, k)ds + dA_s \ge f(s, y, z, k)ds$ and $Y_\tau \ge \xi_\tau$ a.s.

Since f satisfies Assumption 3.1, the comparison theorem for BSDEs can be applied and gives $Y_s \ge X_s(\xi_\tau, \tau)$, $0 \le s \le \tau$ a.s. In particular $Y_S \ge X_S(\xi_\tau, \tau)$. By taking the supremum over $\tau \in \mathcal{T}_S$, we derive that

$$Y_S \ge \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) \quad \text{a.s.} \tag{3.12}$$

It remains to show the converse inequality. We first show the following lemma.

Lemma 3.4. (i) *We have* $Y_{\tau_S^\varepsilon} \le \xi_{\tau_S^\varepsilon} + \varepsilon$ *a.s.*

(ii) *The process* $(Y_t, S \le t \le \tau_S^\varepsilon)$ *is the solution of the BSDE associated with terminal time* τ_S^ε, *terminal condition* $Y_{\tau_S^\varepsilon}$ *and driver* f, *that is*

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad S \le t \le \tau_S^\varepsilon \text{ a.s.} \tag{3.13}$$

Note that Property (ii) implies that for all S and $\tau \in \mathcal{T}_0$ with $S \le \tau \le \tau_S^\varepsilon$, we have $Y_S = \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s. In other words, the process $(Y_t, S \le t \le \tau_S^\varepsilon)$ is an $\mathcal{E}^f$-martingale.

Proof. (i) This follows from the definition of τ_S^ε and the right-continuity of (ξ_t) and (Y_t).

(ii) Note that $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. By definition of τ_S^ε, for a.e. ω, if $t \in [S(\omega), \tau_S^\varepsilon(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that for a.e. ω, the function $t \mapsto A_t^c(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)]$ and $t \mapsto A_t^d(\omega)$ is constant on $[S(\omega), \tau_S^\varepsilon(\omega)[$. Also, $Y_{(\tau_S^\varepsilon)^-} \ge \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$a.s., which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. Hence, the process $(Y_t, S \le t \le \tau_S^\varepsilon)$ is a solution of the BSDE associated with terminal time τ_S^ε, terminal condition $Y_{\tau_S^\varepsilon}$ and driver f. By the uniqueness of the solution of Lipschitz BSDEs, we get (3.13). □

End of proof of Theorem 3.3. Let us prove inequality (3.11). By Lemma 3.4 and by the comparison theorem for BSDEs, we derive that for each $\varepsilon > 0$,

$$Y_S = X_S(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \le X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \quad \text{a.s.} \tag{3.14}$$

Now, by estimates on BSDEs (see Proposition A.4 [23]), we have

$$|X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) - X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon)|^2 \le e^{\beta(T-S)}\varepsilon^2 \quad \text{a.s.}$$

where $\beta := 3C^2 + 2C$. This with (3.14) leads to inequality (3.11). Hence, for each $\varepsilon > 0$,

$$Y_S \le X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + K\varepsilon \le \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) + K\varepsilon \quad \text{a.s.} \tag{3.15}$$

where $K := e^{\frac{\beta T}{2}}$. It follows that $Y_S \le \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau)$ a.s. By (3.12), this inequality is an equality.

Moreover, the $K\varepsilon$-optimality property of τ_S^ε follows from (3.15). □

3.3. Optimal stopping times

By the strict comparison theorem for BSDEs ([23], Th 4.4), we derive the following optimality criterium for the optimal stopping time problem (3.9).

Proposition 3.5 (*Optimality Criterium*)**.** *Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 3.1. Let $S \in \mathcal{T}_0$ and let $\hat{\tau} \in \mathcal{T}_S$. Suppose that in Assumption 3.1, we have*

$$\theta_t^{Y_t, Z_t, k_t, l_t^{\hat{\tau}}} > -1, \quad dt \otimes dP - a.s. \tag{3.16}$$

where $(X^{\hat{\tau}}, \pi^{\hat{\tau}}, l^{\hat{\tau}}) = (X(\xi_{\hat{\tau}}, \hat{\tau}), \pi(\xi_{\hat{\tau}}, \hat{\tau}), l(\xi_{\hat{\tau}}, \hat{\tau}))$ is the solution of the BSDE associated with terminal conditions $(\hat{\tau}, \xi_{\hat{\tau}})$.

The stopping time $\hat{\tau}$ is S-optimal, i.e.

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) = X_S(\xi_{\hat{\tau}}, \hat{\tau}) \quad a.s. \tag{3.17}$$

if and only if

$$Y_s = X_s(\xi_{\hat{\tau}}, \hat{\tau}), \quad S \leq s \leq \hat{\tau} \quad a.s. \tag{3.18}$$

that is if and only if $(Y_s, S \leq s \leq \hat{\tau})$ is the solution of the non reflected BSDE associated with terminal time $\hat{\tau}$ and terminal condition $\xi_{\hat{\tau}}$.

Proof. It is clear that (3.18) $\Rightarrow$ (3.17). Note that this implication does not require condition (3.16). Suppose now (3.17) holds. The process $(Y_s, Z_s, k_s; 0 \leq s \leq \hat{\tau})$ is the solution of the BSDE associated with terminal conditions $\hat{\tau}$, $Y_{\hat{\tau}}$, and driver $f(s, y, z, k)ds + dA_s$. We have $f(s, y, z, k)ds + dA_s \geq f(s, y, z, k)ds$ and $Y_{\hat{\tau}} \geq \xi_{\hat{\tau}}$ a.s. By assumption (3.16), the strict comparison theorem for BSDEs with jumps applied to Y and $X(\xi_{\hat{\tau}}, \hat{\tau})$ leads to (3.18). $\square$

Remark 3.6. In terms of f-conditional expectation $\mathcal{E}^f$, the above criterion can be written as follows: $\hat{\tau}$ is S-optimal if and only if $(Y_s, S \leq s \leq \hat{\tau})$ is an $\mathcal{E}^f$-martingale with $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ a.s. When the driver f does not depend on (y, z), this gives the well-known optimality criterium of the Optimal Stopping Theory: a stopping time $\hat{\tau}$ is S-optimal if and only if $(Y_s + \int_0^s f(r)dr, S \leq s \leq \hat{\tau})$ is a martingale with $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ a.s.

We show below that, under a left regularity condition on the obstacle, τ_S^ε tends to an S-optimal stopping time as ε tends to 0, and we provide two other S-optimal stopping times.

Theorem 3.7. *Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL process in $\mathcal{S}^2$, assumed to be* l.u.s.c. *along stopping times, and let f be a Lipschitz driver satisfying Assumption 3.1. Let $S \in \mathcal{T}_0$.*

(i) *The stopping time $\tilde{\tau}_S$ defined by*

$$\tilde{\tau}_S := \lim_{\varepsilon \downarrow 0} \uparrow \tau_S^\varepsilon,$$

with τ_S^ε given in (3.10), is an S-optimal stopping time.

(ii) *The stopping time τ_S^* defined by*

$$\tau_S^* := \inf\{u \geq S; Y_u = \xi_u\}$$

is an S-optimal stopping time and we have

$$Y_s = X_s(\xi_{\tau_S^*}, \tau_S^*), \quad S \leq s \leq \tau_S^* \text{ a.s.}$$

We also have $\tau_S^* \geq \tilde{\tau}_S$ a.s.

(iii) *The stopping time* $\bar{\tau}_S$ *defined by*

$$\bar{\tau}_S := \inf\{u \geq S;\, A_u - A_S > 0\}$$

is an S-optimal stopping time.

(iv) *Suppose moreover that in Assumption 3.1, for all* x, π, l_1, l_2, *we have*

$$\theta_t^{x,\pi,l_1,l_2} > -1 \quad dt \otimes dP - a.s. \tag{3.19}$$

Then, $\tau_S^* = \tilde{\tau}_S$ *a.s. Moreover* τ_S^* *is the minimal and* $\bar{\tau}_S$ *is the maximal S-optimal stopping time.*

Proof. (i) By letting ε tend to 0 in inequality (3.15), we get

$$Y_S \leq \limsup_{\varepsilon \downarrow 0} X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad \text{a.s.} \tag{3.20}$$

For each ω such that the map $\varepsilon \mapsto \tau_S^\varepsilon(\omega)$ from $\mathbf{R}_+^* \to [0, T]$ is constant for ε sufficiently small, we have

$$\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}(\omega) = \xi_{\tilde{\tau}_S}(\omega).$$

Moreover, since the process (ξ_t) is left-limited, for almost every ω such that for each $\varepsilon > 0$, $\tau_S^\varepsilon(\omega) < \hat{\tau}_S(\omega)$, we have

$$\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}(\omega) = \xi_{\tilde{\tau}_S^-}(\omega).$$

Hence, for almost every ω, $\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}(\omega)$ exists. The continuity property of BSDEs with respect to terminal conditions (see Prop. A6 in [23]), implies

$$\lim_{\varepsilon \downarrow 0} X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = X_S(\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}, \tilde{\tau}_S) \quad \text{a.s.} \tag{3.21}$$

Now, the l.u.s.c. property of the obstacle along stopping times yields $\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon} \leq \xi_{\tilde{\tau}_S}$ a.s. By the comparison theorem, it follows that

$$X_S(\lim_{\varepsilon \downarrow 0} \xi_{\tau_S^\varepsilon}, \tilde{\tau}_S) \leq X_S(\xi_{\tilde{\tau}_S}, \tilde{\tau}_S) \quad \text{a.s.}$$

Hence, by (3.20) and (3.21), we get $Y_S \leq X_S(\xi_{\tilde{\tau}_S}, \tilde{\tau}_S)$ a.s. By using the characterization of Y_S as the value function of the optimal stopping time problem (3.9), we get

$$Y_S = X_S(\xi_{\tilde{\tau}_S}, \tilde{\tau}_S) \quad \text{a.s.} \tag{3.22}$$

Thus, $\tilde{\tau}_S$ is an S-optimal stopping time.

(ii) The right continuity of (Y_t) and (ξ_t) ensures that $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. By definition of τ_S^*, we have that almost surely on $[S, \tau_S^*[$, $Y_t > \xi_t$ and hence the process A is constant on $[S, \tau_S^*[$ and even on $[S, \tau_S^*]$ because A is continuous (see Theorem 2.6). We derive that $(Y_s, S \leq s \leq \tau_S^*)$ is the solution of the BSDE associated with terminal time τ_S^*, terminal condition $\xi_{\tau_S^*}$ and driver f, that is, $Y_s = X_s(\xi_{\tau_S^*}, \tau_S^*)$, $S \leq s \leq \tau_S^*$ a.s. Hence, τ_S^* is an S-optimal stopping time.

Furthermore, for each $\varepsilon > 0$, $\tau_S^\varepsilon \leq \tau_S^*$ a.s. By letting ε tend to 0, we get $\tilde{\tau}_S \leq \tau_S^*$. a.s.

(iii) From the definition of $\bar{\tau}_S$, and the continuity of A, we have $A_{\bar{\tau}_S} - A_S = 0$ a.s. Hence

$$Y_s = X_s(Y_{\bar{\tau}_S}, \bar{\tau}_S), \qquad S \leq s \leq \bar{\tau}_S \text{ a.s.}$$

Also, we have a.s. for all $t > \bar{\tau}_S$, $A_t > A_{\bar{\tau}_S} = A_S$. Since A increases only on the set $\{Y_. = \xi_.\}$, it follows that $Y_{\bar{\tau}_S} = \xi_{\bar{\tau}_S}$. Hence $Y_S = X_S(\xi_{\bar{\tau}_S}, \bar{\tau}_S)$ a.s. In other words, $\bar{\tau}_S$ is S-optimal.

(iv) Let $\hat{\tau}$ be an S-optimal stopping time. By the strict comparison theorem for non reflected BSDEs (or Proposition 3.5), we have $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ a.s. Hence, by definition of τ_S^*, we have $\hat{\tau} \geq \tau_S^*$ a.s.

Thus, $\tilde{\tau}_S \geq \tau_S^*$ a.s.. By (ii), $\tilde{\tau}_S \leq \tau_S^*$ which implies $\tilde{\tau}_S = \tau_S^*$ a.s. We also have proven that τ_S^* is the minimal S-optimal stopping time.

Let us now show that $\bar{\tau}_S$ is the maximal S-optimal stopping time. If $\hat{\tau}$ is S-optimal, by the optimality criterium, $A_{\hat{\tau}_S} - A_S = 0$ a.s. which implies $\hat{\tau} \leq \bar{\tau}_S$ a.s. □

We are interested in exploring this optimal stopping problem in the case of model ambiguity. To this purpose, we first provide some results on optimization problems for RBSDEs. This is the object of the next section.

4. Comparison theorems for RBSDEs and optimization problems

4.1. Comparison theorems for RBSDEs with jumps

Theorem 4.1 (*Comparison*). *Let ξ^1, ξ^2 be two RCLL obstacle processes in $\mathcal{S}^2$. Let f^1 and f^2 be Lipschitz drivers satisfying Assumption 3.1. Suppose that*

- $\xi_t^2 \leq \xi_t^1$, $0 \leq t \leq T$ *a.s.*
- $f^2(t, y, z, k) \leq f^1(t, y, z, k)$, *for all* $(y, z, k) \in \mathbf{R}^2 \times \mathcal{L}_\nu^2$; $dP \otimes dt - a.s.$

Let (Y^i, Z^i, k^i, A^i) be the solution of the RBSDE *associated with (ξ^i, f^i), $i = 1, 2$. Then,*

$$Y_t^2 \leq Y_t^1, \quad \forall t \in [0, T] \text{ a.s.}$$

Proof. We give a simple proof based on the characterization of solutions of RBSDEs (Theorem 3.3) and on the comparison theorem for non reflected BSDEs. Let $t \in [0, T]$. For each $\tau \in T_t$, let us denote by $X^i(\xi_\tau^i, \tau)$ the unique solution of the BSDE associated with (τ, ξ_τ^i, f^i) for $i = 1, 2$. By the comparison theorem for BSDEs, for each τ in $\mathcal{T}_t$, the inequality

$$X_t^2(\xi_\tau^2, \tau) \leq X_t^1(\xi_\tau^1, \tau) \quad \text{a.s.}$$

holds. By taking the essential supremum over τ and using Theorem 3.3, we get

$$Y_t^2 = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} X_t^2(\xi_\tau^2, \tau) \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} X_t^1(\xi_\tau^1, \tau) = Y_t^1 \quad \text{a.s.} \quad \square$$

We now provide a strict comparison theorem. The first assertion addresses the particular case when the obstacle is l.u.s.c. along stopping times and the second one deals with the general case.

Theorem 4.2 (*Strict Comparison*). *Suppose that the assumptions of the comparison theorem (Theorem 4.1) hold and that the driver f^1 satisfies Assumption 3.1 with*

$$\theta_t^{x,\pi,l_1,l_2} > -1 \quad dt \otimes dP - a.s. \tag{4.23}$$

Let S in $\mathcal{T}_0$ and suppose that $Y_S^1 = Y_S^2$ a.s.

1. *Suppose that ξ^1 and ξ^2 are* l.u.s.c. *along stopping times.*
   Let $\tau_i^* = \tau_{i,S}^* := \inf\{s \geq S;\ Y_s^i = \xi_s^i\}$, $i = 1, 2$. Then,
   $$Y_t^1 = Y_t^2, \quad S \leq t \leq \tau_1^* \wedge \tau_2^* \text{ a.s,} \quad \text{and}$$
   $$f^2(t, Y_t^2, Z_t^2, k_t^2) = f^1(t, Y_t^2, Z_t^2, k_t^2) \quad S \leq t \leq \tau_1^* \wedge \tau_2^*, dP \otimes dt - a.s. \tag{4.24}$$
   Moreover if $\xi^1 = \xi^2$ a.s., then $\tau_1^* = \tau_2^*$ a.s. and $Y_{\tau_1^*}^1 = Y_{\tau_1^*}^2 = \xi_{\tau_1^*}^1$ a.s
2. *Consider the general case when ξ^1 and ξ^2 are not supposed to be* l.u.s.c. *along stopping times. For $\varepsilon > 0$, define*
   $$\tau_i^\varepsilon := \inf\{t \geq S, Y_t^i \leq \xi_t^i + \varepsilon\} \quad \text{and} \quad \tilde{\tau}_i := \lim_{\varepsilon \downarrow 0} \uparrow \tau_i^\varepsilon \quad i = 1, 2.$$

Then, $Y_t^1 = Y_t^2$, $S \le t < \tilde{\tau}_1 \wedge \tilde{\tau}_2$ a.s. Moreover,

$$f^2(t, Y_t^2, Z_t^2, k_t^2) = f^1(t, Y_t^2, Z_t^2, k_t^2) \quad S \le t \le \tilde{\tau}_1 \wedge \tilde{\tau}_2, dP \otimes dt - a.s.$$

and if $\xi^1 = \xi^2$ a.s., then for each $\varepsilon > 0$, $\tau_1^\varepsilon = \tau_2^\varepsilon$ a.s. and $\tilde{\tau}_1 = \tilde{\tau}_2$.

Proof. 1. Let $i \in \{1, 2\}$. By the existence theorem (see Theorem 3.7), τ_i^* is optimal for Problem (3.9) with $f = f^i$, $\xi = \xi^i$, that is

$$Y_S^i = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} X_S^i(\xi_\tau^i, \tau) = X_S^i(\xi_{\tau_i^*}^i, \tau_i^*) \quad \text{a.s}$$

where $X^i(\xi_{\tau_i^*}^i, \tau_i^*)$ denotes the solution of the BSDE associated with terminal time τ_i^*, terminal condition $\xi_{\tau_i^*}^i$ and driver f^i. Hence

$$Y_t^1 = X_t^1(Y_{\tau_1^* \wedge \tau_2^*}^1, \tau_1^* \wedge \tau_2^*), \quad \text{and} \quad Y_t^2 = X_t^2(Y_{\tau_1^* \wedge \tau_2^*}^2, \tau_1^* \wedge \tau_2^*), \quad S \le t \le \tau_1^* \wedge \tau_2^* \text{ a.s.}$$

Since $f^1 \ge f^2$ and $\xi^1 \ge \xi^2$, the comparison theorem for RBSDEs (Theorem 4.1) yields that $Y_{\tau_1^* \wedge \tau_2^*}^1 \ge Y_{\tau_1^* \wedge \tau_2^*}^2$ a.s. By hypothesis, $Y_S^1 = Y_S^2$. Now, Assumption (4.23) allows us to apply the strict comparison theorem for non reflected BSDEs with jumps (see [23] Th. 4.4) for terminal time $\tau_1^* \wedge \tau_2^*$. Hence, we get $Y_t^1 = Y_t^2$, $S \le t \le \tau_1^* \wedge \tau_2^*$ a.s., and equality (4.24), which provides the desired result.

Suppose now that $\xi^1 = \xi^2 = \xi$ a.s. Then, using $Y^2 \le Y^1$, we get $\tau_2^* \le \tau_1^*$ a.s. Since we have already shown that $Y_{\tau_2^*}^1 = Y_{\tau_2^*}^2$ a.s., and since $Y_{\tau_2^*}^2 = \xi_{\tau_2^*}$ a.s., we get $Y_{\tau_2^*}^1 = \xi_{\tau_2^*}$ and $\tau_1^* \le \tau_2^*$ a.s. It follows that $\tau_1^* = \tau_2^*$ a.s.

2. Let $\varepsilon > 0$. By a property of τ_1^ε (see Lemma 3.4), we have

$$Y_t^1 = X_t^1(Y_{\tau_1^\varepsilon}^1, \tau_1^\varepsilon), \quad S \le t \le \tau_1^\varepsilon \text{ a.s.}$$

Similarly, $Y_t^2 = X_t^2(Y_{\tau_2^\varepsilon}^2, \tau_2^\varepsilon)$, $S \le t \le \tau_2^\varepsilon$ a.s. By the same arguments as above with τ_1^* and τ_2^* replaced by τ_1^ε and τ_2^ε respectively, we derive the desired result.

Suppose now that $\xi^1 = \xi^2 = \xi$ a.s. Since $Y^2 \le Y^1$, we have $\tau_2^\varepsilon \le \tau_1^\varepsilon$ a.s. Moreover by Lemma 3.4, we have $\xi_{\tau_2^\varepsilon} + \varepsilon \ge Y_{\tau_2^\varepsilon}^2 = Y_{\tau_2^\varepsilon}^1$ a.s. Consequently, $\tau_2^\varepsilon \ge \tau_1^\varepsilon$ a.s. and hence $\tau_2^\varepsilon = \tau_1^\varepsilon$ a.s. By letting ε tend to 0, we get $\tilde{\tau}_1 = \tilde{\tau}_2$ a.s. □

4.2. Optimization problems for RBSDEs

Let ξ in $\mathcal{S}^2$ and let $(f, f^\alpha; \alpha \in \mathcal{A})$ be a family of Lipschitz drivers satisfying Assumption 3.1. In Assumption 3.1, the coefficient associated with f^α (resp. f), is denoted by $\theta^{\alpha,x,\pi,l}$ (resp. $\theta^{x,\pi,l}$). We denote by (Y, Z, k) the solution of the RBSDE associated to obstacle (ξ_t) and driver f, and by $(Y^\alpha, Z^\alpha, k^\alpha)$ the solution of the RBSDE associated with obstacle (ξ_t) and driver f^α.

For each $\tau \in \mathcal{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, we denote by $(X(\zeta, \tau), \pi(\zeta, \tau), l(\zeta, \tau))$ the solution of the BSDE associated with driver f, terminal conditions ζ, τ, and by $(X^\alpha(\zeta, \tau), \pi^\alpha(\zeta, \tau), l^\alpha(\zeta, \tau))$ the solution of the BSDE associated with driver f^α and terminal conditions ζ, τ.

Let $S \in \mathcal{T}_0$. We consider the following optimization problem

$$\operatorname{ess\,inf}_\alpha Y_S^\alpha. \tag{4.25}$$

We first state a characterization of the value function of this problem as well as an existence result, which generalizes a result established in [11] to the case of jumps.

Proposition 4.3 (*Optimization Principle for RBSDEs I*)**.** *Suppose that*

1. *For each $\alpha \in \mathcal{A}$, $f(t, y, z, k) \leq f^\alpha(t, y, z, k)$, for all $(y, z, k) \in \mathbf{R}^2 \times \mathcal{L}^2_\nu$; $dt \otimes dP - a.s.$*
2. *There exists $\bar{\alpha} \in \mathcal{A}$ such that*

$$f(t, Y_t, Z_t, k_t) = f^{\bar{\alpha}}(t, Y_t, Z_t, k_t), \quad 0 \leq t \leq T, dt \otimes dP - a.s. \tag{4.26}$$

Then, for each $S \in \mathcal{T}_0$,

$$Y_S = \operatorname{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\bar{\alpha}} \quad a.s. \tag{4.27}$$

Proof. For each α, since Condition 1. is satisfied and since f^α satisfies Assumption 3.1, the comparison theorem for RBSDEs yields (see Theorem 4.1) that $Y \leq Y^\alpha$. It follows that for each $S \in \mathcal{T}_0$,

$$Y_S \leq \operatorname{ess\,inf}_\alpha Y_S^\alpha \quad \text{a.s.}$$

Now, by Condition 2., Y is a solution of the RBSDE associated with $f^{\bar{\alpha}}$. By uniqueness of the solution of this RBSDE, we have $Y = Y^{\bar{\alpha}}$, which leads to equality (4.27). □

Using an estimate on RBSDEs (see (A.57)), we derive a similar characterization of the value function of the problem (4.25) under weaker hypotheses.

Proposition 4.4 (*Optimization Principle for RBSDEs II*)**.** *Suppose that the drivers f^α, $\alpha \in \mathcal{A}$ satisfy $f \leq f^\alpha$ and are equi-Lipschitz with constant C.*

Suppose moreover that for each $\eta > 0$, there exists $\alpha^\eta \in \mathcal{A}$ such that

$$f(t, Y_t, Z_t, k_t) \geq f^{\alpha^\eta}(t, Y_t, Z_t, k_t) - \eta, \quad 0 \leq t \leq T, dP \otimes dt - a.s. \tag{4.28}$$

Then, for each $S \in \mathcal{T}_0$, we have

$$Y_S = \operatorname{ess\,inf}_\alpha Y_S^\alpha \quad a.s. \tag{4.29}$$

Proof. Since $f \leq f^\alpha$, we have $Y \leq Y^\alpha$ a.s. for each $\alpha \in \mathcal{A}$. It follows that for each $S \in \mathcal{T}_0$, we have $Y_S \leq \operatorname{ess\,inf}_\alpha Y_S^\alpha$ a.s. Since Assumption (4.28) holds, by using estimate (A.57), with $\eta = \frac{1}{C^2}$ and $\beta = 3C^2 + 2C$, we derive that there exists a constant $K \geq 0$, which depends only on C and T, such that, for each $\eta > 0$ and for each $S \in \mathcal{T}_0$,

$$Y_S + K\,\eta \geq Y_S^{\alpha^\eta} \geq \operatorname{ess\,inf}_\alpha Y_S^\alpha \quad \text{a.s.}$$

Equality (4.29) thus follows. □

Remark 4.5. Propositions 4.3 and 4.4 can be seen as *verification theorems* in the following sense: let $f^\alpha, \alpha \in \mathcal{A}$ be a family of drivers. If we are given a driver $f \leq f^\alpha, \alpha \in \mathcal{A}$ satisfying (4.26) or (4.28), then the solution Y of the RBSDE with driver f coincides with the value function of the optimization problem (4.25). Under some conditions on the drivers f^α, f can be explicitly defined in terms of the family $f^\alpha, \alpha \in \mathcal{A}$; see e.g. Section 6.

By using the strict comparison theorem for reflected BSDEs (see Theorem 4.2), we provide now some necessary optimality conditions at a given time $S \in \mathcal{T}_0$.

Proposition 4.6 (*Necessary Optimality Conditions*)**.** *Suppose that the assumptions of Proposition 4.3 or Proposition 4.4 hold. Let $\hat{\alpha} \in \mathcal{A}$, and suppose that in Assumption 3.1 the* *coefficient* $\theta^{\hat{\alpha}}$ *corresponding to driver* $f^{\hat{\alpha}}$ *satisfies* $\theta^{\hat{\alpha},x,\pi,\ell_1,\ell_2} > -1$, *for each* x, π, ℓ_1, ℓ_2. *Let* $S \in \mathcal{T}_0$. *Suppose that* $\hat{\alpha}$ *is S-optimal, i.e.*

$$\operatorname{ess\,inf}_{\alpha} Y_S^{\alpha} = Y_S^{\hat{\alpha}} \quad a.s. \tag{4.30}$$

1. *Suppose* ξ *is* l.u.s.c. *along stopping times. Let* $\tau_S^* := \inf\{t \geq S, Y_t = \xi_t\}$. *Then*

$$\begin{aligned} &Y^{\hat{\alpha}}_{\tau_S^*} = \xi_{\tau_S^*} \quad a.s.; \\ &f(t, Y_t, Z_t, k_t) = f^{\hat{\alpha}}(t, Y_t, Z_t, k_t), \quad S \leq t \leq \tau_S^*, dP \otimes dt - a.s. \end{aligned} \tag{4.31}$$

2. *Consider the case when* ξ *is not supposed to be* l.u.s.c. *along stopping times.*
   For each $\varepsilon > 0$, *let* $\tau_S^{\varepsilon} := \inf\{t \geq S, Y_t \leq \xi_t + \varepsilon\}$. *Then for each* $\varepsilon > 0$,

$$\begin{aligned} &Y^{\hat{\alpha}}_{\tau_S^{\varepsilon}} \leq \xi_{\tau_S^{\varepsilon}} + \varepsilon \quad a.s.; \\ &f(t, Y_t, Z_t, k_t) = f^{\hat{\alpha}}(t, Y_t, Z_t, k_t), \quad S \leq t \leq \tau_S^{\varepsilon}, dP \otimes dt - a.s. \end{aligned} \tag{4.32}$$

Proof. By Proposition 4.3 or Proposition 4.4, we have $Y_S = ess \inf_\alpha Y_S^\alpha$ a.s. From equality (4.30), it follows that $Y_S = Y_S^{\hat{\alpha}}$ a.s.

1. Since $Y \leq Y^{\hat{\alpha}}$, it follows that $\tau_S^* \leq \tau_S^{\hat{\alpha},*}$ where $\tau_S^{\hat{\alpha},*} := \inf\{t \geq S, Y_t^{\hat{\alpha}} = \xi_t\}$. By the strict comparison Theorem 4.2 1. applied to $\xi^1 = \xi^2 = \xi$, $f^1 = f$, $f^2 = f^{\hat{\alpha}}$, $Y^1 = Y$, $Y^2 = Y^{\hat{\alpha}}$, since $Y_S = Y_S^{\hat{\alpha}}$ a.s., we derive equalities (4.31).

2. Let

$$\tau_S^{\hat{\alpha},\varepsilon} := \inf\{t \geq S, Y_t^{\hat{\alpha}} \leq \xi_t + \varepsilon\}. \tag{4.33}$$

Since $Y \leq Y^{\hat{\alpha}}$, it follows that for each $\varepsilon > 0$, we have $\tau_S^{\varepsilon} \leq \tau_S^{\hat{\alpha},\varepsilon}$ a.s. By the strict comparison Theorem 4.2 2. applied to $\xi^1 = \xi^2 = \xi$, $f^1 = f$, $f^2 = f^{\hat{\alpha}}$, $Y^1 = Y$, $Y^2 = Y^{\hat{\alpha}}$, since $Y_S = Y_S^{\hat{\alpha}}$ a.s., we derive (4.32). □

We now provide sufficient conditions of optimality at a given time $S \in \mathcal{T}_0$, which are weaker than those made in Proposition 4.3.

Proposition 4.7 (*Sufficient Optimality Conditions*)**.** *Suppose that for each* $\alpha \in \mathcal{A}$, $f \leq f^\alpha$. *Let* $\hat{\alpha} \in \mathcal{A}$ *and* $S \in \mathcal{T}_0$.

1. *Suppose* ξ *is* l.u.s.c. *along stopping times.*
   If equalities (4.31) *hold, then* $\hat{\alpha}$ *is S-optimal, that is, ess* $\inf_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ *a.s.*
2. *Consider the case when* ξ *is not supposed to be* l.u.s.c. *along stopping times.*
   If for each $\varepsilon > 0$, *conditions* (4.32) *hold, then* $\hat{\alpha}$ *is S-optimal.*

In both cases, we get $Y_S = ess \inf_\alpha Y_S^\alpha$ *a.s.*

Proof. For all α, since $f \leq f^\alpha$, we have $Y_S \leq Y_S^\alpha$ a.s. and thus $Y_S \leq \operatorname{ess\,inf}_\alpha Y_S^\alpha$ a.s.

1. Since $Y \leq Y^{\hat{\alpha}}$, it follows that $\tau_S^* \leq \tau_S^{\hat{\alpha},*}$ where $\tau_S^{\hat{\alpha},*} := \inf\{t \geq S, Y_t^{\hat{\alpha}} = \xi_t\}$. Suppose that equalities (4.31) hold. Then, by the optimality of τ_S^* for Y_S, we have

$$Y_t = X_t(\xi_{\tau_S^*}, \tau_S^*), \quad S \leq t \leq \tau_S^*, \text{a.s.}$$

This with equality (4.31) and the uniqueness result for BSDEs leads to

$$Y_t = X_t(\xi_{\tau_S^*}, \tau_S^*) = X_t^{\hat{\alpha}}(\xi_{\tau_S^*}, \tau_S^*) = X_t^{\hat{\alpha}}(Y^{\hat{\alpha}}_{\tau_S^*}, \tau_S^*), \quad S \leq t \leq \tau_S^*, \text{a.s.},$$

Moreover, according to the previous equalities, $X_t^{\hat{\alpha}}(Y^{\hat{\alpha}}_{\tau_S^*}, \tau_S^*) = Y_t \geq \xi_t$, $S \leq t \leq \tau_S^*$ a.s.

By the uniqueness result for RBSDEs, we get

$$Y_t = X_t^{\hat{\alpha}}(Y^{\hat{\alpha}}_{\tau_S^*}, \tau_S^*) = Y_t^{\hat{\alpha}}, \quad S \le t \le \tau_S^*, \text{ a.s.}$$

By taking $t = S$, we get $Y_S = \operatorname{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ a.s.

2. Since $Y \le Y^{\hat{\alpha}}$, for each $\varepsilon > 0$, we have $\tau_S^\varepsilon \le \tau_S^{\hat{\alpha},\varepsilon}$ a.s. where $\tau_S^{\hat{\alpha},\varepsilon}$ is defined in (4.33). Let us now show that $Y_S \ge Y_S^{\hat{\alpha}}$ a.s. By Lemma 3.4, we have

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \quad S \le t \le \tau_S^\varepsilon, \text{ a.s.}$$

Hence, using equality (4.32), we derive that

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = X_t^{\hat{\alpha}}(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \quad S \le t \le \tau_S^\varepsilon, \text{ a.s.}$$

By the comparison theorem for non reflected BSDEs and inequality $Y_{\tau_S^\varepsilon} \ge \xi_{\tau_S^\varepsilon}$ a.s., we have

$$Y_t = X_t^{\hat{\alpha}}(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \ge X_t^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \quad S \le t \le \tau_S^\varepsilon, \text{ a.s.}$$

Now, by a priori estimates (see [23]), we have

$$Y_S \ge X_S^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \ge X_S^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) - \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.}$$

with $\beta = 3C^2 + 2C$, where C is the Lipschitz constant of $f^{\hat{\alpha}}$. Since by assumption, $\xi_{\tau_S^\varepsilon} + \varepsilon \ge Y^{\hat{\alpha}}_{\tau_S^\varepsilon}$ a.s., the comparison theorem for non reflected BSDEs yields that

$$Y_S + \varepsilon e^{\frac{\beta T}{2}} \ge X_S^{\hat{\alpha}}(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \ge X_S^{\hat{\alpha}}(Y^{\hat{\alpha}}_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad \text{a.s.}$$

By Lemma 3.4, the nondecreasing process associated with $Y_\cdot^{\hat{\alpha}}$ is constant on $[S, \tau_S^{\hat{\alpha},\varepsilon}]$ and hence on $[S, \tau_S^\varepsilon]$, because $\tau_S^\varepsilon \le \tau_S^{\hat{\alpha},\varepsilon}$ a.s. Thus, $(Y_t^{\hat{\alpha}}, S \le t \le \tau_S^\varepsilon)$ is the solution of the non reflected BSDE associated with driver $f^{\hat{\alpha}}$, terminal conditions $(\tau_S^\varepsilon, Y^{\hat{\alpha}}_{\tau_S^\varepsilon})$. We thus get

$$X_S^{\hat{\alpha}}(Y^{\hat{\alpha}}_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = Y_S^{\hat{\alpha}} \quad \text{a.s.}$$

Consequently, for each $\varepsilon > 0$, we have $Y_S + \varepsilon e^{\frac{\beta T}{2}} \ge Y_S^{\hat{\alpha}}$ a.s., and hence, $Y_S \ge Y_S^{\hat{\alpha}}$ a.s. We thus have $Y_S = \operatorname{ess\,inf}_\alpha Y_S^\alpha = Y_S^{\hat{\alpha}}$ a.s., which provides the desired result. □

5. Robust optimal stopping problem

We now consider the optimal stopping problem when there is ambiguity on the risk-measure modeling. Let $\{f^\alpha, \alpha \in \mathcal{A}\}$ be a given family of Lipschitz drivers satisfying Assumption 3.1. For each $\alpha \in \mathcal{A}$, let ρ^α be the risk measure induced by the BSDE with driver f^α, defined as follows: for each terminal time $\tau \in \mathcal{T}_0$ and position $\zeta \in L^2(\mathcal{F}_\tau)$, set

$$\rho_t^\alpha(\zeta, \tau) := -X_t^\alpha(\zeta, \tau), \quad 0 \le t \le T,$$

where $X_t^\alpha(\zeta, \tau)$ denotes the solution of the BSDE associated with driver f^α, terminal condition ζ and terminal time τ. We consider an agent who is averse to ambiguity, and we define her risk measure of position ζ, at each time S in $\mathcal{T}_0$ with $S \le \tau$ a.s., as the supremum over α of the associated risk-measures $\rho_S^\alpha(\zeta, \tau)$ that is,

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\zeta, \tau) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} -X_S^\alpha(\zeta, \tau).$$

Let (ξ_t) be a dynamic position, given by an RCLL adapted process (ξ_t) in $\mathcal{S}^2$. At time $S \in \mathcal{T}_0$, the agent wants to find a stopping time $\tau \in \mathcal{T}_S$ which minimizes her risk measure. At time S, her value function is defined as

$$u(S) := \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_S} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\xi_\tau, \tau). \tag{5.34}$$

Let $S \in \mathcal{T}_0$. Define the *first value function at time S* as

$$\overline{V}(S) := \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} X_S^\alpha(\xi_\tau, \tau), \tag{5.35}$$

and the *second value function at time S* as

$$\underline{V}(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} X_S^\alpha(\xi_\tau, \tau). \tag{5.36}$$

Note that $\underline{V}(S) = -u(S)$ a.s.

By definition, we say that there exists a *value function* at time S for the game problem if $\underline{V}(S) = \overline{V}(S)$ a.s.

Definition 5.1 (*S-Saddle Point*)**.** Let $S \in \mathcal{T}_0$. A pair $(\hat{\tau}, \hat{\alpha}) \in \mathcal{T}_S \times \mathcal{A}$ is called a *S-saddle point* if

- $\underline{V}(S) = \overline{V}(S)$ a.s.,
- the essential infimum in (5.35) is attained at $\hat{\alpha}$,
- the essential supremum in (5.36) is attained at $\hat{\tau}$.

By classical results, for each $S \in \mathcal{T}_0$, $(\hat{\tau}, \hat{\alpha})$ is a S-saddle point if and only if for each $(\tau, \alpha) \in \mathcal{T}_S \times \mathcal{A}$,

$$X_S^{\hat{\alpha}}(\xi_\tau, \tau) \le X_S^{\hat{\alpha}}(\xi_{\hat{\tau}}, \hat{\tau}) \le X_S^\alpha(\xi_{\hat{\tau}}, \hat{\tau}) \quad \text{a.s.} \tag{5.37}$$

Note that for each $S \in \mathcal{T}_0$, the inequality $\underline{V}(S) \le \overline{V}(S)$ a.s. clearly holds. We want to determine when the equality holds, characterize the value function, and address the question of existence of a S-saddle point.

Remark 5.2. Let $S \in \mathcal{T}_0$. If $(\hat{\tau}, \hat{\alpha})$ is an S-saddle point, then $\hat{\tau}$ and $\hat{\alpha}$ attain respectively the infimum and the supremum in $\underline{V}(S)$ that is,

$$\underline{V}(S) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \operatorname*{ess\,inf}_{\alpha} X_S^\alpha(\xi_\tau, \tau) = \operatorname*{ess\,inf}_{\alpha} X_S^\alpha(\xi_{\hat{\tau}}, \hat{\tau}) = X_S^{\hat{\alpha}}(\xi_{\hat{\tau}}, \hat{\tau}).$$

Hence, $\hat{\tau}$ is an optimal stopping time for the agent who wants to minimize over stopping times her risk-measure at time S in the case of ambiguity (see (5.34)).

Also, since $\hat{\alpha}$ attains the essential infimum in (5.35), $\hat{\alpha}$ corresponds at time S to a worst-case scenario. Hence, the robust optimal stopping problem (5.34) reduces to a classical optimal stopping problem associated with a worst-case scenario among the possible ambiguity parameters $\alpha \in \mathcal{A}$.

We relate now the game problem to an optimization problem for RBSDEs.

Let $(Y^\alpha, Z^\alpha, k^\alpha)$ be the solution of the RBSDE with obstacle (ξ_t) and driver f^α. For each $\tau \in \mathcal{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $X^\alpha(\zeta, \tau)$ be the solution of the BSDE with driver f^α and terminal conditions (ζ, τ). By the characterization of RBSDEs (see Theorem 3.3), for each $S \in \mathcal{T}_0$, we have $Y_S^\alpha = \text{ess sup}_{\tau \in \mathcal{T}_S} X_S^\alpha(\xi_\tau, \tau)$ a.s. It follows that

$$\overline{V}(S) = \text{ess} \inf_{\alpha \in \mathcal{A}} Y_S^\alpha \quad \text{a.s.} \tag{5.38}$$

Let f be a Lipschitz driver satisfying Assumption 3.1. Let (Y, Z, k) be the solution of the RBSDE with obstacle (ξ_t) and driver f. For each $\tau \in \mathcal{T}_0$ and $\zeta \in L^2(\mathcal{F}_\tau)$, let $X(\zeta, \tau)$ be the solution of the BSDE with driver f and terminal conditions (ζ, τ).

Theorem 5.3 (*Existence and Characterization of the Common Value Function-I*). *Suppose that the drivers f^α, $\alpha \in \mathcal{A}$ satisfy $f \leq f^\alpha$ and are equi-Lipschitz with constant C. Suppose that there exists $\bar{\alpha}$ such that*

$$f(t, Y_t, Z_t, k_t) = f^{\bar{\alpha}}(t, Y_t, Z_t, k_t), \quad 0 \leq t \leq T, dt \otimes dP - a.s. \tag{5.39}$$

Then, there exists a value function, which is characterized as the solution of the RBSDE *with obstacle (ξ_t) and driver f, that is, for each $S \in \mathcal{T}_0$, we have*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s.$$

In particular, the minimal risk measure, defined by (5.34), satisfies for each $S \in \mathcal{T}_0$

$$u(S) = -Y_S \quad a.s.$$

Proof. Let $S \in \mathcal{T}_0$. Let us prove that $\overline{V}(S) \leq \underline{V}(S)$ a.s. By assumption (5.39) and the optimization principle for RBSDEs (see Proposition 4.3), we have:

$$\overline{V}(S) = \text{ess} \inf_{\alpha \in \mathcal{A}} Y_S^\alpha = Y_S^{\bar{\alpha}} = Y_S \quad \text{a.s.} \tag{5.40}$$

Let $\varepsilon > 0$. By a property of τ_S^ε (see Lemma 3.4), we have

$$Y_t = X_t(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon), \quad S \leq t \leq \tau_S^\varepsilon, \text{a.s.}$$

In other terms, (Y_t, Z_t, k_t) is the solution of the BSDE associated with driver f and terminal conditions $(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon)$. Now Assumption (5.39) holds. By an optimization principle for non reflected BSDEs (see [23]), we thus derive that

$$Y_S = X_S(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = \text{ess} \inf_\alpha X_S^\alpha(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad \text{a.s.} \tag{5.41}$$

Using the comparison theorem for non reflected BSDEs and the inequality $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon$ a.s., we get

$$Y_S = \text{ess} \inf_\alpha X_S^\alpha(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \leq \text{ess} \inf_\alpha X_S^\alpha(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \quad \text{a.s.} \tag{5.42}$$

By a priori estimates for non reflected BSDEs with jumps (see [23]), for each $\varepsilon > 0$ and for each $\alpha \in \mathcal{A}$, we have

$$X_S^\alpha(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \leq X_S^\alpha(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.,}$$

with $\beta = 3C^2 + 2C$, where the constant C is equal to the Lipschitz constant common to all the drivers f^α, $\alpha \in \mathcal{A}$. By taking the essential infimum over α, we derive that for each $\varepsilon > 0$,

$$\text{ess} \inf_\alpha X_S^\alpha(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \leq \text{ess} \inf_\alpha X_S^\alpha(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + \varepsilon e^{\frac{\beta T}{2}} \leq \underline{V}(S) + \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.,}$$

where the last inequality follows from the fact that

$$\underline{V}(S) = \text{ess} \sup_{\tau \in \mathcal{T}_S} \text{ess} \inf_{\alpha} X_S^{\alpha}(\xi_\tau, \tau) \quad \text{a.s.}$$

Using (5.42), we get $Y_S \leq \underline{V}(S) + \varepsilon e^{\frac{\beta T}{2}}$ a.s. Since $\overline{V}(S) = Y_S$ a.s. (see (5.40)), it follows that for each $\varepsilon > 0$, we have

$$\overline{V}(S) = Y_S \leq \underline{V}(S) + \varepsilon e^{\frac{\beta T}{2}} \quad \text{a.s.}$$

Hence, $\overline{V}(S) = Y_S \leq \underline{V}(S)$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., we get $\overline{V}(S) = Y_S = \underline{V}(S)$ a.s. □

Corollary 5.4 (*Existence of Saddle Points*). *Suppose that the assumptions of Theorem 5.3 are satisfied and that the obstacle ξ is* l.u.s.c. *along stopping times. For each $S \in \mathcal{T}_0$, let*

$$\tau_S^* := \inf\{u \geq S; Y_u = \xi_u\}.$$

Then, $(\tau_S^*, \bar{\alpha})$ is an S-saddle point, that is $Y_S = X_S^{\bar{\alpha}}(Y_{\tau_S^*}, \tau_S^*)$ a.s.

In particular, τ_S^* is an optimal stopping time for the agent who wants to minimize her risk measure at time S and $\bar{\alpha}$ corresponds to a worst scenario.

Proof. Since ξ is l.u.s.c. along stopping times, we have

$$Y_S = X_S(Y_{\tau_S^*}, \tau_S^*) = \text{ess} \inf_{\alpha} X_S^{\alpha}(Y_{\tau_S^*}, \tau_S^*) = X_S^{\bar{\alpha}}(Y_{\tau_S^*}, \tau_S^*) \quad \text{a.s.}$$

Hence $(\tau_S^*, \bar{\alpha})$ is an S-saddle point. □

This corollary generalizes a result of [11] obtained in the case of a Brownian framework and a continuous obstacle.

We now show the existence of an S-saddle point under weaker assumptions for fixed S in $\mathcal{T}_0$.

Proposition 5.5 (*Existence of S-Saddle Points*). *Suppose that for each α in $\mathcal{A}$, $f \leq f^{\alpha}$. Let S in $\mathcal{T}_0$. Suppose that the obstacle ξ is* l.u.s.c. *along stopping times. Suppose that there exists $\bar{\alpha}$ such that*

$$\begin{aligned} &Y_{\tau_S^*}^{\bar{\alpha}} = \xi_{\tau_S^*} \quad \text{a.s.} \quad \text{and} \\ &f(t, Y_t, Z_t, k_t) = f^{\bar{\alpha}}(t, Y_t, Z_t, k_t), \quad S \leq t \leq \tau_S^*, dP \otimes dt - a.s. \end{aligned} \tag{5.43}$$

Then, $(\tau_S^*, \bar{\alpha})$ is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

Proof. The result follows from the same arguments as above and the sufficient optimality conditions for RBSDEs optimization (see Proposition 4.7 2.). □

We now show that there exists a value function under weaker hypotheses.

Theorem 5.6 (*Existence and Characterization of the Common Value Function-II*). *Suppose that for each $\alpha \in \mathcal{A}$, $f \leq f^{\alpha}$. Suppose that for each $\eta > 0$, there exists $\alpha^{\eta} \in \mathcal{A}$ such that*

$$f(t, Y_t, Z_t, k_t) \geq f^{\alpha^{\eta}}(t, Y_t, Z_t, k_t) - \eta, \quad 0 \leq t \leq T, dP \otimes dt - a.s. \tag{5.44}$$

Then, for each $S \in \mathcal{T}_0$, the equality $Y_S = \overline{V}(S) = \underline{V}(S)$ holds a.s.

Proof. By Proposition 4.4, we know that $Y_S = \text{ess} \inf_{\alpha} Y_S^{\alpha} = \overline{V}(S)$ a.s.

For each $\varepsilon > 0$, by a property of τ_S^{ε} (see Lemma 3.4), we have

$$Y_S = X_S(Y_{\tau_S^{\varepsilon}}, \tau_S^{\varepsilon}) \quad \text{a.s.}$$

Now, Assumption (5.44) holds. Applying an optimization principle for non reflected BSDE (see [23], Th. 4.6), we derive that

$$Y_S = X_S(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) = \operatorname{ess\,inf}_{\alpha} X_S^\alpha(Y_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \quad \text{a.s.}$$

The end of the proof is the same as that of Theorem 5.3. □

From the above theorems, the following saddle point criterium clearly follows.

Corollary 5.7 (*Saddle Point Criterium*). *Suppose that the assumptions of Theorem 5.3 or Theorem 5.6 are satisfied. Let $S \in \mathcal{T}_0$. For each stopping time $\hat{\tau} \in \mathcal{T}_S$ and for each $\hat{\alpha} \in \mathcal{A}$, the pair $(\hat{\tau}, \hat{\alpha})$ is an S-saddle point if and only if $\hat{\tau}$ is an optimal stopping time for $Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau)$ and $\hat{\alpha}$ is optimal for $Y_S = \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} Y_S^\alpha$.*

Proof. By Theorem 5.3 or 5.6, we have $\underline{V}(S) = \overline{V}(S) = Y_S$ a.s. The result follows from the definition of an S-saddle point (see Definition 5.1). □

Remark 5.8. Proposition 3.5 gives some necessary conditions for a stopping time $\hat{\tau}$ to be S-optimal, and Proposition 4.6 gives some necessary conditions for a coefficient $\hat{\alpha}$ to be S-optimal. Consequently, under the assumptions of Corollary 5.7, we obtain necessary conditions for a pair $(\hat{\tau}, \hat{\alpha})$ to be an S-saddle point.

6. Application to the case of multiple priors

We now apply the results of Section 5 to an optimal stopping problem for dynamic risk-measures in the case of multiple priors. Let A be a Polish space (or a Borelian subset of a Polish space) and let $\mathcal{A}$ be the set of A-valued predictable processes α. With each coefficient $\alpha \in \mathcal{A}$, is associated a model via a probability measure Q^α called *prior* as well as a dynamic risk measure ρ^α. More precisely, for each $\alpha \in \mathcal{A}$, let Z^α be the solution of the SDE:

$$dZ_t^\alpha = Z_{t^-}^\alpha \left(\beta^1(t, \alpha_t) dW_t + \int_U \beta^2(t, \alpha_t, u) d\tilde{N}(dt, du) \right); \quad Z_0^\alpha = 1,$$

where $\beta^1 : (t, \omega, \alpha) \mapsto \beta^1(t, \omega, \alpha)$, is a $\mathcal{P} \otimes \mathcal{B}(A)$-measurable function defined on $[0, T] \times \Omega \times A$ and valued in $[-C, C]$, with $C > 0$, and $\beta^2 : (t, \omega, \alpha, u) \mapsto \beta^2(t, \omega, \alpha, u)$ is a $\mathcal{P} \otimes \mathcal{B}(A) \otimes \mathcal{U}$-measurable function defined on $[0, T] \times \Omega \times A \times U$ which satisfies $dt \otimes dP \otimes d\nu(u)$-a.s.

$$\beta^2(t, \alpha, u) \geq C_1 \quad \text{and} \quad |\beta^2(t, \alpha, u)| \leq \psi(u), \tag{6.45}$$

with $C_1 > -1$ and ψ is a bounded function in L_ν^p for all $p \geq 1$. Hence, $Z_T^\alpha > 0$ a.s. and, by Proposition A1 in [23], $Z_T^\alpha \in L^p(\mathcal{F}_T)$ for all $p \geq 1$. We suppose that β^1 and β^2 are continuous with respect to α. For each $\alpha \in \mathcal{A}$, let Q^α be the probability measure equivalent to P which admits Z_T^α as density with respect to P on $\mathcal{F}_T$. By Girsanov's theorem, under Q^α, the process $W_t^\alpha := W_t - \int_0^t \beta^1(s, \alpha_s) ds$ is a Brownian motion and N is a Poisson random measure with compensated process $\tilde{N}^\alpha(dt, du) = \tilde{N}(dt, du) - \beta^2(t, \alpha_t, u)\nu(du)dt$ independent from W^α.

For each α, we are going to define a dynamic risk measure induced by a BSDE under Q^α and driven by W^α and $\tilde{N}^\alpha$, with a driver defined as follows.

Let $F : [0, T] \times \Omega \times \mathbf{R} \times L_\nu^2 \times A \to \mathbf{R}$; $(t, \omega, \pi, \ell, \alpha) \mapsto F(t, \omega, \pi, \ell, \alpha)$, be a given $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(L_\nu^2) \otimes \mathcal{B}(A)$-measurable function. Suppose F is uniformly Lipschitz with respect to (π, ℓ), continuous with respect to α, and such that $\operatorname{ess\,sup}_{\alpha \in A} |F(\cdot, t, 0, 0, 0, \alpha)| \in \mathbb{H}^{p,T}$, for each $p \geq 2$. Suppose also that

$$F(t, \pi, l_1, \alpha) - F(t, \pi, l_2, \alpha) \geq \langle \theta_t^{\pi, l_1, l_2, \alpha}, l_1 - l_2 \rangle_\nu, \tag{6.46}$$

for some adapted process $\theta_t^{\pi, l_1, l_2, \alpha}(\cdot)$ satisfying $|\theta_t^{\pi, l_1, l_2, \alpha}(u)| \leq \bar{\psi}(u)$, where $\bar{\psi}$ is bounded and in L_ν^p, for all $p \geq 1$, and $\theta_t^{\pi, l_1, l_2, \alpha} \geq -1 - C_1$. For each $\alpha \in \mathcal{A}$, the associated driver is given by $F(t, \omega, \pi, \ell, \alpha_t(\omega))$. Note that these drivers are equi-Lipschitz.

For each $\alpha \in \mathcal{A}$, let ρ^α be the dynamic risk-measure induced by the BSDE associated with $F(., \alpha_t)$ and driven by W^α and $\tilde{N}^\alpha$. More precisely, for each $\tau \in \mathcal{T}_0$ and $\zeta \in L^p(\mathcal{F}_\tau)$ with $p > 2$, there exists a unique solution $(X^\alpha, \pi^\alpha, l^\alpha)$ in $\mathcal{S}_\alpha^2 \times \mathbb{H}_\alpha^2 \times \mathbb{H}_{\alpha,\nu}^2$ of the Q^α-BSDE

$$-dX_t^\alpha = F(t, \pi_t^\alpha, l_t^\alpha, \alpha_t)dt - \pi_t^\alpha dW_t^\alpha - \int_U l_t^\alpha(u) \tilde{N}^\alpha(dt, du); \qquad X_\tau^\alpha = \zeta, \tag{6.47}$$

driven by W^α and $\tilde{N}^\alpha$. This makes sense since we have a Q^α-martingale representation property (see [23], Lemma 5.7).

The dynamic risk-measure $\rho^\alpha(\zeta, \tau)$ is thus well defined by

$$\rho_t^\alpha(\zeta, \tau) := -X_t^\alpha(\zeta, \tau), \quad 0 \leq t \leq \tau, \tag{6.48}$$

with $X^\alpha(\zeta, \tau) = X^\alpha$. Assumption (6.46) yields the monotonicity property of ρ^α.

We consider an ambiguity averse agent. Her risk measure is given, for each $\tau \in \mathcal{T}_S$ and $\zeta \in L^p(\mathcal{F}_\tau)$, $p > 2$, by

$$\operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\zeta, \tau) = -\operatorname{ess\,inf}_{\alpha \in \mathcal{A}} X_S^\alpha(\zeta, \tau) \tag{6.49}$$

at each stopping time $S \in \mathcal{T}_0$. The financial dynamic position is given here by an RCLL process (ξ_t) which belongs to $\mathcal{S}^p$, for some $p > 2$. At fixed time $S \in \mathcal{T}_0$, the agent wants to choose a stopping time in $\mathcal{T}_S$ so that it minimizes (6.49), which leads to the mixed control/optimal stopping problem:

$$u(S) := \operatorname{ess\,inf}_{\tau \in \mathcal{T}_S} \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \rho_S^\alpha(\xi_\tau, \tau) = -\operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} X_S^\alpha(\xi_\tau, \tau), \tag{6.50}$$

which corresponds to that studied in Section 5.

Theorem 6.1. *Let (Y, Z, k) be the solution of the* RBSDE *associated with obstacle (ξ_t) and Lipschitz driver f, defined for each (t, ω, π, ℓ) by*

$$f(t, \omega, \pi, \ell) := \inf_{\alpha \in A} \{F(t, \omega, \pi, \ell, \alpha) + \beta^1(t, \omega, \alpha)\pi + \langle \beta^2(t, \omega, \alpha), \ell \rangle_\nu\}. \tag{6.51}$$

For each $S \in \mathcal{T}_0$, we have

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s.$$

In particular, $u(S) = -Y_S$ a.s.

Proof. In order to prove this result, we express the problem in terms of BSDEs and RBSDEs under probability P and then apply Theorem 5.6.

Fix $\tau \in \mathcal{T}_0$ and $\zeta \in L^p(\mathcal{F}_\tau)$ with $p > 2$. Since $(X^\alpha, \pi^\alpha, l^\alpha)$ is the solution of BSDE (6.47), it clearly satisfies the following P-BSDE driven by W and $\tilde{N}$

$$-dX_t^\alpha = f^\alpha(t, \pi_t^\alpha, l_t^\alpha)dt - \pi_t^\alpha dW_t - \int_U l_t^\alpha(u) \tilde{N}(dt, du); \qquad X_\tau^\alpha = \zeta, \tag{6.52}$$

where the driver is given by

$$f^\alpha(t,\pi,\ell) := F(t,\pi,\ell,\alpha_t) + \beta^1(t,\alpha_t)\pi + \langle \beta^2(t,\alpha_t),\ell\rangle_\nu. \tag{6.53}$$

The process $(X^\alpha,\pi^\alpha,l^\alpha)$ is the solution of P-BSDE (6.52) in $\mathcal{S}^2\times\mathbb{H}^2\times\mathbb{H}^2_\nu$ (see the proof of Th. 5.9 in [23]). Moreover, for each α, f^α satisfies Assumption 3.1, and f, defined by (6.51), is a Lipschitz driver (see [23]). By the definition of f and f^α (see (6.53)), we get that for each $\alpha \in \mathcal{A}$, $f \le f^\alpha$. For each $\eta > 0$ and each $(t,\omega,\pi,l)\in\Omega\times[0,T]\times\mathbf{R}\times L^2_\nu$, there exists $\alpha^\eta\in A$ such that

$$f(t,\omega,\pi,\ell)+\eta \ge F(t,\omega,\pi,\ell,\alpha^\eta) + \beta^1(t,\omega,\alpha^\eta)\pi + \langle\beta^2(t,\omega,\alpha^\eta),\ell\rangle_\nu.$$

By the section theorem of [7], for each $\eta>0$, there exists an A-valued predictable process (α^η_t) such that $f(t,Z_t,k_t)+\eta \ge f^{\alpha^\eta}(t,Z_t,k_t)$. By Theorem 5.6, the result follows. □

Corollary 6.2 (*Existence of Saddle Points*)**.** *Suppose A is compact and that the position (ξ_t) is* l.u.s.c. *along stopping times. Then, there exists $\bar\alpha\in A$ such that*

$$f(t,Y_t,Z_t,k_t) = f^{\bar\alpha}(t,Y_t,Z_t,k_t), \quad 0\le t\le T, dt\otimes dP - a.s. \tag{6.54}$$

Also, for each $S\in\mathcal{T}_0$, the pair $(\tau^*_S,\bar\alpha)$ is an S-saddle point, where $\tau^*_S=\inf\{u\ge S;\, Y_u=\xi_u\}$.

The stopping time τ^* is optimal for the agent who wants to minimize her risk measure at time S, and $Q^{\bar\alpha}$ corresponding to a worst case scenario.

This result still holds in the case when A, instead of being compact, is a bounded, convex and closed subset of a separable Hilbert space, and if F, β^1 and β^2 are convex and lower semicontinuous with respect to α.

Proof. Since A is compact and that F, β^1 and β^2 are continuous with respect to α, the section theorem of [7] provides the existence of $\bar\alpha\in A$ such that (6.54) is satisfied. By Corollary 5.4, $(\tau^*_S,\bar\alpha)$ is thus an S-saddle point. Let us now consider the second case. By convex analysis arguments, one can show the existence of $\bar\alpha\in A$ satisfying equality (6.54) (for details, see the proof of Theorem 5.2 in [23]). The result follows. □

Example. Suppose that L^2_ν is separable and that A is a Borelian of the Hilbert space $\mathbf{R}\times L^2_\nu$ such that $A\subset[-K,K]\times\Upsilon$, where

$$\Upsilon := \{\varphi\in\mathcal{P},\, C_1'\le\varphi(u) \text{ and } |\varphi(u)|\le\psi(u)\nu(du) \text{ a.s.}\},$$

with $C_1'>-1$ and ψ is bounded and in L^p_ν, for all $p\ge1$. For each process $\alpha:=(\alpha^1,\alpha^2)\in\mathcal{A}$, the prior Q^α is defined as the probability measure which admits Z^α_T as density with respect to P, Z^α being the solution of

$$dZ^\alpha_t = Z^\alpha_{t^-}\left(\alpha^1_t dW_t + \int_U \alpha^2_t(u)d\tilde N(dt,du)\right); \qquad Z^\alpha_0=1.$$

Theorem 6.1 and Corollary 6.2 then hold.

Appendix

Proof of Lemma 2.5. For each $S\in\mathcal{T}_0$, define $\overline{Y}(S)$ as

$$\overline{Y}(S) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_S} E\left[\xi_\tau + \int_S^\tau f(t)dt \mid \mathcal{F}_S\right]. \tag{A.55}$$

By classical results of optimal stopping theory, there exists an RCLL adapted process $(\overline{Y}_t)$ such that for each $S \in \mathcal{T}_0$, $\overline{Y}(S) = \overline{Y}_S$ a.s. The process $(\overline{Y}_t + \int_0^t f(s)ds)$ is a supermartingale. By the Doob–Meyer decomposition, it can be uniquely written as $d\overline{Y}_t = -f(t)dt - dA_t + dM_t$, where M is a square-integrable martingale and A is a nondecreasing RCLL predictable process with $E(A_T^2) < \infty$ and $A_0 = 0$. Furthermore, by the theorem of representation [25], there exist unique processes Z in $\mathbb{H}^2$ and k in $\mathbb{H}^2_\nu$ such that $dM_t = Z_t dW_t + \int_U k_t(u)\tilde{N}(dt, du)$. The process A can be uniquely decomposed as $dA_t = dA_t^c + dA_t^d$. We have $\int_0^T (\overline{Y}_t - \xi_t)dA_t^c = 0$ a.s. and $\Delta A_t^d = -\Delta \overline{Y}_t \mathbf{1}_{\{\overline{Y}_{t^-} = \xi_{t^-}\}}$ a.s. (see e.g. Prop. B.11 in [20] or [9]). Hence, $(\overline{Y}, Z, k(), A)$ is a solution of RBSDE (2.3) associated with driver $f(t)$ and obstacle (ξ_t).

When (ξ_t) is l.u.s.c. along stopping times, A is continuous (see e.g. Prop. B.10 in [20]).

Let us now prove the uniqueness. Let $(Y, Z, k(\cdot), A)$ be a solution of the RBSDE associated with driver $f(t)$ and obstacle (ξ_t). Let us show that $Y = \overline{Y}$, with $\overline{Y}$ given by (A.55). By (2.3), the process $Y_\cdot + \int_0^\cdot f(s)ds$ is a supermartingale. Since $Y \geq \xi$, it follows that for each $S \in \mathcal{T}_0$ and $\tau \in \mathcal{T}_S$,

$$Y_S \geq E\left[Y_\tau + \int_S^\tau f(s)ds \mid \mathcal{F}_S\right] \geq E\left[\xi_\tau + \int_S^\tau f(s)ds \mid \mathcal{F}_S\right] \quad \text{a.s.}$$

By taking the supremum over $\tau \in \mathcal{T}_S$, we get $Y_S \geq \overline{Y}(S)$ a.s. It remains to show the converse inequality. Fix $\varepsilon > 0$. By definition of τ_S^ε, $Y_t > \xi_t + \varepsilon$ on $[S, \tau_S^\varepsilon[$ a.s. Hence, the process $A^c_\cdot$ is constant on $[S, \tau_S^\varepsilon]$ and $A^d_\cdot$ is constant on $[S, \tau_S^\varepsilon[$ a.s. Also, $Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s., which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. The process $Y_\cdot + \int_0^\cdot f(s)ds$ is thus a martingale on $[S, \tau_S^\varepsilon]$. Also, by the right-continuity of (ξ_t) and (Y_t), we clearly have $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon$ a.s. We derive

$$\begin{aligned} Y_S &= E\left[Y_{\tau_S^\varepsilon} + \int_S^{\tau^\varepsilon} f(s)ds \mid \mathcal{F}_S\right] \\ &\leq E\left[\xi_{\tau_S^\varepsilon} + \int_S^{\tau^\varepsilon} f(s)ds \mid \mathcal{F}_S\right] + \varepsilon \leq \overline{Y}(S) + \varepsilon \quad \text{a.s.} \end{aligned} \tag{A.56}$$

for each $\varepsilon > 0$, which implies that $Y_S \leq \overline{Y}(S)$ a.s. Hence, $Y_S = \overline{Y}(S)$ a.s. The uniqueness of Y follows. The uniqueness of Z, k, A follows from the uniqueness of the Doob–Meyer decomposition of supermartingales and of the martingale representation.

Moreover, the ε-optimality property of τ_S^ε follows from (A.56). □

We provide below some a priori estimates which are used in the proof of Proposition 4.4. For $\beta > 0$ and $\phi \in \mathbb{H}^{2,T}$, we introduce the norm $\|\phi\|^2_{\beta,T} := E[\int_0^T e^{\beta s}\phi_s^2 ds]$ and for $l \in \mathbb{H}^{2,T}_\nu$, we set $\|l\|^2_{\nu,\beta,T} := E[\int_0^T e^{\beta s}\|l_s\|_\nu^2 ds]$.

Proposition A.1. *Let $T > 0$ and let $\xi \in \mathcal{S}^2$. Let f^1 be a Lipschitz driver with Lipschitz constant C and let f^2 be a driver. For $i = 1, 2$, let (Y^i, Z^i, k^i, A^i) be a solution of the* RBSDE *associated to terminal time T, driver f^i and obstacle ξ. For s in $[0, T]$, denote $\bar{Y}_s := Y_s^1 - Y_s^2$, $\bar{Z}_s := Z_s^1 - Z_s^2$, $\bar{k}_s := k_s^1 - k_s^2$, and $\bar{f}(s) := f^1(s, Y_s^2, Z_s^2, k_s^2) - f^2(s, Y_s^2, Z_s^2, k_s^2)$. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$. If $\eta \leq \frac{1}{C^2}$, then, for each $t \in [0, T]$, we have*

$$e^{\beta t}\bar{Y}_t^2 \leq \eta\, E\left[\int_t^T e^{\beta s}\bar{f}(s)^2 ds \mid \mathcal{F}_t\right] \quad \text{a.s.} \tag{A.57}$$

$$\|\bar{Y}\|_\beta^2 \leq T\eta\|\bar{f}\|_\beta^2. \tag{A.58}$$

Also, if $\eta < \frac{1}{C^2}$, *we then have*

$$\|\bar{Z}\|_\beta^2 + \|\bar{k}\|_{\nu,\beta}^2 \leq \frac{\eta}{1-\eta C^2}\|\bar{f}\|_\beta^2. \tag{A.59}$$

Proof. From Itô's formula applied to the semimartingale $e^{\beta s}\bar{Y}_s$ between t and T, it follows

$$\begin{aligned}
&e^{\beta t}\bar{Y}_t^2 + \beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}\bar{Z}_s^2 ds + \int_t^T e^{\beta s}\|\bar{k}_s\|_\nu^2 ds + \sum_{t<s\leq T} e^{\beta s}(\Delta A_s^1 - \Delta A_s^2)^2 \\
&\quad = 2\int_t^T e^{\beta s}\bar{Y}_s(f^1(s, Y_s^1, Z_s^1, k_s^1) - f^2(s, Y_s^2, Z_s^2, k_s^2))ds \\
&\qquad - 2\int_t^T e^{\beta s}\bar{Y}_s\bar{Z}_s dW_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{Y}_{s^-}\bar{k}_s(u) + \bar{k}_s(u)^2)d\tilde{N}(du, dt) \\
&\qquad + 2\int_t^T e^{\beta s}\overline{Y}_{s^-}dA_s^1 - 2\int_t^T e^{\beta s}\overline{Y}_{s^-}dA_s^2
\end{aligned} \tag{A.60}$$

Now, we have a.s.

$$\overline{Y}_s dA_s^{1,c} = (Y_s^1 - \xi_s)dA_s^{1,c} - (Y_s^2 - \xi_s)dA_s^{1,c} = -(Y_s^2 - \xi_s)dA_s^{1,c} \leq 0$$

and by symmetry, $\overline{Y}_s dA_s^{2,c} \geq 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A_s^{1,d} = (Y_{s^-}^1 - \xi_{s^-})\Delta A_s^{1,d} - (Y_{s^-}^2 - \xi_{s^-})\Delta A_s^{1,d} = -(Y_{s^-}^2 - \xi_{s^-})\Delta A_s^{1,d} \leq 0$$

and $\overline{Y}_{s^-}\Delta A_s^{2,d} \geq 0$ a.s. Consequently, the two last terms of the r.h.s. of (A.60) are non positive. Taking the conditional expectation given $\mathcal{F}_t$, we get

$$\begin{aligned}
&e^{\beta t}\bar{Y}_t^2 + E\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\quad \leq 2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{Y}_s(f^1(s, Y_s^1, Z_s^1, k_s^1) - f^2(s, Y_s^2, Z_s^2, k_s^2))ds \mid \mathcal{F}_t\right].
\end{aligned} \tag{A.61}$$

Moreover,

$$\begin{aligned}
|f^1(s, Y_s^1, Z_s^1, k_s^1) - f^2(s, Y_s^2, Z_s^2, k_s^2)| &\leq |f^1(s, Y_s^1, Z_s^1, k_s^1) - f^1(s, Y_s^2, Z_s^2, k_s^2)| + |\bar{f}_s| \\
&\leq C|\bar{Y}_s| + (C|\bar{Z}_s| + C\|\bar{k}_s\|_\nu + |\bar{f}_s|).
\end{aligned}$$

Now, for all real numbers y, z, k, f and $\varepsilon > 0$

$2y(Cz + Ck + f) \leq \frac{y^2}{\varepsilon^2} + \varepsilon^2(Cz + Ck + f)^2 \leq \frac{y^2}{\varepsilon^2} + 3\varepsilon^2(C^2y^2 + C^2k^2 + f^2)$. Hence, we get

$$\begin{aligned}
&e^{\beta t}\bar{Y}_t^2 + E\left[\beta\int_t^T e^{\beta s}\bar{Y}_s^2 ds + \int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\quad \leq \mathbb{E}\left[\left(2C + \frac{1}{\varepsilon^2}\right)\int_t^T e^{\beta s}\bar{Y}_s^2 ds + 3C^2\varepsilon^2\int_t^T e^{\beta s}(\bar{Z}_s^2 + \|\bar{k}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\qquad + 3\varepsilon^2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{f}_s^2 ds \mid \mathcal{F}_t\right].
\end{aligned} \tag{A.62}$$

Let us make the change of variable $\eta = 3\epsilon^2$. Then, for each $\beta, \eta > 0$ chosen as in the proposition, these inequalities lead to (A.57). We obtain the first inequality of (A.58) by integrating (A.57). Then (A.59) follows from inequality (A.62). □

By classical results on the norms of semimartingales, one similarly shows that $\|\bar{Y}\|_{\mathcal{S}^2} \leq K\|\bar{f}\|_{\mathbb{H}^2}$, where K is a positive constant only depending on T and C.

Proof of Theorem 2.6. Denote by $\mathbb{H}^2_\beta$ the space $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ equipped with the norm $\|Y, Z, k(\cdot)\|^2_\beta := \|Y\|^2_\beta + \|Z\|^2_\beta + \|k\|^2_{\nu,\beta}$. We define a mapping Φ from $\mathbb{H}^2_\beta$ into itself as follows. Given $(U, V, l) \in \mathbb{H}^2_\beta$, let $(Y, Z, k) = \Phi(U, V, l)$ be the solution of the RBSDE associated with driver $f^1(s) := f(s, U_s, V_s, l_s)$. Let A be the associated nondecreasing process. The mapping Φ is well defined by Lemma 2.5. Let us prove that the mapping Φ is a contraction from $\mathbb{H}^2_\beta$ into $\mathbb{H}^2_\beta$. Let (U', V', l') be another element of $\mathbb{H}^2_\beta$ and let $(Y', Z', k') := \Phi(U', V', l')$, that is, the solution of the RBSDE associated with driver process $f(s, U'_s, V'_s, l'_s)$. Set $\bar{U} := U - U'$, $\bar{V} := V - V'$, $\bar{l} := l - l'$, $\bar{Y} := Y - Y'$, $\bar{Z} := Z - Z'$ $\bar{k} := k - k'$. Let $\Delta f_\cdot := f(\cdot, U, V, l) - f(\cdot, U', V', l')$. Using estimates (A.58) and (A.59) with $\eta \leq \frac{1}{2C^2}$ and Lipschitz constant equal to 0 (since the driver f^1 does not depend on the solution), we get

$$\|\bar{Y}\|^2_\beta + \|\bar{Z}\|^2_\beta + \|\bar{k}\|^2_{\nu,\beta} \leq \eta(T+2)\|\Delta f\|^2_\beta \leq \eta(T+2)2C^2(\|\bar{U}\|^2_\beta + \|\bar{V}\|^2_\beta + \|\bar{l}\|^2_{\nu,\beta}),$$

where the second inequality follows from the Lipschitz property of f with constant C. Choosing $\eta = \frac{1}{(T+2)4C^2}$, we deduce $\|(\overline{Y}, \overline{Z}, \bar{k})\|^2_\beta \leq \frac{1}{2}\|(\overline{U}, \overline{V}, \bar{l})\|^2_\beta$. Hence, Φ is a contraction and thus admits a unique fixed point (Y, Z, k) in $\mathbb{H}^2_\beta$, which is the solution of RBSDE (2.3). □

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